# A Development of Set Theory in Fuzzy Logic

Petr Hájek and Zuzana Haniková

Institute of Computer Science
182 07 Praha 8
Czech Republic

**Abstract.** This paper presents an axiomatic set theory FST ('Fuzzy Set Theory'), as a first-order theory within the framework of fuzzy logic in the style of [4]. In the classical ZFC, we use a construction similar to that of a Boolean-valued universe—over an algebra of truth values of the logic we use—to show the nontriviality of FST. We give the axioms of FST. Finally we show that FST interprets ZF.

## 1 Introduction

If anything comes to people's minds on the term 'fuzzy set theory' being used, it usually is the theory (or theories) handling fuzzy sets as real-valued functions on a fixed universe within the classical set-theoretic universe. However, as "fuzzy" or "many-valued" logic has been evolving into a formal axiomatic theory, a few exceptions from this general expectation have emerged: these are formal "set" theories *within* the respective many-valued logics, i.e., theories whose underlying logic is governed by a many-valued semantics.

We rely especially on those works which develop a theory in the language and style of the classical Zermelo-Fraenkel set theory (ZF). We have been inspired by a series of papers developing a theory generalizing ZF in a formally weaker logic—intuitionistic ([7], [2]) and later its strengthening commonly referred to as Gödel logic ([9], [10], [11]); some results and proofs carry over to our system. For an important example, the axiom of foundation, together with a very weak fragment of ZF, implies the law of the excluded middle, which yields the full classical logic (both in Gödel logic and in the logic we use in this paper), and thus the theory developed becomes crisp. For this reason we start with building a non-crisp universe in which we verify our axioms.

However, all these papers fall short in tackling one of the distinguishing traits of many-valued logic, which is the general non-idempotence of the conjunction (conjunction is idempotent in Gödel logic). The non-idempotence of conjunction affects the resulting theory considerably (cf. [3]); in coping with some of the difficulties we appreciated an elegant solution found in [8]. The author works over the so-called phase spaces as algebras of truth values and builds, using an analogy of the construction of a Boolean-valued universe (over a phase-space instead of a Boolean algebra), a class, with class operations evaluating formulas in the language $\{\in, \subseteq, =\}$, in which he verifies the chosen axioms of his set theory. Having observed that the standard Łukasiewicz algebra enriched with the $\Delta$ operator is a particular phase space, we have studied this paper thinking of a more general approach employing (linearly ordered) BL-algebras with $\Delta$; this should form a common generalization of the approach of [9] and [10] and that of [8].

This paper is an extension of [6]; it brings in a simplified definition of the initial universe, and an inner model of ZF in FST. Regarding the nature of this paper we omit most of the logical background, which is to be found mainly in [4], as well as some technical details and some proofs.

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## 2 Prerequisites

**Definition 1.** BL$\forall\Delta$ is a formal logical system with basic connectives $\&$, $\rightarrow$, $\overline{0}$ and $\Delta$, defined connectives $\neg$, $\wedge$, $\vee$ and $\equiv$—where $\neg\varphi$ is $\varphi \rightarrow \overline{0}$, $\varphi \wedge \psi$ is $\varphi\&(\varphi \rightarrow \psi)$, $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, and $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)$—and with two quantifiers $\forall$ and $\exists$. The axioms are as follows:
(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi\&\psi) \rightarrow \varphi$
(A3) $(\varphi\&\psi) \rightarrow (\psi\&\varphi)$
(A4) $(\varphi\&(\varphi \rightarrow \psi)) \rightarrow (\psi\&(\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi\&\psi) \rightarrow \chi)$
(A5b) $((\varphi\&\psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $\overline{0} \rightarrow \varphi$
($\Delta$1) $\Delta\varphi \vee \neg\Delta\varphi$
($\Delta$2) $\Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$
($\Delta$3) $\Delta\varphi \rightarrow \varphi$
($\Delta$4) $\Delta\varphi \rightarrow \Delta\Delta\varphi$
($\Delta$5) $\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$
($\forall$1) $\forall x\varphi(x) \rightarrow \varphi(t)$ ($t$ substitutable for $x$ in $\varphi$)
($\exists$1) $\varphi(t) \rightarrow \exists x\varphi(x)$ ($t$ substitutable for $x$ in $\varphi$)
($\forall$2) $\forall x(\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \forall x\varphi)$ ($x$ not free in $\chi$)
($\exists$2) $\forall x(\varphi \rightarrow \chi) \rightarrow (\exists x\varphi \rightarrow \chi)$ ($x$ not free in $\chi$)
($\forall$3) $\forall x(\varphi \vee \chi) \rightarrow (\forall x\varphi \vee \chi)$ ($x$ not free in $\chi$).
The deduction rules of BL$\forall\Delta$ are modus ponens, generalization, and $\{\varphi/\Delta\varphi\}$.

**Definition 2.** Let $\mathcal{C}$ be an arbitrary schematic extension of BL$\forall\Delta$. A $\mathcal{C}$-algebra is a BL$\Delta$-algebra $L$ in which all the axioms of $\mathcal{C}$ are $L$-tautologies.

**Theorem 3. (Strong completeness)** *Let $\mathcal{C}$ be a schematic extension of* BL$\forall\Delta$, *let $T$ be a theory over $\mathcal{C}$, and let $\varphi$ be a formula of the language of $T$. Then $T$ proves $\varphi$ iff $\varphi$ holds in any safe model $M$ of $T$ over any linearly ordered $\mathcal{C}$-algebra.*

## 3 The Initial Universe

Consider the classical ZFC. Fix $\mathcal{C}$ as a schematic extension of BL$\forall\Delta$, and fix a constant $L$ for an arbitrary linearly ordered complete $\mathcal{C}$-algebra; write $L = (L, *, \Rightarrow, \wedge, \vee, 0, 1)$ (as usual, $L$ denotes both the algebra and its support). In ZFC let us make the following construction: in analogy to the construction of a Boolean-valued universe over a complete Boolean algebra, we build the class $V^L$ by ordinal induction. Define $L^+ = L - \{0\}$.

$$V_0^L = \{\emptyset\}$$

$$V_{\alpha+1}^L = \{f : \mathrm{Fnc}(f) \,\&\, D(f) \subseteq V_\alpha^L \,\&\, R(f) \subseteq L^+\}$$

for any ordinal $\alpha$, and for limit ordinals $\lambda$

$$V_\lambda^L = \bigcup_{\alpha<\lambda} V_\alpha^L$$

Here $\mathrm{Fnc}(x)$ is a unary predicate stating that $x$ is a function, and $D(x)$ and $R(x)$ are unary functions assigning to $x$ its domain and range, respectively.

Note that functions taking the value 0 on any element of their domain are not considered as elements of the universe.

Finally we put

$$V^L = \bigcup_{\alpha\in\mathrm{On}} V_\alpha^L$$

Observe that for $\alpha \leq \beta$, $V_\alpha^L \subseteq V_\beta^L$.

We define two binary functions from $V^L$ into $L$, assigning to any tuple $x, y \in V^L$ the values $||x \in y||$ and $||x = y||$ (representing the "truth values" of the two predicates $\in$ and $=$):
$||x \in y|| = y(x)$ if $x \in D(y)$, otherwise 0,
$||x = y|| = 1$ if $x = y$, otherwise 0.

We now use induction on the complexity of formulas to define for any formula $\varphi(x_1, \ldots, x_n)$—the free variables in $\varphi$ being $x_1, \ldots, x_n$—a corresponding $n$-ary function from $(V^L)^n$ into $L$. The induction steps admit the following cases (we just write $\varphi$ for short):
$\varphi$ is $\overline{0}$: then $||\varphi|| = 0$;
$\varphi$ is $\psi \,\&\, \chi$; then $||\varphi|| = ||\psi|| * ||\chi||$;
$\varphi$ is $\psi \rightarrow \chi$: then $||\varphi|| = ||\psi|| \Rightarrow ||\chi||$;
$\varphi$ is $\psi \wedge \chi$: then $||\varphi|| = ||\psi|| \wedge ||\chi||$;
$\varphi$ is $\psi \vee \chi$: then $||\varphi|| = ||\psi|| \vee ||\chi||$;
$\varphi$ is $\Delta\psi$; then $||\varphi|| = \Delta||\psi||$:
$\varphi$ is $\forall x\psi$: then $||\varphi|| = \bigwedge_{u\in V^L} \psi(x/u)$;
$\varphi$ is $\exists x\psi$: then $||\varphi|| = \bigvee_{u\in V^L} \psi(x/u)$.
Here $\psi(x/u)$ is the result of substituting $u$ for the variable $x$ in $\psi$. We use the symbols $\wedge$, $\vee$, $\Delta$ for both logical connectives and operations in $L$.

**Definition 4.** Let $\varphi$ be a closed formula. We say that $\varphi$ is valid in $V^L$ iff $||\varphi = 1||$ is provable in ZFC.

(As usual, we take closures of formulas containing free variables when considering their validity.)

**Lemma 5.** *(1) ZFC proves for $u \in V^L$:*
$u(x) = ||x \in u||$ *for* $x \in D(u)$, $||u = u|| = 1$
*(2) ZFC proves for $u, v, w \in V^L$:*
(i)$||u = v|| * ||v = w|| \leq ||u = w||$
(ii)$||u \in v|| * ||v = w|| \leq ||u \in w||$
(iii)$||u = v|| * ||v \in w|| \leq ||u \in w||$

*Proof.* (1) immediate,
(2) (i) immediate,
(ii) if $v = w$, then $||v = w|| = 1$, so $||u \in v|| = ||u \in v|| * ||v = w|| = ||u \in w|| * ||v = w|| = ||u \in w||$; otherwise $||v = w|| = 0$, in which case the statement holds trivially.
(iii) analogously. □

**Lemma 6. (Substitution)** *For any formula $\varphi$, ZFC proves: $\forall u, v \in V^L$, $||u = v|| * ||\varphi(u)|| \leq ||\varphi(v)||$.*

**Lemma 7. (Bounded quantifiers)** *ZFC proves $\forall x \in V^L$:*
(i) $||\exists y \in x \varphi(y)|| = ||\exists y(y \in x \,\&\, \varphi(y))|| = \bigvee_{y \in D(x)}(x(y) * ||\varphi(y)||)$
(ii) $||\forall y \in x \varphi(y)|| = ||\forall y(y \in x \rightarrow \varphi(y))|| = \bigwedge_{y \in D(x)}(x(y) \Rightarrow ||\varphi(y)||)$

*Proof.* (i) $||\exists y(y \in x \,\&\, \varphi(y))|| = \bigvee_{y \in V^L}(||y \in x|| * ||\varphi(y)||) = \bigvee_{y \in D(x)}(x(y) * ||\varphi(y)||)$ since $||y \in x||$ is nonzero only if $y \in D(x)$, and in that case it is $x(y)$.
(ii) analogously. □

**Corollary 8.** $||x \subseteq y|| = ||\forall u \in x(u \in y)|| = ||\forall u(u \in x \rightarrow u \in y)||$.

## 4 The Theory FST

We introduce the theory FST in the language $\{\in\}$; we take $=$ to be a logical symbol with the usual axioms (imposing reflexivity, symmetry, transitivity and congruence w.r.t $\in$ on the corresponding relation). The underlying logic of FST is a schematic extension $\mathcal{C}$ (possibly void) of BL$\forall\Delta$; when proving theorems within FST, we only rely on the logical axioms of BL$\forall\Delta$ (thus *any* schematic extension will, for example, interpret ZF). On the other hand, for a given extension $\mathcal{C}$, the universe $V^L$—for $L$ a $\mathcal{C}$-algebra—will yield an interpretation of FST over $\mathcal{C}$.

The reader will have noticed that ours is a crisp equality. This was imposed by the following fact:

**Lemma 9.** *A theory with comprehension (for open formulas) and pairing (or singletons) over a logic which proves the propositional formula* $(\varphi \to \varphi \& \varphi) \to (\varphi \vee \neg\varphi)$ *proves* $\forall x, y(x = y \vee \neg(x = y))$.

*Proof.* Given $x, y$, let $z$ be s. t. $u \in z \equiv (u \in \{x\} \& u = x)$, i. e., $u \in z \equiv (u = x)^2$. Since $(x = x)^2$, we have $x \in z$. If $y = x$ then $y \in z$ by congruence, but then $(y = x)^2$; thus we have proved $y = x \to (y = x)^2$, thus (by assumption on the logic) $(x = y \vee \neg(x = y))$. □

Thus e.g. in Łukasiewicz logic and in product logic we get a crisp $=$. Also, under the usual formulation of extensionality, crispness of $=$ implies crispness of $\in$ in a theory with pairings and unions (cf. [3]). We borrow the elegant solution from [8]: a modification of extensionality uses the $\Delta$ operator, which invalidates the proof of crispness of $\in$ from the crispness of $=$.

**Definition 10.** FST is a first order theory in the language $\{\in\}$, with the following axioms:
(i) (extensionality) $\forall x \forall y(x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$
(ii) (empty set) $\exists x \Delta \forall y \neg(y \in x)$
(iii) (pair) $\forall x \forall y \exists z \Delta \forall u(u \in z \equiv (u = x \vee u = y))$
(iv) (union) $\forall x \exists z \Delta \forall u(u \in z \equiv \exists y(u \in y \,\&\, y \in x))$
(v) (weak power) $\forall x \exists z \Delta \forall u(u \in z \equiv \Delta(u \subseteq x))$
(vi) (infinity)$\exists z \Delta(\emptyset \in z \,\&\, \forall x \in z(x \cup \{x\} \in z))$
(vii) (separation) $\forall x \exists z \Delta \forall u(u \in z \equiv (u \in x \,\&\, \varphi(u, x))$, for any formula not containing $z$ as a free variable
(viii) (collection) $\forall x \exists z \Delta[\forall u \in x \exists v\, \varphi(u, v) \to \forall u \in x \exists v \in z \varphi(u, v))]$ for any formula not containing $z$ as a free variable
(ix) ($\in$-induction) $\Delta \forall x(\forall y \in x \varphi(y) \to \varphi(x)) \to \Delta \forall x \varphi(x)$ for any formula $\varphi$
(x) (support) $\forall x \exists z(\mathrm{Crisp}(z) \& \Delta(x \subseteq z)))$

The $\Delta$'s in the axioms of weak power and $\in$-induction are introduced to weaken the statements; the $\Delta$'s after an existential quantifiers is used to guarantee that an element with the postulated property exists in *every* model of the theory (which does not follow from the semantics of the $\exists$ quantifier).

As usual, in the formulation of some axioms we use the functions of empty set, singleton, and union, which can be introduced using the appropriate axioms as in classical ZF, and also the usual definition of $\subseteq$. For a detailed treatment of introducing functions in BL$\forall$, see [5]. The function introduced by the weak power set axiom will be denoted $WP(x)$ for a set $x$. We use the following definition of crispness of a set:

**Definition 11.** $\mathrm{Crisp}(x) \equiv \forall u(\Delta(u \in x) \vee \Delta\neg(u \in x))$

**Theorem 12.** *Let* $\varphi$ *be a closed formula provable in FST. Then* ZFC *proves* $||\varphi|| = 1$ *in* $V^L$.

Having proved this theorem we shall take the liberty to call $V^L$ a *model* of FST. (More pedantically, we have constructed an interpretation of FST (over $\mathcal{C}$)

in ZFC.) We omit all validity proofs for logical axioms and inference rules. The validity of congruence axioms for $\in$ has been verified in Lemma 5. It remains to verify the set-theoretic axioms.

**Lemma 13.** *The set-theoretical axioms (i)-(x) hold in $V^L$.*

*Proof.* (extensionality) For fixed $x, y \in V^L$, either $x = y$ and then the axiom holds, or $x \neq y$ and $||x = y|| = 0$; then either $D(x) = D(y)$ and w. l. o. g. there is a $z \in D(x)$ s. t. $x(z) < y(z)$, so $||\Delta y \subseteq x|| = 0$, or w. l. o. g. there is a $z \in D(x)$ s. t. $z \notin D(y)$, and then $||z \in x||$ is nonzero while $||z \in y||$ is zero thus $||\Delta x \subseteq y|| = 0$.

(empty set) There is only one candidate for the role of an "empty set in $V^L$", and this is the $\emptyset$ in $V^L_0$. Indeed, for an arbitrary $x$ we get $||x \in \emptyset|| = 0$ since no $x$ can be in the domain of $\emptyset$ (taken as a function).

(pair) For fixed $x, y \in V^L$, there is a $z \in V^L$ such that $D(z) = \{x, y\}$ and $z(x) = z(y) = 1$. The set $z$ has the desired properties: for arbitrary $u \in V^L$, either $u \in D(z)$, then either $u = x$ or $u = y$ and $||u \in z|| = z(u) = 1$, or $u \notin D(z)$, and then $||u \in z|| = ||u = x|| = ||u = y|| = 0$.

(union) For a fixed $x \in V^L$, define (auxiliary) $D^2(x) = \bigcup\{D(v) : v \in D(x)\}$. Define $z$ s. t. $D(z) = \{u \in D^2(x) : \bigvee_{v \in D(x)}(v(u) * x(v)) > 0\}$ (with a nilpotent t-norm, the union of a nonempty set may well be empty), and for $u \in D(z)$ set $z(u) = \bigvee_{v \in D(x)}(v(u) * x(v))$. Then for an arbitrary $u \in V^L$, if $u \in D(z)$ then $||u \in z|| = z(u) = \bigvee_{v \in D(x)}(v(u) * x(v)) = ||\exists y(u \in y \in x)||$. If $u \notin D(z)$, then $||u \in z|| = 0$, and also $||\exists y \in x(u \in y)|| = 0$ by definition of $D(z)$.

(weak power) For a fixed $x \in V^L$, define $z$ s.t. $D(z) = \{u \in V^L : D(u) \subseteq D(x) \,\&\, u(v) \leq x(v)$ for $v \in D(u)\}$, and $z(u) = 1$ for $u \in D(z)$. For $u \in V^L$, either $u \in D(z)$ and then $||u \in z|| = z(u) = 1$, and also (by definition of $D(z)$) $||u \subseteq x|| = 1 = ||\Delta u \subseteq x||$, or $u \notin D(z)$, thus $||u \in z|| = 0$, and (by definition of $D(z)$) either $D(u) \not\subseteq D(x)$, or for some $v \in D(u)$, $u(v) > x(v)$, and in either case $||\Delta \forall v \in u(v \in x)|| = \Delta \bigvee_{v \in D(u)}(u(v) \Rightarrow ||v \in x||) = 0$.

(infinity) Define a function $z$ with $D(z) = V^L_\omega$ and $z(u) = 1$ for $u \in D(z)$. Then $||\emptyset \in z|| = 1$ and, since for $x \in V^L_\alpha$ we have $x \cup \{x\} \in V^L_{\alpha+1}$, also $||\forall x \in z(x \cup \{x\} \in z)|| = 1$.

(separation) For a fixed $x \in V^L$, and a given $\varphi$, define $z$ s. t. $D(z) = \{u \in D(x) : x(u) * ||\varphi(u)|| > 0\}$ and for $u \in D(z)$ set $z(u) = x(u) * ||\varphi(u)||$. Obviously this definition of $z$ demonstrates the validity of separation in $V^L$.

(collection) in [8]

($\in$-induction) Fix a formula $\varphi$ and suppose the axiom does not hold in $V^L$. Then (since $\Delta$ is two-valued) it must be the case that $||\Delta \forall x(\forall y \in x \varphi(y) \rightarrow \varphi(x))|| = 1$ and $||\Delta \forall x \varphi(x)|| = 0$, thus there is a successor ordinal $\alpha$ s. t. $\exists x \in V^L_\alpha(\varphi(x) < 1) \,\&\, \forall \beta < \alpha \forall y \in V^L_\beta(\varphi(y) = 1)$. Suppose first $\alpha = 0$; but $||\forall y \in \emptyset \varphi(y) \rightarrow \varphi(\emptyset)|| = 1 \Rightarrow \varphi(\emptyset) = \varphi(\emptyset) < 1$, thus the antecedent would be 0. Suppose now $\alpha > 0$, and $x \in V^L_\alpha$ is s. t. $\varphi(x) < 1$. From the condition that $||\forall y \in x \varphi(y) \rightarrow \varphi(x)|| = 1$, and since $||\forall y \in x \varphi(y)|| = 1$, we get $||\varphi(x)|| = 1$, a contradiction.

(support) For a fixed $x \in V^L$ take $z$ such that $D(z) = D(x)$ and $\forall u \in D(z)$ $z(u) = 1$. Then $||\mathrm{Crisp}(z)|| = 1$ and $\bigwedge_{u \in D(x)}(x(u) \Rightarrow ||u \in z||) = 1$. □

## 5 An Interpretation of ZF in FST

Within FST, we shall define a class (i.e., we shall give a formula with one free variable, in the language of FST) of hereditarily crisp sets, which will be proved an inner model of ZF in FST.

We start with a bunch of technical statements.

**Lemma 14.** *BL$\forall\Delta \vdash \forall x \Delta\varphi \equiv \Delta\forall x\varphi$.*

*Proof.* The right-to-left implication is easy. We give a BL$\forall\Delta$-proof of the converse one (an analogy to the proof of the Barcan formula in S5). Let $\Diamond\varphi$ stand for $\neg\Delta\neg\varphi$.
(i) $\varphi \to \neg\Delta\neg\varphi$.
In BL$\Delta$, $\Delta\neg\varphi \to \neg\varphi$, thus $(\Delta\neg\varphi \& \varphi) \to \overline{0}$, thus $\varphi \to (\Delta\neg\varphi \to \overline{0})$.
(ii) $\neg\Delta\varphi \to \Delta\neg\Delta\varphi$.
In BL$\Delta$, $\Delta\varphi \vee \neg\Delta\varphi$, thus $\Delta\Delta\varphi \vee \Delta\neg\Delta\varphi$. This gives $\neg\Delta\varphi \to \Delta\neg\Delta\varphi$.
Thus the following two are provable in BL$\Delta$:
(iii) $\varphi \to \Delta\Diamond\varphi$
(iv) $\Diamond\Delta\varphi \to \Delta\varphi$.
Next, (v) $\Delta(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$,
since $\Delta(\varphi \to \psi) \to \Delta(\neg\psi \to \neg\varphi) \to (\Delta\neg\psi \to \Delta\neg\varphi) \to (\Diamond\varphi \to \Diamond\psi)$.
(vi) $\Diamond\forall x\varphi \to \forall x\Diamond\varphi$ is a consequence of (v).
Finally, $\forall x\Delta\varphi \to \Delta\Diamond\forall x\Delta\varphi \to \Delta\forall x\Diamond\Delta\varphi \to \Delta\forall x\Delta\varphi \to \Delta\forall x\varphi$. □

**Lemma 15.** *BL$\forall\Delta \vdash \Delta(\varphi \vee \neg\varphi) \equiv \Delta(\varphi \to \Delta\varphi)$.*

*Proof.* Takes place inside BL$\forall\Delta$. Implication left-to-right: $\Delta(\varphi \vee \neg\varphi) \to (\Delta\varphi \vee \Delta\neg\varphi) \to [\varphi \to (\varphi \& (\Delta\varphi \vee \Delta\neg\varphi))] \to [\varphi \to (\Delta\varphi \vee (\varphi \& \Delta\neg\varphi))]$. In the last formula in the chain, $\varphi \& \Delta\neg\varphi \to \overline{0}$, thus the last formula implies $\varphi \to \Delta\varphi$; we get $\Delta(\varphi \vee \neg\varphi) \to (\varphi \to \Delta\varphi)$, which $\Delta$-generalizes to the desired implication.

Conversely, $\Delta\varphi \to \Delta(\varphi \vee \neg\varphi)$, thus also (i) $\Delta\varphi \to [\Delta(\varphi \to \Delta\varphi) \to \Delta(\varphi \vee \neg\varphi)]$. Moreover $\Delta(\varphi \to \Delta\varphi) \to (\Delta\neg\Delta\varphi \to \Delta\neg\varphi) \to [\Delta\neg\Delta\varphi \to \Delta(\varphi \vee \neg\varphi)]$, thus (ii) $\Delta\neg\Delta\varphi \to [\Delta(\varphi \to \Delta\varphi) \to \Delta(\varphi \vee \neg\varphi)]$. Since BL$\forall\Delta$ proves $\Delta\varphi \vee \Delta\neg\Delta\varphi$, we get the right-to-left implication by putting together (i) and (ii).

□

Recall the definition of a crisp set. We get

**Corollary 16.** *$\forall x(\mathrm{Crisp}(x) \equiv \forall u\Delta(u \in x \to \Delta(u \in x))$.*

Note that $\mathrm{Crisp}(x) \equiv \Delta\forall u(u \in x \to \Delta(u \in x)) \to \Delta\Delta\forall u(u \in x \to \Delta(u \in x))$, so crispness is a crisp property:

**Lemma 17.** *$\mathrm{Crisp}(x) \to \Delta\mathrm{Crisp}(x)$.*

We write $\bowtie \varphi$ for $\Delta(\varphi \vee \neg\varphi)$.

**Lemma 18.** $\mathrm{BL}\Delta \vdash (\bowtie \varphi \&(\varphi \to \Delta\psi)) \to \Delta(\varphi \to \psi)$.

*Proof.* $[\Delta\varphi\&(\varphi \to \Delta\psi)] \to \Delta\psi \to \Delta(\varphi \to \psi)$, and $\Delta\neg\varphi \to \Delta(\varphi \to \psi)$, so $\bowtie \varphi \equiv (\Delta\varphi \vee \neg\Delta\varphi) \to ((\varphi \to \Delta\psi) \to \Delta(\varphi \to \psi))$. □

**Lemma 19.** $\mathrm{BL}\forall\Delta \vdash \forall xy((\mathrm{Crisp}(x)\&\mathrm{Crisp}(y)\&x \subseteq y) \to \Delta(x \subseteq y))$.

*Proof.* $(\mathrm{Crisp}(x)\&\mathrm{Crisp}(y)\&(u \in x \to u \in y)) \to$
$(\bowtie (u \in x)\&(u \in x \to \Delta u \in y)) \to \Delta(u \in x \to u \in y)$. □

**Definition 20. (Hereditarily crisp transitive set)** $HCT(x) \equiv \mathrm{Crisp}(x) \,\&\, \forall u \in x(\mathrm{Crisp}(u)\&u \subseteq x)$

The formula $HCT(x)$ defines a class, and we adopt the habit of writing $x \in HCT$ instead of $HCT(x)$ and approaching classes in a similar way we approach sets. A "crisp class" $C$ is a class for which $\forall x\Delta(x \in C \vee \neg(x \in C))$, or equivalently, $\forall x\Delta(x \in C \to \Delta(x \in C))$.

**Lemma 21. (Crispness of $HCT$)** $x \in HCT \to \Delta(x \in HCT)$.

*Proof.* $x \in HCT \equiv \big[\mathrm{Crisp}(x)\&\forall u(u \in x \to (\mathrm{Crisp}(u)\&u \subseteq x))\big] \to$
$\big[\Delta\mathrm{Crisp}(x)\&\forall u(\bowtie (u \in x)\&(u \in x \to \Delta(\mathrm{Crisp}(u)\&u \subseteq x)))\big] \to$
$\big[\Delta\mathrm{Crisp}(x)\&\forall u\Delta(u \in x \to (\mathrm{Crisp}(u)\&u \subseteq x))\big] \to$
$\Delta(x \in HCT)$. □

**Definition 22. (Hereditarily crisp set)** $\mathrm{H}(x) \equiv \mathrm{Crisp}(x) \,\&\, \exists x' \in \mathrm{HCT}(x \subseteq x')$.

**Lemma 23. (Crispness of $H$)** $x \in H \to \Delta(x \in H)$.

*Proof.* By definition of $H$, we are to prove
$(\mathrm{Crisp}(x)\&\exists y \in \mathrm{HCT}(x \subseteq y)) \to \Delta(\mathrm{Crisp}(x)\&\exists y \in \mathrm{HCT}(x \subseteq y))$.
Since $\mathrm{Crisp}(x) \to \Delta\mathrm{Crisp}(x)$, it suffices to prove
$(\mathrm{Crisp}(x)\&\exists y \in \mathrm{HCT}(x \subseteq y)) \to \Delta(\exists y \in \mathrm{HCT}(x \subseteq y))$
(we may use the presumption $\mathrm{Crisp}(x)$ in each of the two implications since it is idempotent).
First, $(\mathrm{Crisp}(x)\&y \in \mathrm{HCT}\&x \subseteq y) \to \Delta(y \in \mathrm{HCT}\&x \subseteq y)$ by Lemma 21 and Lemma 19. Now generalize: $\forall y((\mathrm{Crisp}(x)\&y \in \mathrm{HCT}\&x \subseteq y) \to \Delta(y \in \mathrm{HCT}\&x \subseteq y))$; hence $(\mathrm{Crisp}(x)\&\exists y(y \in \mathrm{HCT}\&x \subseteq y)) \to \exists y\Delta(y \in \mathrm{HCT}\&x \subseteq y)$; and the succedent implies $\Delta\exists y(y \in \mathrm{HCT}\&x \subseteq y)$. □

We show that FST proves H to be an inner model of ZF. In more detail, for $\varphi$ a formula in the language of ZF, define $\varphi^H$ inductively as follows:
$\varphi^H = \varphi$ for $\varphi$ atomic;
$\varphi^H = \varphi$ for $\varphi = \overline{0}$;
$\varphi^H = \psi^H\&\chi^H$ for $\varphi = \psi\&\chi$;
$\varphi^H = \psi^H \to \chi^H$ for $\varphi = \psi \to \chi$;
$\varphi^H = (\forall x \in H)\psi^H$ for $\varphi = (\forall x)\psi$;
$\varphi^H = (\exists x \in H)\psi^H$ for $\varphi = (\exists x)\psi$.

**Theorem 24.** *Let $\varphi$ be a theorem of ZF. Then* $\mathrm{FST} \vdash \varphi^{\mathrm{H}}$.

To prove this theorem, we first show that the law of the excluded middle (LEM) holds in H; since BL$\forall$ together with LEM yield classical logic, we will have proved that all logical axioms of ZF are provable when relativized to H. Then we prove the H-relativized versions of all the axioms of the ZF set theory.

**Lemma 25.** *Let $\varphi(x_1, \ldots, x_n)$ be a ZF-formula whose free variables are among $x_1, \ldots, x_n$. Then FST proves* $\forall x_1 \in H \ldots \forall x_n \in H(\varphi^H(x_1, \ldots, x_n) \vee \neg\varphi^H(x_1, \ldots, x_n))$.

*Proof.* We consider a formula $\varphi$ with (at most) one free variable $x$, the modification for multiple free variables being easy. The formula to be proved in FST is $\forall x(x \in H \to (\varphi^H(x) \vee \neg\varphi^H(x))$; it suffices to prove $\forall x \Delta(x \in H \to (\varphi^H(x) \vee \neg\varphi^H(x))$, and by Lemma 18 and Lemma 15, it suffices to prove $\forall x(x \in H \to \Delta(\varphi^H(x) \to \Delta\varphi^H(x))$.

The proof proceeds by induction on the complexity of $\varphi$.

Atomic subformulas: $=$ is a crisp predicate, for $\in$ we have to prove $x \in y \to \Delta x \in y$ assuming $x, y \in H$. In fact $y \in H$ implies $\mathrm{Crisp}(y)$, which entails $\forall x(x \in y \to \Delta x \in y)$.

Conjunction: for a subformula $\psi_1(x, y) \& \psi_2(x, z)$ of $\varphi$ (we assume one free variable in common, and one distinct free variable for each subformula, and we henceforth omit their explicit listings, for legibility's sake) assume $x, y \in H \to (\psi_1^H \to \Delta\psi_1^H)$ and $x, z \in H \to (\psi_2^H \to \Delta\psi_2^H)$. Then $(x \in H)^2 \& (y, z \in H) \to ((\psi_1^H \& \psi_2^H) \to \Delta(\psi_1^H \& \psi_2^H))$, and since $x \in H$ is idempotent, this completes the induction step for conjunction.

Implication: for a subformula $\psi_1(x, y) \to \psi_2(x, z)$ of $\varphi$, assume $x, y \in H \to (\psi_1^H \to \Delta\psi_1^H)$ and $x, z \in H \to (\psi_2^H \to \Delta\psi_2^H)$. Thus $(\psi_1^H \to \psi_2^H) \& (x, z \in H) \to (\psi_1^H \to \Delta\psi_2^H)$, and since $x, y \in H$ implies crispness of $\psi_1^H$, we get $x, y \in H \to ((\psi_1^H \to \Delta\psi_2^H) \to \Delta(\psi_1^H \to \psi_2^H))$. Thus $x, y, z \in H \to ((\psi_1^H \to \psi_2^H) \to \Delta(\psi_1^H \to \psi_2^H))$.

The universal quantifier: for a subformula $\forall y \psi(x, y)$ of $\varphi$, the induction hypothesis is $x, y \in H \to (\psi^H(x, y) \to \Delta\psi^H(x, y))$. Generalize in $y$: $x \in H \to \forall y(y \in H \to (\psi^H(x, y) \to \Delta\psi^H(x, y))$; now since for a crisp $\alpha$, BL proves $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$, we may modify the succedent to $\forall y((y \in H \to \psi^H(x, y)) \to (y \in H \to \Delta\psi^H(x, y))$, and distributing $\forall y$, we have proved: $x \in H \to (\forall y \in H \psi^H(x, y) \to \forall y \in H \Delta\psi^H(x, y))$. To flip the $\Delta$ and the $\forall y \in H$ in the succedent, use Lemma 18.

The existential quantifier: for a subformula $\forall y \psi(x, y)$ of $\varphi$, the induction hypothesis is $x, y \in H \to (\psi^H(x, y) \to \Delta\psi^H(x, y))$. Generalize in $y$: $x \in H \to \forall y(y \in H \to (\psi^H(x, y) \to \Delta\psi^H(x, y))$; now $(y \in H \to (\psi^H(x, y) \to \Delta\psi^H(x, y))) \to (y \in H \to (y \in H \& \psi^H(x, y) \to y \in H \& \Delta\psi^H(x, y))) \to (y \in H \& \psi^H(x, y) \to y \in H \& \Delta\psi^H(x, y))$. We get $x \in H \to (\exists y(y \in H \& \psi^H(x, y)) \to \exists y \Delta(y \in H \& \psi^H(x, y)))$, and the last succedent implies $\Delta\exists y(y \in H \& \psi^H(x, y))$. □

**Definition 26.** $WPC(x) = \{u \in WP(x); \mathrm{Crisp}(u)\}$.

**Lemma 27.** $\forall x(x \subseteq H \& \mathrm{Crisp}(x) \to x \in H)$.

*Proof.* Fix an $x$. $x \subseteq H$ is by definition $\forall u \in x \exists u'(u' \in HCT \& u \subseteq u')$. Thus $\exists v_0 \forall u \in x \exists u' \in v_0(u' \in HCT \& u \subseteq u')$. Fix $v_0$ and separate: $v = \{u \in v_0 : \Delta u \in v_0 \& u \in HCT \& \exists u_0 \in x(u_0 \subseteq u)\}$. Note that $\forall u \in x \exists u' \in v(u' \in HCT \& u \subseteq u')$, because $x$ is crisp. $v$ is a crisp set and all its elements are crisp (all of them are in $HCT$); hence $\bigcup v$ is crisp; its elements are crisp since $a \in \bigcup v$ implies $\exists b \in HCT(a \in b \in v)$, thus $a$ is crisp. $\bigcup v$ is also transitive: $b \in a \in \bigcup v$ implies $\exists y(b \in a \in y \in v)$, and since $y \in HCT$ is transitive, $b \in y \in v$ and $b \in \bigcup v$. Now consider $WPC(\bigcup v)$ (the crisp elements of the weak power set of $\bigcup v$); this is a crisp set, and it is a subset of $WPC(\bigcup v) \cup \bigcup v$, which is a crisp transitive set of crisp elements, thus in $HCT$, thus $WPC(\bigcup v) \in H$. Since $v \subseteq WPC(\bigcup v)$, we get $\forall u \in x \exists u' \in WPC(\bigcup v)(u \subseteq u')$, thus $\forall u \in x(u \in WPC(\bigcup v)$; this means $x \subseteq WPC(\bigcup v) \in HCT$, so $x \in H$. □

We consider ZF with the following axioms: empty set, pair, union, power set, infinity, separation, collection, extensionality, $\in$-induction. The exact spelling of these axioms is given separately when proving in FST their H-ed versions.

**Theorem 28.** *For $\varphi$ being any of the abovementioned axioms of ZF, FST proves $\varphi^H$.*

*Proof.* **(empty set)** $\exists z \forall u \neg(u \in z)$.
The H-translation, which reads $\exists z \in H \forall u \in H \neg u \in z$, is provable since $0 \in HCT$.
**(pair)** $\forall x, y \exists z \forall u(u \in z \equiv (u = x \vee u = y))$.
The H-translation $\forall x, y \in H \exists z \in H \forall u \in H(u \in z \equiv (u = x) \vee (u = y))$ is absolute: the set $\{x, y\}$ is crisp (since $=$ is crisp); to show that it is a subset of a set which is in HCT, consider $x' \in HCT$ a witness for $x \in H$ and $y' \in HCT$ a witness for $y \in H$. Then $\{x, y\} \cup x' \cup y'$ is a crisp transitive set with crisp elements, hence in HCT, and thus $\{x, y\}$ is in H.
**(union)** $\forall x \exists z \forall u(u \in z \equiv \exists y(u \in y \in x))$.
The H-translation $\forall x \in H \exists z \in H \forall u \in H(u \in z \equiv \exists y \in H(u \in y \in x))$ is absolute: let $x' \in HCT$ witness $x \in H$. Then $\bigcup x$ is a crisp set with crisp elements (since $x \subseteq x'$), and $\bigcup x \subseteq \bigcup x' \in HCT$, which witnesses $\bigcup x \in H$.
**(power set)** $\forall x \exists z \forall u(u \in z \equiv u \subseteq x)$
The $H$-translation reads $\forall x \in H \exists z \in H \forall u \in H(u \in z \equiv \forall y \in H(y \in u \to y \in x))$. Let $x'$ be a witness for $x \in H$. then for $x \in H$ it holds that $\forall u \in H(u \in WPC(x) \equiv \Delta(u \subseteq x)^H \equiv (u \subseteq x)^H)$, $WPC(x)$ is crisp, and $WPC(x) \subseteq WPC(x') \cup x'$, which is a transitive crisp set with crisp elements, thus in $HCT$ and a witness for $WPC(x) \in H$.
**(separation)** $\forall x \exists z \forall u(u \in z \equiv (u \in x \& \varphi(u))$ for a ZF-formula not containing $z$ freely.
The H-translation is absolute: let $x' \in HCT$ witness $x \in H$ and set $z = \{u \in x; \varphi^H(u)\}$, then $z$ is a crisp set and $z \subseteq x \subseteq x' \in HCT$ (i.e., $x'$ is a witness for $z \in H$).
**(infinity)** $\exists z(0 \in z \& \forall u \in z(u \cup \{u\} \in z))$.

It suffices to prove that there is a set $z \in H$ s.t. $0 \in z \& \forall u \in z(u \cup \{u\} \in z)$. Let $z_0$ be any set satisfying the axiom of infinity in FST and define $z_1 = \{x \in z_0 : \Delta(x \in z_0) \& \mathrm{Crisp}(x)\}$. Then $z_1$ is a subset of $z_0$ and $0 \in z_1$ and $u \in z_1 \to u \cup \{u\} \in z_1$. Now let $z = \{x \in z_1 : x \subseteq z_1\}$, i.e., a transitive subset of $z_1$. Obviously $0 \in z$, let us prove $x \in z \to (x \cup \{x\}) \in z$, that is by definition of $z$, $[x \in z_1 \& \forall y \in x(y \in z_1)] \to [x \cup \{x\} \in z_1 \& \forall a(a \in x \vee a = x \to a \in z_1)]$. We know $x \in z_1 \to (x \cup \{x\} \in z_1)$. Also, $x \in z_1 \& (y \in x \to y \in z_1) \to (y \in x \vee y = x \to y \in z_1)$. (Note that we made multiple use of the presumption $x \in z_1$; we may do that because the formula is crisp, thus $x \in z_1 \equiv (x \in z_1)^2$.) Finally, $z$ is a crisp transitive set with crisp elements, so $z \in HCT$.
**(extensionality)** $\forall xy(x = y \equiv \forall z(z \in x \equiv z \in y))$.
The H-translation $\forall x, y \in H(x = y \equiv \forall z \in H(z \in x \equiv z \in y))$ follows from extensionality in FST by H being transitive and by the crispness of its elements (the $\Delta$'s may be left out).
**(collection)** $\forall u(\forall x \in u \exists y \varphi(x,y) \to \exists v \forall x \in u \exists y \in v \varphi(x,y))$ for $\varphi$ not containing $v$ freely.
The H-translation reads $\forall u \in H(\forall x \in H(x \in u \to \exists y \in H \varphi^H(x,y)) \to \exists v \in H \forall x \in H(x \in u \to \exists y \in H(y \in v \& \varphi^H(x,y))))$. Fix $u \in H$ and a ZF-formula $\varphi$; we want to find a corresponding $v \in H$ s.t. the above is true. Define $v_0$ by collection in FST for $u$ and the formula $y \in H \& \varphi^H(x,y)$; separate $v = \{x \in v_0 : \Delta(x \in v_0) \& x \in H\}$. Then $v \subseteq H$ is a crisp set and, since $u$ is crisp, satisfies the collection axiom. By Lemma 27, $v \in H$.
**($\in$-induction)** $\forall x(\forall y \in x \varphi(y) \to \varphi(x)) \to \forall x \varphi(x)$ for any ZF-formula $\varphi$.
The H-translation is $\forall x \in H(\forall y \in H(y \in x \to \varphi^H(y)) \to \varphi^H(x)) \to \forall x \in H \varphi^H(x)$. Fix a ZF-formula $\varphi$, and consider the instance of $\in$-induction in FST for the formula $x \in H \to \varphi^H(x)$:
$\Delta \forall x(\forall y \in x(y \in H \to \varphi^H(y)) \to (x \in H \to \varphi^H(x))) \to \Delta \forall x(x \in H \to \varphi^H(x))$.
This formula is our aim except for the $\Delta$'s. Let us denote $A$ the antecedent and $S$ the succedent in the implication, with the $\Delta$'s omitted. Since $\Delta S \to S$, it remains to be proved that $A \to \Delta A$. Let us further denote $\forall y \in H(y \in x \to \varphi^H(y))$ with $I$. First, it is obvious that
$x \in H \to \forall y(y \in x \to \bowtie \varphi^H(y))$, and
$x \in H \to \forall y \bowtie (y \in x)$. Thus,
$x \in H \to \forall y(\bowtie (y \in x) \& (y \in x \to \bowtie \varphi^H(y)))$, hence
$x \in H \to \forall y(\bowtie (y \in x \to \varphi^H(y)))$, and
$x \in H \to \bowtie \forall y(y \in x \to \varphi^H(y))$, thus $x \in H \to \bowtie I$. Since $x \in H \equiv \bowtie (x \in H)$, we get
$A \to \forall x \in H(\bowtie I \& (I \to \Delta \varphi^H(x)))$, hence by use of Lemma 18
$A \to \forall x \in H \Delta(I \to \varphi^H(x))$, so
$A \to \Delta \forall x \in H(I \to \varphi^H(x))$, which is $A \to \Delta A$. □

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