Charles University in Prague
Faculty of Mathematics and Physics

# Mathematical and Metamathematical Properties of Fuzzy Logic

Zuzana Haniková

Ph. D. Thesis

December 2003

Thesis Advisor:
Prof. RNDr. Petr Hájek, DrSc.

Branch M1 — Algebra, Theory of Numbers and Mathematical Logic

# Contents

# Acknowledgments

I thank my advisor Petr Hájek for his help, advice and support. I have greatly enjoyed the excellent scientific background he provided and the opportunity he gave me to meet many other outstanding researchers in the field.

My thanks go to people from the Department of Theoretical Computer Science, Institute of Computer Science, Academy of Sciences of the Czech Republic, where this thesis has been written. Their lively interest in my work and their encouragement has been of vital importance. Especially, I would like to thank Věra Kůrková and Petr Savický.

I appreciated the stimulating environment for research provided by the wider logical community in Prague, where I had the luck to conduct my Ph.D. studies.

I thank my husband Jirka for many interesting comments and discussions on this thesis and on other issues in logic, for his patience, help and good cheer. I am grateful to all my family for supporting my interest in mathematics.

Petr Cintula, Libor Běhounek and Jirka Hanika have read parts of the manuscript of this thesis and suggested many improvements.

I acknowledge continual support of grant IAA 103 0004, Grant Agency of the Academy of Sciences of the Czech Republic, which enabled me to travel and meet other researchers.

# Chapter 1

# Introduction

Fuzzy[I)] logic is a logic of vague notions. It originates with the attempt to handle concepts which admit many (more than two) *degrees of truth*; standard examples are concepts like 'young', 'beautiful' or—after all—'true'. A nice criterion of fuzziness of a concept is its ability to admit comparison: one person may be *more* beautiful, one statement may be *more* true than another one.[II)] Not all concepts are fuzzy; for example 'dead' is not fuzzy. The opposite term to 'fuzzy' is 'crisp'.

The path from initial considerations about fuzziness to a full-fledged logical system is by no means straightforward, and walking it one has to make a lot of *choices*. Naturally then, one will find oneself inhabiting a small niche within the world of fuzzy logic (broadly speaking), determined by the choices one has made. One of the distinctive traits of our niche is that our logic is truth-functional, i. e., the truth value of a compound formula can be computed from the truth values of its subformulas, using the function determined by the connective combining these subformulas. Another trait is its syntactical resemblance to the classical logic, and the fact that the system has the classical truth values 0 (absolutely false) and 1 (absolutely true) and behaves classically on these.

[I)] meaning 1. having a soft, light, hairy texture; 2. not clear in shape, sound, etc. (cf. [Hor95]). The word 'fuzzy' is of provincial English origin, with cognate forms *voos*—Dutch 'spongy', *fussig*—Low German 'loose', 'weak', etc. A *fuzz-ball* (also called *puff-ball*) is a light, spongy ball resembling a mushroom (cf. [Ske10]). Unfortunately, the rather popular Czech "cognate form" *fousatá* ('bearded') does not actually seem to be related etymologically.

[II)] Note that the ordering thus induced on truth degrees need not be linear. This is reflected in the algebras of truth values, which in general have a lattice ordering.

This work is a contribution to the niche of Hájek's *Basic Fuzzy Logic* and some closely related logical systems, discussed in [Háj98b], where motivating explanations can be found and the issues of adequacy, usefulness and possible applications of the calculus are addressed. Readers familiar with Hájek's works will no doubt find these paragraphs superfluous. Our purpose is to stress that the logic discussed here is not *the* fuzzy logic, and even within the limits given by the abovementioned choices it has many alternatives. The fact that this work is so strongly anchored within Hájek's system may also render it difficult to comprehend for readers who come from very different backgrounds. We try to make up for this by including the basic facts.

## 1.1 About this thesis

We have hopefully explained that in this work, 'fuzzy logic' will have a very specific meaning. We study some of its metamathematical properties (proving theorems *about* the calculi) and mathematical properties (proving theorems *within* the calculi).

What the abovementioned choices give us mathematically is the existence and some properties of *algebras of truth values* for our logic. These are above all algebras given by continuous t-norms on the real interval $[0, 1]$. These motivated the propositional calculus BL. This system turns out to admit much more general algebraic semantics given by the so-called BL-algebras. For readers' convenience Chapter 2 tries to bring in the main definitions and theorems.

Of the metamathematical properties of fuzzy logic we focus mainly on the computational complexity of logics given by continuous t-norms, and related issues. Namely, in Chapter 3 we establish some terminology and prepare technical means for the following chapter. We show that algebras given by continuous t-norms generate only countably many subvarieties of the variety of BL-algebras. In Chapter 4 we cover available results on computational complexity of propositional logics given by t-norms. After a brief overview of known facts, its main result is the coNP-completeness of each propositional logic given by a continuous t-norm.

Chapter 5 develops an axiomatic set theory over a particular many-valued predicate logic. It includes a brief history of the topic and an overview of the predicate calculus used. It then defines the axiomatic theory and discusses the choice of axioms. It shows that the theory admits many-valued interpretations, and, defining a crisp "core" of its universe, shows it to be an inner model of the classical Zermelo-Fraenkel set theory.

## 1.2 Bibliographical remarks

As mentioned, one whole chapter of this work is a compilation of known results which are in some way relevant. Moreover, we give references to other works throughout the text as needed. Known results on individual topics like computational complexity of many-valued logics or set theory within many-valued logics are discussed in their appropriate chapters; in particular, Section 4.1 gives results about propositional complexity of BL and its extensions Ł, G, and Π; Section 5.1 is a brief glimpse into the history of set theory in many-valued logic and Section 5.2 contains the definition of and some theorems on the predicate calculus, which are mostly covered elsewhere except for the added $\Delta$ connective.

Chapter 5 is based on a joint paper with Hájek.

Of the many works on many-valued and/or fuzzy logic, let us mention a few monographs. [Got88] has long been the basic work on many-valued logic. [CDM00] is a thorough study on MV-algebras. [MNP00] is a development of formal fuzzy logic in the style of Pavelka. [KMP00] is a detailed study of t-norms.

On a more general scope, non-classical logics are discussed in [GG94].

## 1.3 Some definitions and notation

We use basic definitions and results from logic, computational complexity, algebra and set theory; these notions are used in a standard way.

Here we introduce some notation and point out some particular terminology used in this work.

Closed intervals are denoted by square brackets (e. g., $[0, 1]$); open intervals by round brackets (e. g., $(0, 1)$). Unless specified otherwise, the intervals $[0, 1]$, $(0, 1)$ are intervals of reals.

$N$ denotes the set of natural numbers (including 0).

### 1.3.1 Logic

A standard and comprehensive reference is [Bar77]. There are two recent monographs in Czech, covering classical propositional and predicate logic: [Soc01] and [Šve02].

Formulas are always denoted with Greek letters.

For simplicity we assume that the language of classical propositional logic has basic connectives &, $\rightarrow$, 0 (as in BL), and the definitions of other connectives are as in BL; in particular, $\neg\varphi$ is an abbreviation for $\varphi \rightarrow 0$.

Regarding parentheses, the precedence of logical symbols is as follows: quantifiers bind stronger than any connective; negation binds stronger than any binary connective; &, $\wedge$, $\vee$ bind stronger than $\rightarrow$, $\equiv$. The formula $\forall x\varphi \,\&\, \neg\psi \rightarrow \chi$ should be parsed as $((\forall x\varphi) \,\&\, (\neg\psi)) \rightarrow \chi$. We try to employ parentheses so as to make formulas legible and prevent ambiguity, and there are minor aberrations from any rigid rules of usage.

For a natural number $n$, $\varphi^n$ stands for $\varphi \& \varphi \ldots \& \varphi$, $n$ times.

A *complete evaluation* of a formula in an algebra of truth values is the set of values of all subformulas (often considered as the *partial subalgebra* induced by it, see Section 1.3.3).

Structures (algebras) are denoted with boldface Roman capitals. We distinguish a structure (e. g., $\mathbf{L}$) and its universe (e. g., $L$), always using the same (capital) letter for both.

We extend the notion of *idempotence* of the operation $*$ in BL-algebras and the corresponding logical connective & to relations and formulas (e. g., $=$) and say that a formula $\varphi$ is idempotent iff $\varphi \equiv \varphi \,\&\, \varphi$ (i. e., iff & is idempotent on the formula in question).

### 1.3.2 Computational complexity

In Chapter 4 we work with the P, NP and coNP classes from the polynomial hierarchy, and the notion of polynomial-time reducibility.

We use (a modification of) Cook's theorem, stating that the SAT problem of satisfiability of propositional formulas in our basic propositional language is NP-complete, and the TAUT problem is coNP-complete.

A good general reference is [Pap94].

### 1.3.3 Algebra

We use the notions of homomorphism, isomorphism, embedding, etc. of algebras; this means that the respective mappings preserve all operations of the algebra. The formulation 't-norm $*_a$ on $[x_a, y_a]$ is isomorphic to $*_b$ on $[x_b, y_b]$' means that there is a bijection on the underlying intervals which preserves the t-norm operation. We are more explicit where confusion could arise.

We clarify the following notion.

**Definition 1.3.1** *Let* **A** *and* **B** *be two algebras of the same type.*
*(i) A* partial subalgebra *of* **A** *is a finite subset $X \subseteq A$, with the restriction of all its (functional) relations to the elements of $X$.*
*(ii)* **A** *is* partially embeddable *into* **B** *if every partial subalgebra of* **A** *is embeddable into* **B** *(i. e., there is a finite $Y \subseteq B$ and a $1-1$ mapping $f : X \longrightarrow Y$ which is a partial isomorphism).*

We use a few basic notions and results from universal algebra, which will be found in any book on universal algebra, e. g., [Grä79a].

Varieties of algebras are denoted with bold roman uppercase (e. g., **BL**, **Ł**, etc.).

### 1.3.4 Set theory

A good reference is [Jec02]. A comprehensive alternative in Czech is [BŠ86]. Throughout, ZF denotes the classical Zermelo-Fraenkel set theory and ZFC is ZF with the axiom of choice.

# Chapter 2

# BL and BL-algebras

Basic Fuzzy Logic (BL) and BL-algebras, its algebraic counterpart, were introduced by Hájek in [Háj98b]. This chapter contains a selection of basic notions and known results, relevant to subsequent chapters. It gives a brief glimpse of the state of the art and is intended primarily for reference, making this thesis reasonably self-contained.

The material has been selected and arranged so as to suit the purpose of this chapter and the author's point of view. A considerable part of the material in this chapter has been taken from [Háj98b], but naturally it has been extended with regard to other works, especially more recent results. The outcome is rather technical and can hardly be recommended as a primer on the topic. For the sake of briefness we omit many important explanations and discussions and also proofs, unless they carry indispensable information.

The topics discussed are the propositional calculi given by continuous t-norms, related algebraic issues, and some closely related calculi in an expanded language.

## 2.1 Basic facts about BL-algebras

We follow [Háj98b], Definition 2.3.2 and 2.3.3, which presents a BL-algebra as a residuated lattice with some additional properties.

**Definition 2.1.1 (BL-algebra)**
(1) *A* residuated lattice *is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *with four binary operations and two constants such that:*
(i) $\langle L, \wedge, \vee, 0, 1\rangle$ *is a lattice with the greatest element* 1 *and the least element*

0 *(with respect to the lattice ordering $\leq$)*
(ii) $\langle L, *, 1\rangle$ *is a commutative monoid, i. e., $*$ is commutative, associative, and* $1 * x = x$ *for all* $x \in L$
(iii) $*$ *and* $\Rightarrow$ *form an adjoint pair, i. e., for all* $x, y, z \in L$,
$z \leq (x \Rightarrow y)$ *iff* $x * z \leq y$ (adjointness)
(2) *A residuated lattice is a* BL-algebra *iff the following identities hold for all* $x, y \in L$*:*
(i) $x \wedge y = x * (x \Rightarrow y)$ (divisibility)
(ii) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ (prelinearity)

A BL-algebra is linearly ordered iff its lattice ordering is linear, i. e., for all $x, y \in L$, $x \wedge y = x$ or $x \wedge y = y$. Linearly ordered BL-algebras are called *BL-chains.*

From the adjointness condition, we get $x \Rightarrow y = 1$ iff $x \leq y$. Moreover, $x \vee y = ((x \Rightarrow y) \Rightarrow y) \wedge ((y \Rightarrow x) \Rightarrow x)$.

In a BL-algebra we define the operation $-$ of *precomplement*: $-x = x \Rightarrow 0$.

An element $x \in L$ is *idempotent* iff $x * x = x$.

BL algebras form a variety (see [Háj98b], Lemma 2.3.10). The variety of BL-algebras will be denoted **BL**.

The following theorem shows that any subvariety of **BL** is fully characterized by its linearly ordered elements. It was proved in [Háj98b], Lemma 2.3.16.

**Theorem 2.1.2 (Subdirect representation for BL)** *Each BL-algebra is a subdirect product of a family of BL-chains.*

The following lemma is (a part of) [Háj98a], Lemma 1:

**Lemma 2.1.3** *Let* $\mathbf{L} = (L, \vee, \wedge, *, \Rightarrow, 0, 1)$ *be a BL-algebra and* $x, y, u \in L$*. Then*
(i) *if* $x \leq u \leq y$ *and* $u$ *is idempotent then* $x * y = x$
(ii) *if* $x < u \leq y$ *and* $u$ *is idempotent then* $y \Rightarrow x = x$

### 2.1.1 T-algebras

T-algebras are an important subclass of BL-algebras, as they determine a *standard* interpretation of the logical language. Besides, t-algebras play an important part in applications.

**Definition 2.1.4** *A t-norm is a binary operation* $*$ *on* $[0,1]$*, satisfying the conditions:*
(i) $*$ *is commutative and associative*
(ii) $*$ *is non-decreasing in both arguments*
(iii) $1 * x = x$ *and* $0 * x = 0$ *for all* $x \in [0,1]$.

The operation $\Rightarrow$ satisfying, for all $x, y, z \in [0,1]$, $x * z \leq y$ iff $z \leq x \Rightarrow y$, is determined in a unique way by $*$ iff $*$ is left continuous. In that case $x \Rightarrow y = \max\{z \mid x * z \leq y\}$. If $\Rightarrow$ is unique, it is called the *residuum* of $*$ (cf. [Háj98b], Lemma 2.1.4) and $(*, \Rightarrow)$ is called the *residuated pair*.

Note that the residuated pair $(*, \Rightarrow)$ satisfies the *divisibility condition* $x \wedge y = x * (x \Rightarrow y)$ iff $*$ is continuous. In that case also $x \vee y = ((x \Rightarrow y) \Rightarrow y) \wedge (y \Rightarrow x) \Rightarrow x)$.

**Definition 2.1.5 (t-algebra)** *A t-algebra given by a continuous t-norm* $*$ *is the algebra* $[0,1]_* = \langle [0,1], *, \Rightarrow, 0 \rangle$*, where* $*$ *is a continuous t-norm and* $\Rightarrow$ *is its residuum.*

Note that each continuous t-norm $*$ determines fully its t-algebra $[0,1]_*$. It is an easy observation that each t-algebra is a BL-chain. The underlying lattice is fixed and therefore can be omitted from the signature. We choose the signature so as to correspond to the basic propositional language (see 2.2.1). *Terminology:* the terms 't-algebra', 'standard algebra' and 't-norm algebra' are all used frequently and have the same meaning. In this work we prefer 't-algebras', but sometimes we use collocations like 'standard completeness' or 'standard interpretation'; these all refer to the semantics on $[0,1]$.

It is natural to ask whether any BL-algebra on $[0,1]$ must be a t-algebra. An affirmative answer was given in [CEGT00], Lemma 5.1: in any dense BL-chain the operation $*$ is continuous w. r. t. the order topology; and if the universe is $[0,1]$, then $*$ satisfies all the conditions defining a t-norm.

Three examples of continuous t-norms stand out (cf. 2.1.6): the Łukasiewicz t-norm, the Gödel t-norm, and the product t-norm; their definitions, with the respective residua for $x > y$, are listed below. By the above definition, $x \Rightarrow y = 1$ iff $x \leq y$.

| | $x * y$ | $x \Rightarrow y$ |
|---|---|---|
| Łukasiewicz | $x + y - 1$ | $1 - x + y$ |
| Gödel | $\min(x, y)$ | $y$ |
| product | $x.y$ | $y/x$ |

The following theorem explains their importance immediately. Because of the light it sheds on the structure of continuous t-norms, it is often referred to as the 'characterization', 'decomposition' or 'representation' theorem.

The theorem was first proved in [MS57]. Here we quote the version from [Háj98b], Theorem 2.1.16 and the preceding Remark. The set of all idempotents of $*$ is a closed subset of $[0,1]$. Its complement is a union of countably many pairwise disjoint open intervals; denote this set of intervals $\mathcal{I}_o$. Let $[a,b] \in \mathcal{I}$ iff $(a,b) \in \mathcal{I}_o$ (so $\mathcal{I}$ is the set of corresponding closed intervals for the intervals in $\mathcal{I}_o$).

**Theorem 2.1.6 (Representation theorem for continuous t-norms)** *Let $*$ be any continuous t-norm.*
(i) *For each $[a,b] \in \mathcal{I}$, $*$ on $[a,b]$ is isomorphic either to the product t-norm (on $[0,1]$) or to Łukasiewicz t-norm (on $[0,1]$).*
(ii) *If for $x, y \in [0,1]$ there is no $[a,b] \in \mathcal{I}$ such that $x, y \in [a,b]$, then $x * y = \min(x,y)$.*

As proved in Chapter 3, isomorphisms of continuous t-norms also preserve the respective residua, so if two continuous t-norms are isomorphic, so are the t-algebras they generate.

## 2.2 The propositional calculus BL

We define a propositional calculus BL naturally corresponding to t-algebras.

### 2.2.1 Language

The alphabet has countably many propositional variables, basic connectives 0 (constant), &, $\rightarrow$ (binary), and defined connectives 1 (constant), $\neg$ (unary), $\wedge$, $\vee$, and $\equiv$ (binary). Formulas are defined as usual.

Propositional connectives are defined from &, $\rightarrow$, and 0 as follows:

$$\begin{array}{rll} \neg\varphi & \text{is} & \varphi \rightarrow 0 \\ \varphi \wedge \psi & \text{is} & \varphi \,\&\, (\varphi \rightarrow \psi) \\ \varphi \vee \psi & \text{is} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \varphi \equiv \psi & \text{is} & (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi) \\ 1 & \text{is} & 0 \rightarrow 0 \end{array}$$

Let $\mathbf{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ be a BL-algebra. An $\mathbf{L}$-evaluation of propositional variables is a mapping $e$, assigning to each propositional variable $p$ an element of $L$. Each evaluation of propositional variables extends uniquely to propositional formulas as follows:

$$
\begin{aligned}
e(0) &= 0 \\
e(\varphi \,\&\, \psi) &= e(\varphi) * e(\psi) \\
e(\varphi \to \psi) &= e(\varphi) \Rightarrow e(\psi)
\end{aligned}
$$

Then also $e(1) = 1$, $e(\neg\varphi) = -e(\varphi)$, $e(\varphi \wedge \psi) = e(\varphi) \wedge e(\psi)$, $e(\varphi \vee \psi) = e(\varphi) \vee e(\psi)$.

There is a 1-1 correspondence between formulas of propositional BL and terms in the language of BL-algebras; the term results from the formula by replacing all connectives with the operations which evaluate them and by replacing propositional variables by object variables, and is called the *associated term* of the formula. Conversely, if operation symbols in a term are replaced by propositional connectives and object variables are replaced by propositional variables, the result is referred to as the *associated formula* of the term.

**Definition 2.2.1** (i) *A formula* $\varphi$ *is a* 1-tautology *of a BL-algebra* $\mathbf{L}$ *(an* $\mathbf{L}$*-tautology) iff* $e(\varphi) = 1$ *for all* $\mathbf{L}$*-evaluations* $e$ *(i. e., iff its associated term always has the value* 1*).*
(ii) *A formula is a* t-tautology *iff it is a* 1*-tautology of each t-algebra.*

### 2.2.2 The Basic Fuzzy Logic

**Definition 2.2.2** *The following formulas are the axioms of the Basic Fuzzy Logic (BL):*
(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \,\&\, \psi) \to \varphi$
(A3) $(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$
(A4) $(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$
(A5b) $((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $0 \to \varphi$
*The deduction rule of BL is modus ponens. Proofs are defined as usual.*

Soundness of BL w. r. t. BL-algebras is proved in the usual way, i. e., by induction on the length of proofs. It is very easy to verify that all axioms of

BL are **L**-tautologies of each BL-algebra **L**, and that modus ponens preserves this property.

### 2.2.3 Completeness theorem

The following is [Háj98b], Theorem 2.3.19. The implication (iii) to (ii) is an immediate consequence of the subdirect representation theorem for BL-algebras.

**Theorem 2.2.3 (Completeness theorem for BL)**
*BL is complete w. r. t. the variety of BL-algebras; the following three conditions are equivalent:*
(i) BL $\vdash \varphi$
(ii) *for each BL-algebra* **L**, $\varphi$ *is an* **L**-*tautology*
(iii) *for each BL-chain* **L**, $\varphi$ *is an* **L**-*tautology.*

## 2.3 Subvarieties and schematic extensions

**Definition 2.3.1** *A propositional calculus $\mathcal{C}$ (in the BL-language) is a schematic extension of BL if it results from BL by adding some axiom schemata.*

Let $\mathcal{C}$ be a schematic extension of BL. A BL-algebra **L** is a $\mathcal{C}$-algebra iff any $\varphi \in \mathcal{C}$ is an **L**-tautology.

The proof of the completeness theorem for BL also yields completeness for schematic extensions (in the BL language): a schematic extension $\mathcal{C}$ of BL proves $\varphi$ iff $\varphi$ is an **L**-tautology for each $\mathcal{C}$-algebra **L** iff $\varphi$ is an **L**-tautology for each $\mathcal{C}$-chain **L**.

Some extensions of BL receive special attention: we discuss Łukasiewicz logic, Gödel logic, product logic and SBL.

### 2.3.1 MV algebras and Łukasiewicz logic

Łukasiewicz logic was introduced as a formal system of many-valued logic for the semantics given by $[0,1]_{\text{Ł}}$, during the twenties. Depending on context, Łukasiewicz logic may be introduced with various sets of basic connectives. Here we view it as an extension of BL.

**Definition 2.3.2** Łukasiewicz logic Ł *is a schematic extension of BL with the propositional schema* $\neg\neg\varphi \to \varphi$.

Alternatively, $((\varphi \to \psi) \to \psi) \vee ((\psi \to \varphi) \to \varphi)$ together with BL yields the Łukasiewicz logic.

The BL-algebras corresponding to this schematic extensions are usually called *MV-algebras*, but they lurk in many contexts under a bunch of names. The calculus introduced by Łukasiewicz has been investigated by Wajsberg; *Wajsberg algebras* are termwise equivalent to MV-algebras. The term MV-algebras comes from Chang. A thorough study of MV-algebras has been carried out in [CDM00], where references and historical remarks are also to be found.

We list a few facts of interest. First, it is to be noted that the t-algebra $[0,1]_{\text{Ł}}$ is an MV-algebra. It has been shown in [dN91] that any MV-chain can be embedded in an ultrapower of $[0,1]_{\text{Ł}}$. The paper [dNL99] shows that each subvariety of MV can be described by a single equation in two variables; thus the lattice of subvarieties of **MV** is countable.

Subdirect representation of MV-algebras by MV-chains follows from [Háj98b], Theorem 2.3.16, but was first proved by Chang in [Cha59]. Thus Ł is complete w. r. t. MV-chains.

The famous Chang's completeness theorem ([Cha58] and [Cha59]) states that Łukasiewicz logic is complete w. r. t. $[0,1]_{\text{Ł}}$ (i. e., it is the standard completeness theorem for Ł). By associating to every MV-chain a linearly ordered Abelian group, he proved that every MV-chain is partially embeddable into $[0,1]_{\text{Ł}}$. It immediately follows that $[0,1]_{\text{Ł}}$ generates the variety **MV**. There are many proofs of this fact available, see [CDM00], Chapter 2 for references. One may also consult [Háj98b], Chapter 10.

### 2.3.2 Gödel algebras and Gödel logic

The study of Gödel logic starts with Gödel's paper [Göd32], in which he introduces finite Gödel chains in order to show that the intuitionistic propositional calculus is not complete w. r. t. any many-valued semantics. Later, Dummett extended the intuitionistic calculus with the prelinearity axiom, thus making it complete w. r. t. $[0,1]_{\text{G}}$ (and w. r. t. any infinite Gödel chain).

Again here we introduce Gödel logic as an extension of **BL**.

**Definition 2.3.3** Gödel logic G *is a schematic extension of BL with the propositional schema* $\varphi \to (\varphi \,\&\, \varphi)$.

Thus in Gödel logic the connectives & and $\wedge$ coincide.

The members of the corresponding subvariety **G** of **BL** are called *Gödel algebras*. They have subdirect representation by chains on the ground of being

BL-algebras. The fact that operations on a Gödel chain are order-determined implies that any subset of a Gödel chain which contains 0 and 1 is a Gödel subchain and any finite Gödel chain can be embedded into any Gödel chain of greater cardinality. Consequently, $[0, 1]_G$ generates the variety **G** of Gödel algebras.

### 2.3.3 Product algebras and product logic

Product logic has been axiomatized in [EGH96]. Although the truth functions of $[0, 1]_\Pi$ have been used previously (e. g., the standard interpretation of the product implication is sometimes referred to as *Goguen* implication), a systematic study of product logic and algebras has not been undertaken until BL emerged.

**Definition 2.3.4** Product logic $\Pi$ *is a schematic extension of BL with the propositional schemata*

$$\neg\neg\chi \to [(\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi)] \qquad (\Pi 1)$$

*and*

$$\varphi \wedge \neg\varphi \to 0 \qquad (\Pi 2)$$

Alternatively, [Cin01a] shows that the axiom $\neg\neg\varphi \to ((\varphi \to \varphi \,\&\, \psi) \to \psi \,\&\, \neg\neg\psi)$ together with BL yields the product logic and it is impossible to obtain it by adding a single schema in one variable to BL.

It is interesting to note that there are no non-trivial finite product algebras. In fact, as pointed out in [AFM03], the variety **P** of product algebras has only one proper non-trivial subvariety, namely, the variety of Boolean algebras.

Standard completeness has been proved in [Háj98b], Chapter 4. Again the proof is carried out by establishing partial embeddability of every product chain into $[0, 1]_\Pi$.

Łukasiewicz logic has a faithful interpretation in product logic. The proof rests on the following ([Háj98b], Lemma 4.1.14).

**Theorem 2.3.5** *Let* $0 < a < 1$*. Then* $[0, 1]_Ł$ *is isomorphic to the BL-algebra* $\langle [a, 1], \min, \max, *_a, \Rightarrow, a, 1\rangle$ *(the* restricted product algebra*), where for* $x, y \in [a, 1]$*,* $x *_a y = \max(a, x * y)$ *(where* $a \in (0, 1)$*;* $*$ *and* $\Rightarrow$ *are the product t-norm and its residuum).*

### 2.3.4 SBL

SBL is an extension of BL introduced in [EGHN00]. The *chains* (linearly ordered algebras) in the variety generated by SBL are exactly those chains whose precomplement $-$ is *strict*, i. e., $-x = 1$ for $x = 0$, otherwise $-x = 0$. One way to achieve this is to add the schema $\varphi \wedge \neg\varphi \to 0$ (the axiom $\Pi 2$) to BL.

**Definition 2.3.6** *SBL is a schematic extension of BL with the axiom schema* $\Pi 2$.

Standard completeness of SBL follows from [CEGT00].

It is easy to verify that a saturated BL-chain is an SBL-chain iff either it has a first segment which is not an MV-segment (thus it is a Gödel or product segment), or it has no first segment (see Section 2.4 for definitions).

## 2.4 The structure of BL-chains

The paper [Háj98a] was the first to come with the idea that a theorem analogous to the representation theorem for continuous t-norms might be provable for (saturated) BL-chains. The paper sketches steps necessary to proving such theorem, leaving some of them as open problems. The proof was completed in [CEGT00].

In the following we reproduce some key notions and theorems from [Háj98a].

**Definition 2.4.1** *Let $\langle I, \leq \rangle$ be a linearly ordered set with the least element* $0$ *and greatest element* $1$. *For $i \in I$ let $i^+$ be the upper neighbour of $i$ in $I$ if it exists, otherwise $i^+ = i$. For each $i \in I$, let $\mathbf{L}_i$ be a BL-chain such that its least element is $i$, its greatest element is $i^+$ and the non-extremal elements do not belong to any $L_j, j \neq i$.*
*The* ordered sum $\bigoplus_{i \in I} \mathbf{L}_i$ *is defined as follows:*
(i) *the domain is* $\bigcup_{i \in I} L_i$
(ii) *define $x \leq y$ iff $x \leq_i y$ or $x \in L_i$, $y \in L_j$, $j > i$*
(iii) *$x * y = x *_i y$ for $x, y \in L_i$, otherwise* $x * y = \min(x, y)$
(iv) *$x \Rightarrow y = 1$ iff $x \leq y$*
(v) *$x \Rightarrow y = x \Rightarrow_i y$ for $x, y \in L_i$ and $x > y$*
(vi) *$x \Rightarrow y = y$ for $x \in L_i$, $y \in L_j$, $y \neq j^+$ and $i > j$.*

**Lemma 2.4.2** *In the above notation, $\mathbf{L} = \bigoplus_{i \in I} \mathbf{L}_i$ is a BL-chain.*

Instead of 'ordered' sum $\mathbf{L}$ some authors say 'ordinal'; the algebras $\mathbf{L}_i$ are often are called *components*; here we call them *segments*. In particular, we refer to the isomorphic copies of $[0, 1]_{Ł}$, $[0, 1]_{G}$, and $[0, 1]_{\Pi}$ in t-algebras as Ł-segments, G-segments, and $\Pi$-segments respectively.

**Definition 2.4.3** *Let* $\mathbf{L}$ *be a BL-chain.*
(i) $\mathbf{L}$ *is* (sum-)reducible *iff there are* $\mathbf{L}_1$, $\mathbf{L}_2$ *(each with at least two elements), and* $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$.
(ii) *A pair* $X, Y \subseteq L$ *is a* cut *in* $\mathbf{L}$ *iff* $X \cup Y = L$, $x \in X$ *and* $y \in Y$ *implies* $x \leq y$ *for all* $x, y \in L$, $Y$ *is closed under* $*$, *and* $x * y = x$ *for each* $x \in X$, $y \in Y$.
(iii) $\mathbf{L}$ *is* saturated *iff for each cut* $X, Y \in L$ *there is an idempotent* $c$ *such that* $x \in X$ *and* $y \in Y$ *implies* $x \leq c \leq y$.

**Theorem 2.4.4** *Each BL-chain is a dense subalgebra of a saturated BL-chain.*

**Theorem 2.4.5** *Each saturated BL-chain is an ordered sum of saturated irreducible BL-chains.*

A question posed by Hájek in [Háj98a] was whether each irreducible saturated BL-chain was an MV- or product chain. In fact he suggested two additional axioms B1 and B2 which (imposed on BL-algebras) would guarantee this. In [CEGT00] the authors showed that B1 and B2 were redundant (provable in BL).

The variant of the representation theorem for saturated BL-chains as presented in [CEGT00] takes each maximal interval of idempotent elements as one segment, a Gödel chain (which is not irreducible unless it is trivial). The authors prove that for a saturated BL-chain the set of its idempotents $\mathcal{E}$ is closed, thus the complement is a union of a set $\mathcal{I}_o$ of open intervals. Let $I$ be the set of corresponding closed intervals. $[a, b]_L$ denotes the algebra on $[a, b]$ with restricted operations of $\mathbf{L}$. $G(\mathcal{E})$ is the set of maximal non-trivial intervals of idempotents; $\mathcal{E}_{\text{is}}$ denotes the set of *isolated* idempotents.

**Theorem 2.4.6 (Representation theorem for saturated BL-chains)** *Let* $\mathbf{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be a saturated BL-chain. Then*
(i) *For each* $[a, b] \in I$, *the algebra* $[a, b]_L$ *is either an MV-chain or a product chain;*
(ii) *If for* $x < y \in L$ *there is no* $[a, b] \in I$ *s. t.* $x, y \in [a, b]$, *then* $x * y = x$ *and* $y \Rightarrow x = x$. *In particular, for each* $[a, b] \in G(\mathcal{E})$, $[a, b]_L$ *is a Gödel chain;*
(iii) *Let* $J = \{a : a \in \mathcal{E}_{\text{is}} \vee \exists b\,([a, b] \in I \cup G(\mathcal{E}) \vee [b, a] \in I \cup G(\mathcal{E}))\}$ *(ordered as in* $\mathbf{L}$*). For each* $j \in J$ *let* $\mathbf{L}_j$ *be the trivial one-element BL-algebra if* $j$ *has no successor in* $J$, *or the algebra* $[j, j^+]_L$. *Then* $\mathbf{L} = \bigoplus_{j \in J} \mathbf{L}_j$ *(sum defined as in Definition 2.4.1).*

### 2.4.1 Standard completeness of BL

Thanks to the partial embeddability results for all three types of segments, Theorem 2.4.6 has most interesting consequences; above all, it shows that the variety BL is generated by its subclass of t-algebras.

**Theorem 2.4.7** *Let $\sigma = \tau$ be an identity valid in all t-algebras. Then it is valid in all BL-algebras.*

This theorem was proved in [Háj98a], Theorem 7, with the reservation that the BL-algebras must satisfy the two axioms B1 and B2. The proof can be briefly paraphrased as follows: assume there is a BL-algebra $\mathbf{A}$ in which $\sigma = \tau$ does not hold. W. l. o. g. we may assume $\mathbf{A}$ is a saturated chain. Moreover, we may assume that $\mathbf{A} = \bigoplus_{i=1}^{n} \mathbf{A}_i$ (i. e., that $\mathbf{A}$ is a finite sum of BL-chains). Fix a particular evaluation $e$ under which the identity does not hold. Now define a t-algebra $\mathbf{B}$ with $n$ segments, such that if $\mathbf{A}_i$ is an MV-algebra, then the segment $\mathbf{B}_i$ is isomorphic to $[0, 1]_{\text{Ł}}$, and analogously for $\Pi$ and G. Then for each $i$, $\mathbf{A}_i$ is partially embeddable into $\mathbf{B}_i$, thus $\mathbf{A}$ is partially embeddable into $\mathbf{B}$ (cf. Lemma 2.1.3). Thus the complete evaluation $e(\varphi)$ in $\mathbf{A}$ determines a complete evaluation $e'(\varphi)$ in $\mathbf{B}$, under which $\sigma = \tau$ does not hold.

**Theorem 2.4.8** BL *is complete w. r. t. t-algebras: a formula $\varphi$ is a theorem of* BL *iff it is a t-tautology.*

It is an interesting consequence of Theorem 2.3.5 that the variety **BL** is generated by all t-algebras which are finite ordered sums of Ł-segments. This was proved in [BHMV02], serving as a basis for the main result that t-tautologies are in coNP. Thus one gets

**Theorem 2.4.9** *A formula is a t-tautology iff it holds in all t-algebras which are finite sums of Ł-segments.*

## 2.5 The importance of hoops

For their illuminative effect on BL-algebras, we include a short and selective survey of hoops. A thorough algebraic study of hoops has been carried out in [Fer92], a work containing also historical remarks on the evolution of hoops; perhaps a more accessible material on hoops is [BF00]. Recent studies of hoops motivated by research on BL include [EGHM03], where the propositional calculi given by hoops are studied, [AFM03], where hoops are studied from the point of view of continuous t-norms, and [AM03], which obtains crucial results by representing BL-chains as ordered sums of Wajsberg hoops.

The material in this section is taken from [Fer92], [AFM03], [AM03] and [EGHM03]. We start with a definition which makes it obvious that hoops form a variety (cf. [Fer92], Corollary 1.17).

**Definition 2.5.1** *A* hoop *is an algebra* $\mathbf{H} = \langle H, *, \Rightarrow, 1\rangle$*, such that* $\langle H, *, 1\rangle$ *is a commutative monoid and for all* $x, y, z \in H$*,*
(i) $x \Rightarrow x = 1$
(ii) $x * (x \Rightarrow y) = y * (y \Rightarrow x)$
(iii) $x \Rightarrow (y \Rightarrow z) = (x * y) \Rightarrow z$.

For any $x, y \in H$ we define $x \leq y$ iff $x \Rightarrow y = 1$. This ordering is a meet-semilattice and $x \wedge y = x * (x \Rightarrow y)$ for all $x, y \in H$ (cf. [Fer92], Prop. 1.12).

**Definition 2.5.2** *Let* $\mathbf{H} = \langle H, *, \Rightarrow, 1\rangle$ *be a hoop.*
(i) $\mathbf{H}$ *is* Wajsberg[I] *iff* $(x \Rightarrow y) \Rightarrow y) = (y \Rightarrow x) \Rightarrow x)$ *for all* $x, y \in H$*.*
(ii) $\mathbf{H}$ *is* basic *iff* $(((x \Rightarrow y) \Rightarrow z) \Rightarrow (((y \Rightarrow x) \Rightarrow z) \Rightarrow z)) = 1$ *for all* $x, y, z \in H$*.*
(iii) $\mathbf{H}$ *is* cancellative *iff* $x = y \Rightarrow (x * y)$ *for all* $x, y \in H$*.*
*An algebra* $\langle H, *, \Rightarrow, 0, 1\rangle$ *is a* bounded hoop *iff* $\langle H, *, \Rightarrow, 1\rangle$ *is a hoop and* $0 \leq x$ *for all* $x \in H$*.*

The following theorem is [Fer92], Prop. 4.4 (i) and 4.6 (ii), [BP94], Theorem 1.19 (iii) and [AFM03], Corollary 3.4 (iv).

**Theorem 2.5.3** (i) *Every cancellative hoop is Wajsberg.*
(ii) *Let* $\mathbf{H}$ *be a linearly ordered hoop and* $|H| > 1$*. Then* $\mathbf{H}$ *is cancellative iff it is an unbounded Wajsberg hoop.*
(iii) *Bounded Wajsberg hoops are termwise equivalent to MV-algebras.*
(iv) *Bounded basic hoops are termwise equivalent to BL-algebras.*

We define the notion of ordered sum of hoops along the lines of [AM03], which differs from ordered sum of BL-algebras by its treatment of the unit element.

[I] Wajsberg hoops were called Łukasiewicz hoops in [Fer92]

**Definition 2.5.4** *Let $(I, \leq)$ be a linearly ordered set, and for $i \in I$ let $\mathbf{H}_i$ be a hoop s. t. for $i \neq j$, $H_i \cap H_j = \{1\}$. The* ordered sum *$\bigoplus_{i\in I} \mathbf{H}_i$ is defined as follows:*
(i) *the domain is $\bigcup_{i\in I} H_i$*
(ii) *define $x \leq y$ iff either $x, y \in H_i$ and $x \leq_i y$, or $x \in H_i \setminus \{1\}$ and $y \in H_j, j > i$*
(iii) *$x * y = x *_i y$ for $x, y \in H_i$, otherwise $x * y = \min(x, y)$*
(iv) *$x \Rightarrow y = 1$ iff $x \leq y$*
(v) *$x \Rightarrow y = x \Rightarrow_i y$ for $x, y \in H_i, i \in I$*
(vi) *$x \Rightarrow y = y$ for $y \in H_j \setminus \{1\}$, $x \in H_i, i > j$.*

We say that a hoop or BL-algebra is *(sum-)irreducible* iff it cannot be decomposed as $\bigoplus_{i\in I} \mathbf{H}_i$, where $|I| \geq 2$ and $\mathbf{H}_i, i \in I$ are nontrivial hoops. By [AM03], Theorem 3.6, a linearly ordered hoop (BL-algebra) is sum-irreducible iff it is a Wajsberg hoop (an MV-algebra). The authors prove the following theorem (cf. [AM03], Theorem 3.7):

**Theorem 2.5.5** *Every linearly ordered hoop (BL-algebra) is an ordered sum of Wajsberg hoops (where the first segment is an MV-algebra).*

By Proposition 7.2 of [AFM03], $\mathbf{A}$ is a subdirectly irreducible product algebra iff $\mathbf{A} = 2$ or $\mathbf{A} = 2 \oplus \mathbf{C}$ where 2 is the Boolean algebra on $\{0, 1\}$ and $\mathbf{C}$ is a subdirectly irreducible cancellative hoop. Therefore, given a decomposition of a saturated BL-chain into MV-, G-, and Π-chains, the hoop decomposition is obtained by decomposing each product chain into $2 \oplus \mathbf{C}$ where $\mathbf{C}$ is a cancellative hoop, and regarding each G-segment as an ordered sum of 2's.

Using the hoop decomposition of BL-chains, [AM03] obtain a characterization of those BL-chains which generate the whole variety $\mathbf{BL}$. In the following, $\mathbf{Wa}_n$ is the MV-algebra on $n + 1$ elements and $\mathbf{Q}$ is the (MV-)subalgebra of $[0, 1]_{\text{Ł}}$ on the rationals.

**Theorem 2.5.6** *Let $\mathbf{L} = \bigoplus_{i\in I} \mathbf{H}_i$ be a decomposition of a BL-chain into Wajsberg hoops (thus the decomposition is unique, $I$ has a minimum $0$ and $\mathbf{H}_0$ is an MV-chain). Then $\mathbf{L}$ is BL-generic iff:*
(i) *the MV-algebra $\mathbf{Q}$ is embeddable in $\mathbf{H}_0$*
(ii) *for each $n$, there are infinitely many $i$ such that $\mathbf{Wa}_n$ can be embedded into $\mathbf{H}_i$.*

As a corollary one obtains the following characterization of t-algebras which generate $\mathbf{BL}$:

**Corollary 2.5.7** *A t-algebra generates the variety* **BL** *iff its first segment is Ł and it has infinitely many Ł-segments.*

## 2.6 Some expansions of language

This section contains very brief information on some calculi obtained by expanding the propositional language of BL. We shall work with the unary propositional connectives $\Delta$ (called the $\Delta$-projection or Baaz's $\Delta$[II)]) and $\sim$ (called the involutive negation).

The material on $\Delta$ is a brief excerpt from [Háj98b], Section 2.4, the material on $\sim$ is taken from [EGHN00].

### 2.6.1 Adding the $\Delta$ connective

In a BL-chain $\mathbf{L}$, the semantics of $\Delta$ is a function $\Delta : L \longrightarrow L$ such that $\Delta(1) = 1$ and $\Delta(a) = 0$ for $a < 1$. Correspondingly, one expands the language of BL with the unary connective $\Delta$.

**Definition 2.6.1** *The logic* BL$\Delta$ *(with language of BL expanded by* $\Delta$*) has the following axioms:*
($\Delta$1) $\Delta\varphi \vee \neg\Delta\varphi$
($\Delta$2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
($\Delta$3) $\Delta\varphi \to \varphi$
($\Delta$4) $\Delta\varphi \to \Delta\Delta\varphi$
($\Delta$5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$
*Deduction rules of* BL$\Delta$ *are modus ponens and* $\Delta$*-generalization: from* $\varphi$ *derive* $\Delta\varphi$*.*

A *schematic extension* of BL$\Delta$ is a calculus $\mathcal{C}\Delta$ obtained from a schematic extension $\mathcal{C}$ of BL by adding the above axioms and deduction rule for $\Delta$.

**Definition 2.6.2** *A* BL$\Delta$*-algebra is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1, \Delta\rangle$ *such that* $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *is a BL-algebra and the following is true:*
(i) $\Delta x \vee -\Delta x = 1$
(ii) $\Delta(x \vee y) \leq \Delta x \vee \Delta y$
(iii) $\Delta x \leq x$
(iv) $\Delta x \leq \Delta\Delta x$

II) In fact, a similar connective with the same interpretation on $[0, 1]$ appears in [Sko57], [TT92] and other works.

(v) $(\Delta x) * \Delta(x \Rightarrow y) \leq \Delta y$
(vi) $\Delta 1 = 1$

Let $\mathcal{C}\Delta$ be a schematic extension of BL$\Delta$. A $\mathcal{C}\Delta$-algebra is a BL$\Delta$-algebra **L** such that all the additional schemata of $\mathcal{C}$ are **L**-tautologies.

**Theorem 2.6.3 (Completeness theorem for BL$\Delta$ and extensions)** *Let $\mathcal{C}\Delta$ be a schematic extension of BL$\Delta$. The following conditions are equivalent:*
(i) $\mathcal{C}\Delta \vdash \varphi$
(ii) *for each $\mathcal{C}\Delta$-algebra **L**, $\varphi$ is an **L**-tautology*
(iii) *for each $\mathcal{C}\Delta$-chain **L**, $\varphi$ is an **L**-tautology*

### 2.6.2 Adding an involutive negation

Any t-algebra can be enriched with a decreasing involution, i. e., a function $\sim: [0,1] \longrightarrow [0,1]$, s. t. for all $x, y \in [0,1]$, $\sim\sim x = x$ and if $x \leq y$, then $\sim y \leq \sim x$. A particular and important example of a decreasing involution is the function $1 - x$ on $[0,1]$. In $[0,1]_{\text{Ł}}$, this is the truth function of the definable negation $-x = x \Rightarrow 0$. In other t-algebras however, the decreasing involution is distinct from the definable negation. As the interplay of the two negation functions is best observed when the defined negation is strict, usually involutive negations are added to calculi whose natural negation is the strict negation, i. e., to extensions of SBL.

Interestingly, if $\sim$ is added to SBL or its extensions, the $\Delta$ connective is definable[III]: $\Delta\varphi$ is $\neg{\sim}\varphi$ (cf. [EGHN00], p. 109). The calculus SBL with $\sim$ will be denoted SBL$_\sim$.

**Definition 2.6.4** *Axioms of* SBL$_\sim$ *are the axioms of* SBL *plus*[IV]
($\sim$1) $\sim\sim\varphi \equiv \varphi$
($\sim$2) $\neg\varphi \to \sim\varphi$
($\sim$3) $\Delta(\varphi \to \psi) \to \Delta(\sim\psi \to \sim\varphi)$
($\Delta$1) $\Delta\varphi \vee \neg\Delta\varphi$
($\Delta$2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
($\Delta$5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$
*Deduction rules of* SBL$_\sim$ *are modus ponens and $\Delta$-generalization: from $\varphi$ derive $\Delta\varphi$.*

---

[III] i. e., the following definition of $\Delta$ from $\sim$ and $\neg$ defines the $\Delta$-projection in all chains
[IV] It has been proved in [Cin01b] that the axioms $\Delta$1, $\Delta$2, $\Delta$5 are redundant in the definition.

A *schematic extension* of $\mathrm{SBL}_\sim$ is a calculus $\mathcal{C}_\sim$ obtained from a schematic extension $\mathcal{C}$ of SBL by adding the above axioms and deduction rule for $\sim$.

**Definition 2.6.5** *An* $\mathrm{SBL}_\sim$*-algebra is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1, \sim\rangle$ *where* $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *is an* SBL*-algebra, and the axioms (*$(\sim 1)$*–*$(\sim 3)$*,* $(\Delta 1)$*,* $(\Delta 2)$*,* $(\Delta 5)$ *are* $\mathbf{L}$*-tautologies.*

Let $\mathcal{C}_\sim$ be a schematic extension of $\mathrm{SBL}_\sim$. A $\mathcal{C}_\sim$-algebra is an $\mathrm{SBL}_\sim$-algebra $\mathbf{L}$ such that all the additional schemata of $\mathcal{C}$ are $\mathbf{L}$-tautologies.

[EGHN00] proves subdirect representation by chains for $\mathrm{SBL}_\sim$-algebras, and consequently the following completeness theorem holds:

**Theorem 2.6.6 (Completeness theorem for** $\mathrm{SBL}_\sim$ **and extensions )** *Let* $\mathcal{C}_\sim$ *be a schematic extension of* $\mathrm{SBL}_\sim$*. The following conditions are equivalent:*
(i) $\mathrm{SBL}_\sim \vdash \varphi$
(ii) *for each* $\mathcal{C}_\sim$*-algebra* $\mathbf{L}$*,* $\varphi$ *is an* $\mathbf{L}$*-tautology*
(iii) *for each* $\mathcal{C}_\sim$*-chain* $\mathbf{L}$*,* $\varphi$ *is an* $\mathbf{L}$*-tautology*

As for standard completeness, one has to specify the "standard" interpretation of the involutive negation first; a more restrictive approach is to take the function $1 - x$ on $[0, 1]$ as the standard interpretation, a liberal one is to allow any decreasing involution on $[0, 1]$. The paper [EGHN00] takes the former approach; we adopt it and call the algebras obtained from a standard BL-algebra by adding any decreasing involution $\sim$*-standard.*[V)]

[EGHN00] shows standard completeness for $\mathrm{G}_\sim$ (Gödel logic with involutive negation), i. e., completeness w. r. t. the standard Gödel algebra with $1 - x$ for the involutive negation. (In fact, all standard Gödel algebras with arbitrary decreasing involutions are isomorphic).

[EGHN00] also shows that the calculus $\Pi_\sim$ fails to capture the tautologies of $[0, 1]_\Pi$ with $1 - x$. For $\Pi_\sim$ it obtains $\sim$*-standard completeness* (w. r. t. all $\sim$-standard algebras obtained from $[0, 1]_\Pi$ by adding an arbitrary decreasing involution). The result is based on the following lemma (cf. [EGHN00], Lemma 10):

**Lemma 2.6.7** *Let* $0 < a_0 < \cdots < a_k < 1$ *be reals. Then there is a decreasing involution* $n$ *on* $[0, 1]$ *such that* $n(a_i) = a_{k-i}$ *for* $i = 0, \ldots, k$*.*

---

[V)] [EGHN00] used the term 'semi-standard'.

# Chapter 3

# On t-algebras

This chapter contains some observations on isomorphisms of t-algebras, derives some interesting consequences of the BL-chain and hoop representation of t-algebras, and shows that t-algebras generate only countably many subvarieties of **BL**.

The material in this chapter comes from [Han01] and [Han02].

## 3.1 Isomorphisms of t-algebras

We investigate isomorphisms of continuous t-norms and of t-algebras.

As shown in this section, isomorphisms of continuous t-norms also preserve the respective residua, so if two continuous t-norms are isomorphic, so are the t-algebras they generate.

Let $*_1$ and $*_2$ be two continuous t-norms, and $[a_1, b_1]$, $[a_2, b_2]$ be two subintervals of $[0, 1]$. We say that $*_1$ on $[a_1, b_1]$ is isomorphic to $*_2$ on $[a_2, b_2]$ iff there is a bijection $f : [a_1, b_1] \longrightarrow [a_2, b_2]$, such that $\forall x, y \in [a_1, b_1](f(x *_1 y) = f(x) *_2 f(y))$. If $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$, we say that $*_1$ and $*_2$ are isomorphic.

We sometimes say that two subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[0, 1]$ are isomorphic: this is always in the context of there being $*_1$ and its residuum $\Rightarrow_1$ defined on $[a_1, b_1]$, and similarly $*_2$ and $\Rightarrow_2$ on $[a_2, b_2]$, and it means that there is a bijection $f$ preserving the operation $*$ *and* the operation $x \Rightarrow y$ for $x > y$.

### 3.1.1 Classes of isomorphism of t-norms

The relation of being isomorphic is an equivalence on the class of continuous t-norms. As continuous functions are determined by their values on the rationals, there are at most $2^\omega$ continuous t-norms. We present several ways to conceive $2^\omega$ distinct continuous t-norms. Thus the class of continuous t-norms has the cardinality $2^\omega$.

Let $*$ be an arbitrary continuous t-norm. How to obtain all of its isomorphic copies? Any isomorphism is increasing on $[0,1]$, since if $x < y$, then by [Háj98b], 2.1.7 (1), $\exists z(z * y = x)$, so $f(z) *_1 f(y) = f(x)$, so $f(x) < f(y)$. Thus any isomorphism of two continuous t-norms is continuous.

**Lemma 3.1.1** *Let $f$ be a continuous, increasing, bijective mapping of $[0,1]$ onto itself. Let $*$ be a continuous t-norm. For each $a, b \in [0,1]$, define $a *_1 b = f(f^{-1}(a) * f^{-1}(b))$; then $*_1$ is a continuous t-norm (isomorphic to $*$).*

*Proof.* From the definition of $*_1$ we get $f(a * b) = f(a) *_1 f(b)$. Observe that $f^{-1}$ is also increasing and $f^{-1}(c *_1 d) = f^{-1}(c) * f^{-1}(d)$. Indeed, put $a = f^{-1}(c)$, $b = f^{-1}(d)$; then $f^{-1}(c *_1 d) = f^{-1}(f(a) *_1 f(b)) = f^{-1}(f(a * b)) = a * b = f^{-1}(c) * f^{-1}(d)$.

Now let us check that $*_1$ as defined above is a t-norm. Commutativity is clear; as for associativity,
$a *_1 (b *_1 c) = f[f^{-1}(a) * f^{-1}(b *_1 c)] = f[f^{-1}(a) * f^{-1}(f(f^{-1}(b) * f^{-1}(c)))] = f[f^{-1}(a) * (f^{-1}(b) * f^{-1}(c))] = f[(f^{-1}(a) * f^{-1}(b)) * f^{-1}(c)] = f[f^{-1}(f(f^{-1}(a) * f^{-1}(b))) * f^{-1}(c)] = f[f^{-1}(a *_1 b)] *_1 f(f^{-1}(c)) = (a *_1 b) *_1 c$.

Since $f$ and $f^{-1}$ are increasing, if $c \le d$, then $c *_1 a = f(f^{-1}(c) * f^{-1}(a)) \le f(f^{-1}(d) * f^{-1}(a)) = a *_1 d$; the rest follows from commutativity.

$1 *_1 x = f(1 * f^{-1}(x)) = f(f^{-1}(x)) = x$, and $0 *_1 x = f(0 * f^{-1}(x)) = f(0) = 0$.

Since $*_1$ is a composition of two continuous mappings on $[0,1]$, it is continuous. $\mathcal{QED}$

The following lemma determines cardinalities of the equivalence classes.

**Lemma 3.1.2** *Let $*$ be a continuous t-norm. If $a, b \in [0,1]$ exist so that $a * b < \min(a, b)$, then there are $2^\omega$ distinct continuous t-norms isomorphic to $*$; otherwise there are no continuous t-norms isomorphic to $*$ (and distinct from $*$).*

*Proof.* Take two elements $a$ and $b$ for which $a * b < \min(a, b)$; obviously $0 < a, b < 1$, and w. l. o. g. we may assume $a < b$. Fix $0 < f(a) < f(b) < 1$ and define the rest of a homeomorphism $f$ on $[0, 1]$ by choosing $f(a * b)$ arbitrarily in $(0, f(a))$ (there is a continuum of values to choose from) and letting $f$ be, e. g., piecewise linear connecting 0, $f(a * b)$, $f(a)$, $f(b)$, 1.
This defines $2^\omega$ distinct homeomorphisms $f$, and these in turn define $2^\omega$ distinct continuous t-norms $*'$ isomorphic to $*$ (the t-norms differ (at least) in the value of $f(a) *' f(b)$).
If no such $a$ and $b$ exist, then $*$ is the Gödel t-norm. It is easy to check that the Gödel t-norm has no isomorphic copies distinct from itself. $\mathcal{QED}$

Thus each continuous t-norm has $2^\omega$ isomorphic copies, except for the Gödel t-norm, the equivalence class of which is a singleton.

It is an easy consequence of the Representation theorem 2.1.6 that there are $2^\omega$ equivalence classes of isomorphism.

**Lemma 3.1.3** *There are $2^\omega$ pairwise non-isomorphic continuous t-norms.*

*Proof.* Define pairwise non-isomorphic continuous t-norms for distinct infinite sequences of 0's and 1's. Take the sequence $1 - 1/2^i, i \in N$ of reals. The elements of the sequence, together with 1, will be (the only) idempotents of the t-norms. Pick a sequence of 0's and 1's and define $*$ on each of the intervals $[1 - 1/2^{i-1}, 1 - 1/2^i]$, $i \in N \setminus \{0\}$, to be isomorphic to the Łukasiewicz t-norm if the $i$-th element of the sequence is 0, and to the product t-norm otherwise.

We show that t-norms defined by distinct sequences are pairwise non-isomorphic. If an isomorphism existed between two t-norms $*_1$ and $*_2$ defined by distinct sequences, then clearly it would have to map each of the idempotents $1 - 1/2^i$ onto itself, and hence the t-norms would have to be isomorphic on each interval $[1 - 1/2^{i-1}, 1 - 1/2^i]$. Suppose the sequences differ on $i$-th position; if $*_1$ and $*_2$ were isomorphic on $[1 - 1/2^{i-1}, 1 - 1/2^i]$, then we would get an isomorphism between $*_{\mathrm{L}}$ and $*_\Pi$ on $[0, 1]$. $\mathcal{QED}$

### 3.1.2 Residuation

We prove that if two continuous t-norms are isomorphic via $f$, then so are their residua.

**Theorem 3.1.4** *Let $*_a$, $*_b$ be two continuous t-norms, let $0 \leq a_1 < a_2 \leq 1$, where $a_1$ and $a_2$ are idempotents of $*_a$, let $0 \leq b_1 < b_2 \leq 1$ and let $f$ be a bijective mapping of $[a_1, a_2]$ onto $[b_1, b_2]$, such that* $(\forall x, y \in [a_1, a_2])\, f(x *_a y) =$

*$f(x) *_b f(y)$. Then*
(i) *$f$ is increasing (and hence continuous) on $[a_1, a_2]$, and*
(ii) *$(\forall x, y \in [a_1, a_2])$, $y > x$ implies $f(y \Rightarrow_a x) = f(y) \Rightarrow_b f(x)$.*

*Proof.* (i) Suppose $x < y$, $x, y \in [a_1, a_2]$. Then by [Háj98b], 2.1.7 (1), $\exists z(z *_a y = x)$. In this case $a_1 \leq z \leq a_2$, because $a_1 *_a z = (a_1 *_a y) *_a z = a_1 *_a x = a_1$ and if $z > a_2$, then $z *_a y = y$. Thus $f(z) *_b f(y) = f(x)$ and therefore $f(x) \leq f(y)$. Because $f(x) = f(y)$ is impossible, $f(x) < f(y)$.
(ii) By definition, $f(y) \Rightarrow_b f(x) = \max z : z *_b f(y) \leq f(x)$. Because $f$ is increasing, this is equal to $f(\max u : u *_a y \leq x)$, i. e., $f(y \Rightarrow_a x)$. $\mathcal{QED}$

Note, however, that an analogous statement cannot be formed for homomorphisms. For example, consider two t-algebras $[0,1]_{*_1}$ and $[0,1]_{*_2}$, where $*_1$ has a non-extremal idempotent $1/2$, and $*_1$ on $[1/2, 1]$ is isomorphic to $*_2$ on $[0,1]$. Then a mapping $f : [0,1] \longrightarrow [0,1]$, such that $f(x) = 0$ for $x \in [0, 1/2]$ and $f(x) = 2x - 1$ otherwise, is a homomorphism of $*_1$ and $*_2$, but not of $\Rightarrow_1$ and $\Rightarrow_2$.

## 3.2 T-algebras as ordered sums

Since t-algebras are considered here mainly as semantics for propositional calculi in the BL-language, it is usually unnecessary to distinguish two isomorphic t-algebras. It is common to consider each class of isomorphism of t-algebras as a single *type of sum.*

As already indicated in Chapter 2, we consider two slightly different notions of ordered sum: first, a sum of BL-chains, second, a sum of hoops.

Below we give some useful observations based on these sum representations of t-algebras.

### 3.2.1 Ordered sums of Ł, G, and Π segments

**Definition 3.2.1** *Let $[0,1]_*$ be a t-algebra. An idempotent $x$ of $[0,1]_*$ is called a* cutpoint *iff there is no interval $[a, b]$, $a < b$, of idempotents such that $x \in (a, b)$.*

For any t-algebra, its cutpoints form a closed subset of $[0,1]$. For $I$, $J$ two sets of cutpoints on $[0,1]$ we say that they are isomorphic iff they are isomorphic as ordered sets $\mathbf{I} = \langle I, \leq \rangle$, $\mathbf{J} = \langle J, \leq \rangle$, where $\leq$ is the natural ordering of reals. For $i \in I$ we define $i^+$ to be its successor in $I$, otherwise

$i^+ = i$. Naturally if two t-algebras are isomorphic, then so are their respective sets of cutpoints.

We take each *maximal* (w. r. t. inclusion) interval of idempotents as an isomorphic copy of $[0,1]_G$. If $[u,v]$, $u < v$, is an interval of idempotents of $[0,1]_*$, then it is isomorphic to $[0,1]_G$, as is easily verified.

The *layout* of a t-algebra $[0,1]_*$ with cutpoints $\mathbf{I}$ is an assignment of (one of the symbols) Ł, G, and Π to each $[i,i^+]$, $i < i^+$, $i \in \mathbf{I}$.

**Observation 3.2.2** *Let $[0,1]_*$ be a t-algebra with cutpoints $\mathbf{I}$ and a layout $\lambda$ on $\mathbf{I}$, and let $[0,1]_{*_1}$ be a t-algebra with cutpoints $\mathbf{J}$ and a layout $\lambda_1$. Then $[0,1]_*$ and $[0,1]_{*_1}$ are isomorphic iff $\mathbf{I}$ and $\mathbf{J}$ are isomorphic (as ordered sets) via some $f$ and for each $i \in \mathbf{I}$, $i < i^+$, $\lambda([i,i^+]) = \lambda_1([f(i), f(i^+)])$.*

*Proof.* It is obvious that if two t-algebras are isomorphic, so are their sets of cutpoints, and (by the argument of the proof of Lemma 3.1.3), the layouts must match on each corresponding pair of non-trivial intervals. On the other hand, it is easy to define isomorphism of t-algebras on the basis of isomorphism of their sets of cutpoints and their matching layouts. $\mathcal{QED}$

Each equivalence class of isomorphic standard algebras is thus determined by an equivalence class of pairwise isomorphic sets of cutpoints and a layout for these sets.

Throughout this work we use a rather informal terminology and notation based on the representation theorems for continuous t-norms and for BL-chains. Often it is said that some BL-chain is an ordered sum of MV, Gödel and product *segments*. In t-algebras the three types of segments are denoted (according to their type of isomorphism) with symbols Ł, G, and Π (the same symbols also denote the three schematic extensions of BL corresponding to the respective t-norms; we use a more detailed notation wherever confusion could arise). When decomposing a t-algebra into BL-chains, each copy of Gödel counts as one segment (unless explicitly stated otherwise), thus, e. g., $[0,1]_{Ł\oplus G\oplus \Pi}$ is a t-algebra with three segments, namely a sum of a copy of the Łukasiewicz algebra, a copy of the Gödel algebra and a copy of the product algebra; the type of the sum is Ł⊕G⊕Π.

### 3.2.2 Ordered sums of hoops

By Theorem 3.7 of [AM03] (reproduced as Theorem 2.5.5), every BL-chain is an ordered sum of Wajsberg hoops. In case the BL-chain is a t-algebra, the summands are (isomorphic copies of) one of the three following Wajsberg hoops: the standard MV-algebra $[0,1]_Ł$ (denoted Ł), the algebra $\langle(0,1], *_\Pi, \Rightarrow_\Pi, 1\rangle$ on the semi-open unit interval with the t-norm and residuum of the standard product algebra (denoted $C$), and the Boolean algebra on $\{0,1\}$ (denoted 2). Note that the hoop decomposition of a t-algebra always has a first element and this is either 2 or Ł.

By [AFM03], Prop. 7.2, $\mathbf{A}$ is a subdirectly irreducible product algebra iff it is an ordered sum of 2 and a subdirectly irreducible cancellative hoop; this shows how to obtain a hoop decomposition from a BL-chain decomposition of a t-algebra. As a consequence of the proposition, every subdirectly irreducible cancellative hoop is partially embeddable into $C$.

Moreover, because of the isomorphism of the restricted product algebra with $[0,1]_Ł$ (see Lemma 4.1.14 of [Háj98b]), $C$ is partially embeddable into Ł. 2 is a subalgebra of Ł and thus embeddable into Ł.

Note that by definition of ordered sum of hoops, the values of operations $*$ and $\Rightarrow$ for two arguments belonging to different hoops are determined by the ordering.

### 3.2.3 Embedding partial subalgebras

**Definition 3.2.3** *Let $\mathbf{L} = \bigoplus_{i\in I} \mathbf{H}_i$ be a BL-chain, $\mathbf{H}_i$, $i \in I$ be Wajsberg hoops and let $a_1 < \cdots < a_n$ be elements of $L$. The* subsum of hoops given by *$a_1, \ldots, a_n$ is a BL-chain which is an ordered sum of $\mathbf{H}_0$ and each $\mathbf{H}_i$ s. t. for some $j \in \{1, \ldots, n\}$ we have $a_j \in \mathbf{H}_i$, in the ordering given by $\mathbf{L}$.*

Note that from the assumption that $\mathbf{L}$ is a BL-chain, $I$ has the least element, which we denote 0. If $\mathbf{H}_0 \oplus \cdots \oplus \mathbf{H}_m$, $m \leq n$ is the subsum of hoops of $\mathbf{L}$ given by $a_1, \ldots, a_n$, then it is a subalgebra of $\mathbf{L}$ and (thus) a BL-chain.

This implies the following. Let $\mathbf{L} = \bigoplus_{i\in I} \mathbf{H}_i$ be a BL-chain, $\varphi(p_1, \ldots, p_n)$ a formula and $e$ an arbitrary evaluation in $\mathbf{L}$. Let $\mathbf{H}_0 \oplus \cdots \oplus \mathbf{H}_m$ be the subsum of hoops of $\mathbf{L}$ given by $e(p_1), \ldots, e(p_n)$. Then the subsum $\mathbf{H}_0 \oplus \cdots \oplus \mathbf{H}_m$ contains the complete evaluation $e(\varphi)$ in $\mathbf{L}$, i. e., the values of all subformulas.

Now we prove several lemmas.

**Lemma 3.2.4** *Let $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$ be an SBL t-algebra. If there are infinitely many i's s. t. $\mathbf{A}_i$ is* Ł*, then $\mathbf{A}$ generates the variety* **SBL**.

*Proof.* We show that if $\varphi$ is not an SBL-tautology, then it is not a 1-tautology in $\mathbf{A}$.

In the formulation of the statement it is not important whether we view **A** as a sum of BL-chains or a sum of hoops since only the Ł-segments are important and these can be viewed as both. In the proof we prefer to view **A** as a sum of Wajsberg hoops (Ł, $C$, and 2).

If $\varphi(p_1 \dots, p_k)$ is not an SBL-tautology, then by standard completeness of SBL, there is an SBL t-algebra **B** and an evaluation $e(\varphi) < 1$ in **B**. **B** is an ordered sum of Wajsberg hoops Ł, $C$, and 2; note that $\mathbf{B}_0$ is not Ł (thus it is 2). Let $\mathbf{B}_0 \oplus \cdots \oplus \mathbf{B}_{k'}$, $k' \leq k$, be the subsum of hoops in **B** given by $e(p_1), \dots, e(p_k)$. This is a subalgebra of **B** and hosts the complete evaluation $e(\varphi) < 1$.

Recall that $C$ is partially embeddable into Ł. Since there are infinitely many Ł-segments in **A**, it is possible to find a subsum $\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{k'}$ in **A**, such that $\mathbf{A}_0$ is the initial hoop in **A** (which is *not* Ł and thus it is 2), and each of the $\mathbf{A}_i$, $i = 1, \dots, n$ is Ł, and a partial embedding of the complete evaluation $e(\varphi)$ in $\mathbf{B}_0 \oplus \cdots \oplus \mathbf{B}_{k'}$ into $\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{k'}$. The image then yields an evaluation $e'$ in **A** such that $e'(\varphi) < 1$. Thus $\varphi$ is not a 1-tautology of **A**. $\mathcal{QED}$

We note that the assumptions in the lemma *characterize* t-algebras which generate **SBL**, as proved in [EGM03].

**Lemma 3.2.5** *If $\varphi(p_1, \dots, p_k)$ is not a t-tautology, then it is not a 1-tautology of some t-algebra which is an ordered sum of at most $k+1$ BL-chains.*

*Proof.* If $\varphi(p_1 \dots, p_k)$ is not a t-tautology, then there is a t-algebra **A** and an evaluation $e(\varphi) < 1$ in **A**. Let $\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{k'}$, $k' \leq k$, be the subsum of hoops of **A** given by $e(p_1), \dots, e(p_k)$. This is a BL-chain consisting of at most $k+1$ hoops, which hosts a complete evaluation $e(\varphi) < 1$. It need not be a t-algebra but can be suitably "padded" so that the result is a t-algebra which is a sum of at most $k+1$ Ł, G, and Π-segments. (First, replace each $2 \oplus C$ with a Π; then replace each 2 with a G, then replace each $C$ with a Π.)

As the resulting algebra has $\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{k'}$ as a subalgebra, $e$ gives a complete evaluation of $\varphi$ in the resulting algebra for which $e(\varphi) < 1$, thus $\varphi$ is not a 1-tautology in this algebra. $\mathcal{QED}$

One may avoid using G-segments during the "padding" and thus obtain a finite sum of Ł- and Π-segments only, which may be useful (as that is also a finite sum of hoops).

By combining the ideas of the above proofs, we get the following (cf. [BHMV02], Theorem 3, where the bound was $4|\varphi| + 1$):

**Lemma 3.2.6** *If $\varphi(p_1, \dots, p_k)$ is not a t-tautology, then it is not a tautology of some t-algebra which is a sum of at most $k + 1$ Ł-segments.*

*Proof.* If $\varphi(p_1 \dots, p_k)$ is not an t-tautology, then there is an t-algebra $\mathbf{A}$ and an evaluation $e(\varphi) < 1$ in $\mathbf{A}$. Let $\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{k'}$, $k' \leq k$, be the subsum of hoops in $\mathbf{A}$ given by $e(p_1), \dots, e(p_k)$. This is a subalgebra of $\mathbf{A}$ and hosts the complete evaluation $e(\varphi) < 1$. $\mathbf{A}_0$ is either 2 or Ł.

Since 2 is embeddable into Ł and $C$ is partially embeddable into Ł, $\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{k'}$ is partially embeddable into $k' + 1$-potent sum of Ł's. On this basis, from $e(\varphi)$ in $\mathbf{A}$ one can obtain an evaluation $e'$ in the $k+1$-potent sum of L's, such that $e'(\varphi) < 1$. $\mathcal{QED}$

In the following section we will need a lemma, the idea of which comes from [Han01], 3.6.5:

**Lemma 3.2.7** *Let $[0, 1]_*$ and $[0, 1]_{*'}$ be two t-algebras which are infinite sums of* G- *and* Π*-segments (without* Ł*-segments). Then* $\mathrm{TAUT}^{[0,1]_*} = \mathrm{TAUT}^{[0,1]_{*'}}$.

*Proof.* An observation to be made is that if a t-algebra is an infinite sum without Ł-segments, then there are infinitely many Π-segments in the sum. Seen as a sum of hoops, there is an infinite alternating sequence of hoops of type 2 and $C$ in the sum. Observe that the presence/absence of G-segments inbetween Π-segments and the ordering type of the index set for the sum have no influence on the set of tautologies.

Let $\varphi(p_1, \dots, p_k)$ be a formula which does not hold in $[0, 1]_*$, fix an evaluation $e$ in $[0, 1]_*$ s. t. $e(\varphi) < 1$. Let $\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{k'}$, $k' \leq k$ be the sum of hoops in $[0, 1]_*$ given by $e(p_1), \dots, e(p_k)$. As $[0, 1]_{*'}$ has an infinite alternating sequence of hoops 2 and $C$, one can find a subsum $\mathbf{A}'_0 \oplus \cdots \oplus \mathbf{A}'_{k'}$ in $[0, 1]_{*'}$ such that $\mathbf{A}_i$ and $\mathbf{A}'_i$ are of the same type for $i = 0, \dots, k'$. Thus there is an evaluation $e'$ in $[0, 1]_{*'}$ such that $e'(\varphi) < 1$.

The other inclusion is analogous. $\mathcal{QED}$

This can be generalized: let $[0, 1]_*$ be an arbitrary t-algebra with two non-extremal idempotents $0 < c_1 < c_2 < 1$. Define two new t-algebras by substituting copies of two arbitrary infinite sums without Ł-segments into the interval $[c_1, c_2]$. Then the resulting two t-algebras will have the same sets of tautologies.

## 3.3 Varieties generated by t-algebras

In this section we show that the set of tautologies of each t-algebra can be fully described by a finite string in a simple finite alphabet; thus there are only countably many subvarieties of **BL** generated by a single t-algebra. This result appeared first in [Han02], where the aim was to prove coNP-completeness of the set of tautologies of each t-algebra.

**Theorem 3.3.1** *There are only countably many subvarieties of* **BL** *which are generated by a single t-algebra.*

One ingredient to this claim is Corollary 2.5.7 which is a direct consequence of the general theorem of [AM03]. It characterizes those t-algebras which generate the whole variety **BL**, as ones that have a first segment Ł and contain infinitely many Ł-segments. The finite description of the tautologies of each of these algebras is $\infty$Ł.

Further, Lemma 3.2.4 tells us that there is a large class of t-algebras each of which generates the variety **SBL**. These are t-algebras with infinitely many Ł-segments but without an initial Ł-segment. The finite description for members of this class is $\Pi \oplus \infty$Ł.

It thus remains to build the finite descriptions for t-algebras which are sums of Ł, G, and Π-segments with only finitely many Ł-segments. We may immediately turn our attention to infinite sums, since for finite sums the description is the sum itself.

Let **A** be a t-algebra with $n$ Ł-segments. Inbetween each two consecutive Ł-segments there can be either a finite (possibly void) subsum of G- and Π-segments, or an infinite subsum of Π-segments including possibly some G-segments. The encoding of a finite subsum is the sum itself. The encoding of an infinite subsum consisting of Π- and possibly G-segments is $\infty\Pi$, which, by Lemma 3.2.7 and subsequent generalization, captures all we need to know about the sum as far as tautologies are concerned.

As each of the encodings is a finite word in the alphabet $\{$Ł$, \mathrm{G}, \Pi, \infty$Ł$, \infty\Pi\}$, there are only countably many subvarieties of **BL** which are generated by a single t-algebra.

We remark that [Han02] has been followed by the paper [EGM03], which characterizes conditions under which, given two t-algebras **A** and **B**, one has $\mathrm{Var}(\mathbf{A}) \subseteq \mathrm{Var}(\mathbf{B})$ and axiomatizes the tautologies of each t-algebra.

# Chapter 4

# Complexity of t-norm logics

This chapter studies the computational complexity of propositional logics given by t-algebras. In Section 4.1 we introduce the topic and go over relevant known results. These concern especially BL and its three extensions Ł, G, and Π. Section 4.2 determines the complexity of 1-tautologies of individual t-algebras. The material therein comes mostly from [Han02]. Section 4.3 is based on [Han03] and considers the complexity of t-norm logics with an involutive negation.

## 4.1 Known results for BL and extensions

### 4.1.1 Prelude: classical logic and complexity

We shall work with the NP and coNP classes from the polynomial hierarchy. In the scope of this chapter, saying 'reducible' without attributes means polynomial-time (many-one) reducibility.

To be able to compare results on classical and many-valued logic, let us suppose that the (basic) connectives of formulas of the classical propositional logic are $\&$, $\to$ and $0$.

Then we define

$$\mathrm{TAUT} = \{\varphi; \forall e(e(\varphi) = 1)\}$$
$$\mathrm{SAT} = \{\varphi; \exists e(e(\varphi) = 1)\}$$

where $\varphi$ runs over all formulas in the basic language.

Obviously $\mathrm{TAUT} = \{\neg\varphi; \varphi \notin \mathrm{SAT}\}$ and (consequently, considering the semantic equivalence of $\varphi$ and $\neg\neg\varphi$,) $\mathrm{SAT} = \{\neg\varphi; \varphi \notin \mathrm{TAUT}\}$. Here $\neg\varphi$ is as usual, $\varphi \to 0$. Thus TAUT is reducible to the complement of SAT and vice versa.

By Cook's theorem the SAT problem is NP-complete, thus TAUT is coNP-complete.

As any reasonable axiomatics of the classical propositional logic is complete w. r. t. TAUT, it is pointless to distinguish the set of tautologies of the two-element Boolean algebra and the set of provable formulas.

### 4.1.2 SAT and TAUT in many-valued setting

With a many-valued logic $\mathcal{C}$, or a $\mathcal{C}$-algebra $\mathbf{A}$, impediments occur which have no analogy in the classical case.

*First*, one might wonder about the definition of the SAT and TAUT problems, as the classical dichotomy is no longer at hand to solve such difficult questions like what we mean by 'satisfied'.

For a fixed semantics given by an algebra $\mathbf{A}$, it makes sense to distinguish the following sets of formulas (cf. [Háj98b], Section 6.1). In all cases $\varphi$ stands for propositional formulas in the BL-language and $e_{\mathbf{A}}$ runs over evaluations in $\mathbf{A}$.

$$\begin{aligned}
\mathrm{TAUT}_1^{\mathbf{A}} &= \{\varphi : \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{TAUT}_{\mathrm{pos}}^{\mathbf{A}} &= \{\varphi : \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\} \\
\mathrm{SAT}_1^{\mathbf{A}} &= \{\varphi : \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{SAT}_{\mathrm{pos}}^{\mathbf{A}} &= \{\varphi : \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\}
\end{aligned}$$

These sets are referred to as 1-tautologies, positive tautologies, 1-satisfiable formulas and positively satisfiable formulas of $\mathbf{A}$.

For a class $K$ of algebras of the same type, one may generalize:

$$\begin{aligned}
\mathrm{TAUT}_1^{K} &= \{\varphi : \forall \mathbf{A} \in K\, \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{TAUT}_{\mathrm{pos}}^{K} &= \{\varphi : \forall \mathbf{A} \in K\, \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\} \\
\mathrm{SAT}_1^{K} &= \{\varphi : \exists \mathbf{A} \in K\, \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{SAT}_{\mathrm{pos}}^{K} &= \{\varphi : \exists \mathbf{A} \in K\, \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\}
\end{aligned}$$

Let us recall here that a *t-tautology* is a formula which is a 1-tautology of each t-algebra.

*Second*, for particular fixed semantics, there need not be a simple relationship between its TAUT and SAT problems. The 1-tautologies will probably be most of interest; however, the other sets present themselves to be investigated as well.

*Third*, it may be that a calculus $\mathcal{C}$ we have in mind is complete w. r. t. a class of algebras, but individual members of this class generate their own (stronger) calculi; perhaps the calculus $\mathcal{C}$ is not complete w. r. t. the set of tautologies of any particular algebra.

For a calculus $\mathcal{C}$, it then makes sense to ask about the complexity of the set of $\mathcal{C}$-provable formulas.

We now give a short overview of known propositional complexity results for BL, Ł, G, and Π. All these logics enjoy standard completeness, thus it is unnecessary to distinguish provability and standard 1-tautologousness. We write $\mathrm{TAUT}_1^{\text{Ł}}$ for $\mathrm{TAUT}_1^{[0,1]_{\text{Ł}}}$ etc.

### 4.1.3 Łukasiewicz logic

For the Łukasiewicz t-algebra $[0,1]_{\text{Ł}}$, whose negation is involutive, it is easy to establish the following relationships: $\mathrm{TAUT}_1^{\text{Ł}} = \{\neg\varphi : \varphi \notin \mathrm{SAT}_{\mathrm{pos}}^{\text{Ł}}\}$ and $\mathrm{TAUT}_{\mathrm{pos}}^{\text{Ł}} = \{\neg\varphi : \varphi \notin \mathrm{SAT}_1^{\text{Ł}}\}$.

The following theorem comes from [Mun87]; proofs are readily available also in [CDM00], Section 9.3, and in [Háj98b], Section 6.2.

**Theorem 4.1.1** *The sets* $\mathrm{SAT}_{pos}^{\text{Ł}}$, $\mathrm{SAT}_1^{\text{Ł}}$ *are NP-complete. Thus the sets* $\mathrm{TAUT}_1^{\text{Ł}}$, $\mathrm{TAUT}_{\mathrm{pos}}^{\text{Ł}}$ *are coNP-complete.*

Containment in NP can be shown, as it is in [Háj98b], by reduction to the *mixed integer programming* (MIP) problem. Also a slight modification of the SAT problem for $[0,1]_{\text{Ł}}$ can be shown to be in NP via reduction to MIP: namely, given a set of equations in variables $0 = x_0 < \cdots < x_n = 1$, of the form $x_i * x_j = x_k$ or $x_i \Rightarrow x_j = x_k$, where $i, j, k \in \{0, 1, \ldots, n\}$ and $*$ and $\Rightarrow$ are operations on $[0,1]_{\text{Ł}}$, do these equations have a solution preserving the prescribed ordering? This more general reduction is described in detail in [BHMV02], and reproduced here in the proof of Lemma 4.1.5.

### 4.1.4 Gödel and product logics

For Gödel and product logics one cannot establish relationships analogous to the case of Łukasiewicz logic between tautologies and satisfiable formulas.

Actually, as the following theorem states, both 1-satisfiable and positively satisfiable formulas of G and of Π coincide with classically satisfiable formulas:

**Theorem 4.1.2** *The sets* $\mathrm{SAT}_1^{\mathrm{G}}$*,* $\mathrm{SAT}_{\mathrm{pos}}^{\mathrm{G}}$*,* $\mathrm{SAT}_1^{\Pi}$*,* $\mathrm{SAT}_{\mathrm{pos}}^{\Pi}$ *are all NP-complete. Moreover, they are all equal to the classical* SAT*.*

This result was conceived in [BHKŠ98], where authors show that $\mathrm{SAT}_1^{\Pi}$, $\mathrm{SAT}_1^{\mathrm{G}}$ are NP-complete, and received the final touch in [Háj98b], Section 6.2. It holds more generally for any BL-chain with the strict negation (cf. also Theorem 4.2.2).

The sets of 1-tautologies for G and Π are distinct from the classical case and from each other. However, the situation resembles the classical case insofar as the 1-tautology problem for both these logics is coNP-complete.

**Theorem 4.1.3** (i) $\mathrm{TAUT}_{\mathrm{pos}}^{\mathrm{G}} = \mathrm{TAUT}_{\mathrm{pos}}^{\Pi} = \mathrm{TAUT}$
(ii) $\mathrm{TAUT}_1^{\mathrm{G}}$ *is coNP-complete and* $\mathrm{TAUT}_1^{\Pi}$ *is coNP-complete.*

(i) comes from [Háj98b] and (ii) has been proved in [BHKŠ98] (but the authors disclaim the results on Gödel logic as folklore).

### 4.1.5 BL

For BL, provable formulas coincide with t-tautologies thanks to the standard completeness theorem of [CEGT00].

The following result comes from [BHMV02]. It is important to note that by Theorem 3 of [BHMV02], a formula is a t-tautology iff it is a 1-tautology of all t-algebras which are finite sums of Ł-segments (cf. also Lemma 3.2.6).

**Theorem 4.1.4** *The set of t-tautologies is coNP-complete.*

The coNP-hardness of the t-tautology problem has been established via reduction to $\mathrm{TAUT}_1^{\text{Ł}}$ (cf. also Theorem 4.2.1).

The coNP-containment of the problem is shown by presenting a nondeterministic algorithm working in polynomial time and accepting the complement of t-tautologies. Here we give a slightly modified version of the algorithm, planning to use its various modifications later. To make the presentation self-contained, below we also quote from [BHMV02] Hähnle's reduction of the problem of satisfiability of a system of equations and inequalities to the MIP problem.

`Preliminaries.` For a propositional formula $\varphi$, let $m$ denote the number of subformulas in $\varphi$. In case $\varphi$ only contains binary connectives, $m$ is $2m' - 1$, where $m'$ is the number of occurrences of propositional variables and of the constant 0 in $\varphi$.

Further, $k$ denotes the number of propositional variables in $\varphi$.

`// algorithm T-TAUT accepting non-t-tautologies`

`input:` $\varphi$

`begin`

`guessCardinality()` Pick at random a natural $n$ s. t. $0 < n \leq k + 1$.[I)]

Let $\mathbf{A}$ be the t-algebra which is a sum of $n$ Ł-segments.

`nameSubformulas()` Fix an enumeration of all subformulas of $\varphi$; assume $\varphi$ gets the index 1. For $m$ being the number of subformulas in $\varphi$, introduce variables $x_1 \ldots, x_m$, and assign the variable $x_i$ to the subformula $\varphi_i$ of $\varphi$ (thus $x_1$ is assigned to $\varphi$).

`cutpointVariables()` Introduce variables $z_0, \ldots, z_n$ for the cutpoints of $\mathbf{A}$, enumerated so that $z_i$ represents the $i$-th cutpoint of $\mathbf{A}$ in the ordering of reals; $z_0$ is intended for 0 and $z_n$ is intended for 1.

Set $V = \{z_0, \ldots, z_n\} \cup \{x_1, \ldots, x_m\}$.

`guessOrder()` Guess a linear ordering $\preceq$ of variables in $V$, such that $x_1 \prec z_n$. (Write the variables down in a sequence, for each consequent pair $a$ and $b$ decide whether $a \approx b$ or $a \prec b$.)

`checkOrder()` Check that $\preceq$ satisfies basic natural conditions: first, that preserves the natural strict ordering of the $z$-variables, given by the cutpoints they represent, i. e., if $i < j$, then $z_i \prec z_j$. Second, any variable assigned to the constant 0 must be $\approx$-equal to $z_0$, the variable denoting the least cutpoint.

`We` say that variables $x_j$ s. t. $z_i \preceq x_j \preceq z_{i+1}$ *belong to* $i$ or *are in* $i$.

`checkExternal()` Check external soundness of $\preceq$: for $\varphi_i$, $\varphi_j$ subformulas of $\varphi$ ( $1 \leq i, j \leq m$),

– if $\varphi_i \,\&\, \varphi_j$ is a subformula $\varphi_k$ of $\varphi$ for some $k \in \{1, \ldots, m\}$ and, for some $l \in \{0, \ldots, n\}$, we have $x_i \preceq z_l \preceq x_j$, then $x_k \approx x_i$;

– if $\varphi_i \to \varphi_j$ is a subformula $\varphi_k$ of $\varphi$ for some $k \in \{1, \ldots, m\}$ and $x_i \preceq x_j$, then $x_k \approx z_n$;

– if $\varphi_i \to \varphi_j$ is a subformula $\varphi_k$ of $\varphi$ for some $k \in \{1, \ldots, m\}$ and for some $l \in \{0, \ldots, n\}$, we have $x_j \prec z_l \preceq x_i$, then $x_k \approx x_j$.

[I)] See Lemma 3.2.5.

`checkInternal()` Check internal soundness of $\preceq$ for each interval $[z_i, z_{i+1}]$, $i = 0, \ldots, n-1$ in $\preceq$. Consider variables in $i$. Construct a system $\mathcal{S}_i$ of equations and inequalities; $\mathcal{S}_i$ is initially empty. For each subformula $\varphi_l$ which is $\varphi_j \mathbin{\&} \varphi_k$, if $x_j$ and $x_k$ are in $i$, check $x_l$ is also in $i$ and put equation $x_j * x_k = x_l$ into $\mathcal{S}_i$. For each subformula $\varphi_l$ which is $\varphi_j \to \varphi_k$, such that $x_k \prec x_j$, if $x_j$ and $x_k$ are in $i$, check $x_l$ is also in $i$ and put equation $x_j \Rightarrow x_k = x_l$ into $\mathcal{S}_i$.

Further, put all equations and inequalities defined by $\preceq$ for the variables in $i$ into $\mathcal{S}_i$. Check whether the system $\mathcal{S}_i$ has a solution in the $i$-th segment of $\mathbf{A}$, such that $z_i$ and $z_{i+1}$ evaluate to the cutpoints delimiting the $i$-th segment of $\mathbf{A}$.

`end`

Since each segment of $\mathbf{A}$ is isomorphic to $[0,1]_{\text{Ł}}$ it is obvious that the system $\mathcal{S}_i$ in the `checkInternal()` step is solvable in the $i$-th segment of the algebra iff it is solvable in $[0,1]_{\text{Ł}}$, in such a way that $z_i$ evaluates to 0 and $z_{i+1}$ evaluates to 1.

The ordering $\preceq$ contains clusters of mutually $\approx$-equivalent variables, so to simplify things, we choose one representative from each cluster and replace with it all the occurrences in $\mathcal{S}_i$ of all the other variables in the cluster; accordingly in $\preceq$ we consider only the strict ordering of these representatives. We assume w. l. o. g. the chosen representatives are $z_i = x_{i_0} < \cdots < x_{i_{j-1}} = z_{i+1}$, their total number being $j \leq m+2$. We proceed to call the system $\mathcal{S}_i$. Thus $\mathcal{S}_i$ is a system of at most $m$ equations in $j$ variables, with a strict ordering of length $j$.

**Lemma 4.1.5** [II] *The problem of solvability of each $\mathcal{S}_i$ in $[0,1]_{\text{Ł}}$ is in NP.*

*Proof.* To reduce the strict inequalities in $\mathcal{S}_i$ to one strict and many non-strict inequalities, add a new variable $q$, postulate $q < 1$ and replace each strict inequality $x_{i_l} < x_{i_{l+1}}$ by $x_{i_l} \Rightarrow x_{i_{l+1}} \leq q$.

Then eliminate $*$: since $x*y = \neg(x \Rightarrow \neg y)$ and $\neg x = x \Rightarrow 0$, each condition $x*y = z$ can be replaced by $y \Rightarrow 0 = y_1$, $x \Rightarrow y_1 = z_1$ and $z_1 \Rightarrow 0 = z$.

Replace each equality by two (non-strict) inequalities ($\leq$ and $\geq$). Then use the following replacement: $x \Rightarrow y \geq z$ iff there are $s_1, s_2 \in [0,1]$ such that $x \leq s_1$, $y \geq s_2$ and $z + s_1 + s_2 = 1$. $x \Rightarrow y \leq z$ iff there are $s_1, s_2 \in [0,1]$ and $t \in \{0,1\}$ such that $x \geq s_1$, $y \leq s_2$, $t - z \leq 0$, $t + s_1 \leq 1$, $t \leq s_2$ and $t + z + s_1 - s_2 = 1$. (Finally add for each variable $x$, the conditions $x \geq 0$ and $x \leq 1$).

[II] see the proof of Theorem 4 of [BHMV02]

This reduces solvability of $\mathcal{S}_i$ in $[0,1]_{\text{Ł}}$ to a particular Mixed Integer Programming problem of the general form

$$\Sigma_k a_{kl} x_k \leq b_l \text{ for each } l, \text{ and } \Sigma_k c_k x_k > d$$

where all $a_{kl}$, $b_k$, $c_k$, $d$ are rational; the problem is to decide whether there is a rational solution $(x_k)$, with some coordinates demanded to be Boolean. This problem is in NP. $\mathcal{QED}$

Having established that the `checkInternal()` step is an NP-subroutine, we may conclude the algorithm runs in polynomial time in $m$.

It remains to add that the output of a computation on $\varphi$ is 'yes' whenever there is an algebra $\mathbf{A}$ which is finite sum of Ł-segments and an evaluation $e_{\mathbf{A}}$ such that $e(\varphi) < 1$ (i. e., there is a branch of computation on which all checks are positive); if there is no successful branch of computation on $\varphi$, the output is 'no'. Thus the algorithm accepts the complement of the set of t-tautologies, hence the t-tautology problem is in coNP.

In [BHMV02] it is also shown that 1-satisfiable formulas in all t-algebras coincide with the classical SAT and thus are NP-complete.

## 4.2 t-algebras

This section determines the complexity classes for the four abovementioned decision problems (but especially $\text{TAUT}_1^{[0,1]_*}$) for an arbitrary t-algebra $[0,1]_*$.

We start with some easy results on individual t-algebras. These concern the sets $\text{TAUT}_{\text{pos}}^{[0,1]_*}$, $\text{SAT}_1^{[0,1]_*}$ and $\text{SAT}_{\text{pos}}^{[0,1]_*}$ for an arbitrary $[0,1]_*$. Subsequently we focus on 1-tautologies of individual t-algebras.

Since the 1-tautologies of any given semantics are always the most interesting class of formulas, we shall conveniently drop the index 1 and write $\text{TAUT}^{\mathbf{A}}$ or $\text{TAUT}^K$ instead of $\text{TAUT}_1^{\mathbf{A}}$ or $\text{TAUT}_1^K$, where $\mathbf{A}$ is an algebra and $K$ is a class of algebras which determine the semantics. We stick to the original notation where confusion could arise.

We stress that within this section we decompose t-algebras into BL-chains (Ł-,G and Π-segments), so $[0,1]_{\text{Ł}\oplus\text{G}\oplus\Pi}$ is a finite sum.

### 4.2.1 Preliminary considerations

Thanks to already existing results on the complexity of some of the decision problems for the Łukasiewicz, Gödel, and product logics, it is easy to determine the complexity of $\mathrm{TAUT}^{[0,1]_*}_{\mathrm{pos}}$, $\mathrm{SAT}^{[0,1]_*}_{1}$ and $\mathrm{SAT}^{[0,1]_*}_{\mathrm{pos}}$ for an arbitrary $[0,1]_*$.

Note that a t-algebra either has a first segment Ł, or has the strict negation.

**Theorem 4.2.1** *Let* $[0,1]_*$ *be a t-algebra which has a first segment* Ł*. Then*
(i) $\mathrm{TAUT}^{[0,1]_*}_{\mathrm{pos}} = \mathrm{TAUT}^{\text{Ł}}_{\mathrm{pos}}$
(ii) $\mathrm{SAT}^{[0,1]_*}_{1} = \mathrm{SAT}^{\text{Ł}}_{1}$
(iii) $\mathrm{SAT}^{[0,1]_*}_{\mathrm{pos}} = \mathrm{SAT}^{\text{Ł}}_{\mathrm{pos}}$
(iv) $\mathrm{TAUT}^{[0,1]_*}_{1}$ *is coNP-hard.*

*Proof.* (i)–(iii) Observe that $[0,1]_{\text{Ł}}$ is a subalgebra of $[0,1]_*$ and the mapping sending $x$ to $--x$ in $[0,1]_*$ is a homomorphism of $[0,1]_*$ onto $[0,1]_{\text{Ł}}$, such that $--x = 0$ iff $x = 0$. (iv) Use a reduction of $\mathrm{TAUT}^{\text{Ł}}_{1}$ to $\mathrm{TAUT}^{[0,1]_*}_{1}$ as in [BHMV02], Theorem 4: for any formula $\varphi$, form its translation $\varphi^{\neg}$ by prefixing a negation to each occurrence of a propositional variable. Then $\varphi \in \mathrm{TAUT}^{\text{Ł}}$ iff $\varphi^{\neg} \in \mathrm{TAUT}^{[0,1]_*}$. $\mathcal{QED}$

**Theorem 4.2.2** *Let* $[0,1]_*$ *be a t-algebra with strict negation. Then*
(i) $\mathrm{TAUT}^{[0,1]_*}_{\mathrm{pos}} = \mathrm{TAUT}$
(ii) $\mathrm{SAT}^{[0,1]_*}_{1} = \mathrm{SAT}$
(iii) $\mathrm{SAT}^{[0,1]_*}_{\mathrm{pos}} = \mathrm{SAT}$
(iv) $\mathrm{TAUT}^{[0,1]_*}_{1}$ *is coNP-hard.*

*Proof.* (i)–(iii) Observe that $\{0,1\}_{\mathrm{Bool}}$ is a subalgebra of $[0,1]_*$ and the mapping sending $x$ to $--x$ in $[0,1]_*$ is a homomorphism of $[0,1]_*$ onto $\{0,1\}_{\mathrm{Bool}}$, such that $--x = 0$ iff $x = 0$. (iv) In particular, TAUT can be reduced to $\mathrm{TAUT}^{[0,1]_*}$ forming the translation $\varphi^{\neg\neg}$ of $\varphi$ by prefixing double negation to each occurrence of a propositional variable, exactly as described for the Gödel and product logics in [Háj98b], Lemma 6.2.8. $\mathcal{QED}$

The last mentioned reduction in the proof can be also used for *classes* of algebras with the strict negation. Take for example SBL: forming a translation $\varphi^{\neg\neg}$ as above, we have $\varphi \in \mathrm{TAUT}$ iff $\varphi^{\neg\neg}$ is a 1-tautology in all standard SBL-algebras (thus a theorem of SBL).

An analogous consideration with the reduction in the previous proof would yield coNP-hardness of the set of 1-tautologies of all t-algebras whose first segment is Ł; but this set is, as shown in [Han01], exactly BL.

Further we need coNP-hardness for some calculi in the language of BL expanded with additional connectives. In particular this connective would be the involutive negation, but the result holds generally under conditions given.

Trivially if $\mathcal{C}$ is a calculus in the language of BL expanded with additional connectives, and $\mathcal{C}'$ is the reduct of $\mathcal{C}$ to the BL-language, then if $\mathcal{C}'$ is coNP-hard then $\mathcal{C}$ is coNP-hard. The translation function for the reduction is the identity.

The presumption of coNP-hardness of the reduct $\mathcal{C}'$ holds if $\mathcal{C}'$ is the logic given by some t-algebra, or if $\mathcal{C}'$ is BL or SBL. This is always the case with calculi we consider in Section 4.3.

### 4.2.2 Finite sums

The coNP-hardness of the set of 1-tautologies of any t-algebra which is a finite sum has been shown in Theorems 4.2.1 and 4.2.2. It remains to show containment in the class coNP.

**Theorem 4.2.3 (coNP containment for finite sums)** *Let* $\mathbf{A}$ *be a t-algebra which is a finite sum. Then* $\mathrm{TAUT}^{\mathbf{A}}$ *is in coNP.*

The proof forms the rest of this section. The algorithm `FIN` (for finite sums) presented here is a slight modification of one that has been used for the same purpose in [Han02]. Generally, it has been inspired by the algorithm accepting the complement of t-tautologies, presented in [BHMV02] and reproduced here in Section 4.1.5.

In this approach there are different algorithms for different finite sums and the type and cardinality of the sum is used as a built-in information. Throughout a t-algebra $\mathbf{A}$ which is a finite ordered sum is fixed. By an easy modification the type and cardinality of the sum could be regarded as a second input to a general version that would work for any finite sum.

We claim the algorithm solves the required problem, i. e., decides whether or not an input formula $\varphi$ has an evaluation in $\mathbf{A}$ s. t. $e_{\mathbf{A}}(\varphi) < 1$. The output is 'yes' if such $e_{\mathbf{A}}$ exists, i. e., the formula is *not* an $\mathbf{A}$-tautology, otherwise it is 'no'. So the set of formulas accepted by `FIN` is $\{\varphi : \varphi \notin \mathrm{TAUT}^{\mathbf{A}}\}$. Further, the algorithm `FIN` is nondeterministic and runs in polynomial time w. r. t. the length $m$ of $\varphi$. The reader will find that the only step requiring a thorough inspection from the latter aspect is the last, `checkInternal()` step; we will show that this step is an NP subroutine.

`Preliminaries.` Let $n$ denote the cardinality of the sum $\mathbf{A}$ (the number of segments in the sum). For a propositional formula $\varphi$, the symbol $m = |\varphi|$

denotes the number of subformulas in $\varphi$. In case $\varphi$ only contains binary connectives, $m$ is $2m' - 1$, where $m'$ is the number of occurrences of propositional variables and of the constant 0 in $\varphi$.

```
// algorithm FIN accepting non-tautologies of finite sum A
input:  φ
begin
nameSubformulas()
cutpointVariables()
guessOrder()
checkOrder()
checkExternal()
checkInternal()
end
```

We discuss the NP nature of the `checkInternal()` step, considering the situation in the $i$-th segment, for $i$ fixed. The step defines a system $\mathcal{S}_i$ of equations of type $x * y = z$ and of type $x \Rightarrow y = z$, and of equations and inequalities imposed by $\preceq$. We assume w. l. o. g. representatives are chosen for clusters of $\approx$-equivalent variables, and these are $z_i = x_{i_0} < \cdots < x_{i_{j-1}} = z_{i+1}$. Thus $\mathcal{S}_i$ is a system of at most $m$ equations in $j$ variables, with a strict ordering of length $j$ (see notes in 4.1.5).

An NP routine which checks solvability of $\mathcal{S}_i$ in $[0,1]_{\text{Ł}}$ has been reproduced from [BHMV02] in 4.1.5, so it remains to show how to perform the check for $[0,1]_{\text{G}}$ and for $[0,1]_{\Pi}$.

**Observation 4.2.4** *The solvability of the system $\mathcal{S}_i$ in $[0,1]_{\text{G}}$ can be checked in time linear in $m$.*

*Proof.* Obviously the system is solvable in $[0,1]_{\text{G}}$ iff it is solvable in a Gödel chain on $j$ elements. It is enough therefore to consider the variables $x_{i_0} < \cdots < x_{i_{j-1}}$ as a Gödel chain and to find out whether all the equations hold in it. The number of equations in $\mathcal{S}_i$ is bounded by $m$, the number of subformulas of $\varphi$ (it is in fact bounded by $m' - 1$). $\mathcal{QED}$

For the product t-algebra we use the following lemma.

**Lemma 4.2.5** *The system $\mathcal{S}_i$ is solvable in $[0,1]_\Pi$ iff it is solvable in a t-algebra $[0,1]_{Ł\oplus Ł}$, so that $x_{i_0}$ is evaluated by $0_{Ł\oplus Ł}$, $x_{i_{j-1}}$ is evaluated by $1_{Ł\oplus Ł}$ and $x_{i_1}, \dots, x_{i_{j-2}}$ are evaluated in $(1/2, 1)$, where $1/2$ is the non-extremal cut-point.*

*Proof.* By 2.3.5 $[0,1]_Ł$ is isomorphic to the *restricted product* algebra, resulting from the product t-algebra by choosing an element $c \in (0,1)$ and defining the $*$-operation on $[c,1]$ as $x *_c y = \max(c, x *_\Pi y)$, the constant 0 as $c$ and retaining $\Rightarrow_\Pi$.

Let $0 = a_0 < \dots < a_{j-1} = 1$ be a $j$-tuple of elements of $[0,1]$ which solve $\mathcal{S}_i$ in $[0,1]_\Pi$. Introduce a cut $c$ so that $a_0 < c < a_1$. Let $g$ be an isomorphism of the restricted product algebra $[c,1]$ with $[1/2,1]$ in the algebra $[0,1]_{Ł\oplus Ł}$. Then the values $0_{Ł\oplus Ł} < g(a_1) < \dots < g(a_{j-1}) = 1_{Ł\oplus Ł}$ solve $\mathcal{S}_i$ in $[0,1]_{Ł\oplus Ł}$. Conversely, to transfer a solution $0_{Ł\oplus Ł} < a_1 < \dots < a_{i_{j-1}} = 1_{Ł\oplus Ł}$ to $[0,1]_\Pi$, introduce an arbitrary cut $0 < c < 1$ and use an isomorphism $g'$ from $[1/2,1]$ in $[0,1]_{Ł\oplus Ł}$ to $[c,1]$; then $0_\Pi < g'(a_1) < \dots < g'(a_{i_{j-1}} = 1)$ is a solution of $\mathcal{S}_i$ in $[0,1]_\Pi$. $\mathcal{QED}$

Thus, to check solvability in the product t-algebra, we first eliminate all equations involving $y_{i0}$; the soundness of any such equation can be, and indeed has been in part, checked "externally"; for the remaining cases, check, for any $u, v$ belonging to $i$, that if $u * v = y_{i0}$ then either $u$ or $v$ is $y_{i0}$, that if $u \Rightarrow v = y_{i0}$ (and $u > v$) then $v$ is $y_{i0}$, and that $u \Rightarrow y_{i0} = y_{i0}$. Then we consider the remaining equations and strict inequalities in Ł, introducing a new inequality $0 < y_{i1}$, and check solvability of this system of equations and inequalities using the algorithm for solvability in Ł.

We conclude that the `checkInternal()` step for product segments is a nondeterministic subroutine running in time polynomial in $m$.

Finally, it is obvious from the construction of the algorithm that the output of a computation over $\varphi$ is 'yes' (on at least one branch) iff the formula $\varphi$ has a counterexample evaluation in **A**, i. e., is not an **A**-tautology. Thus the set of **A**-tautologies is in coNP.

### 4.2.3 Infinite sums

As proved in [AM03], a t-algebra generates **BL** iff it is an infinite sum starting with an Ł-segment and containing infinitely many Ł-segments. Since BL is coNP-complete, so are the tautologies of each t-algebra which generates **BL**.

It remains to investigate infinite sums which do not generate **BL**. As shown in 3.2.4, a large subclass of these algebras generate **SBL**, and the rest generate only countably many subvarieties of **BL** (cf. also axiomatics for these subvarieties, recently given in [EGM03]).

### SBL-generic t-algebras

The propositional logic SBL is shortly introduced here in Section 2.3.4.

As shown in Lemma 3.2.4, any t-algebra with infinitely many Ł-segments, whose initial segment (if there is one) is not an Ł-segment, generates the whole variety **SBL**. We are presently concerned with the set of 1-tautologies of any such algebra (i. e., with theorems of SBL).

We prove that SBL is in coNP. The coNP-hardness of the problem was discussed in Section 4.2.1.

**Theorem 4.2.6** *The propositional logic* SBL *is in coNP.*

The proof forms the rest of this section. If $\varphi$ is not an SBL-tautology, then it has a counterexample evaluation in a t-algebra which is a finite sum whose first element is not an Ł. Moreover, the cardinality of the finite sum is bounded (polynomially in the number of propositional variables in $\varphi$, cf. Lemma 3.2.5). Thus we may modify the algorithm `FIN` by adding steps guessing the cardinality of the sum and its type.

`Preliminaries.` For a propositional formula $\varphi$, let $m$ denote the number of subformulas in $\varphi$. Further, $k$ denotes the number of propositional variables in $\varphi$.

`// algorithm accepting non-SBL-tautologies`

`input:` $\varphi$

`begin`

`guessCardinality()` Pick at random a natural $n$, $0 < n \leq k + 1$[III)].

`guessLayout()` Assign to each $i = 1, \ldots, n$ one of the symbols Ł, G, Π, signifying the type of the $i$-th segment of the sum, in such a way that the first symbol is not an Ł.

We use the term 'constructed sum' and the symbol **C** to denote this finite sum. From now on the algorithm works with **C**.

`nameSubformulas()`

[III)] see Lemma 3.2.5

```
cutpointVariables()
guessOrder()
checkOrder()
checkExternal()
checkInternal()
end
```

It is obvious that this modification is a nondeterministic algorithm working in time polynomial in $m$ and accepting the complement of the logic SBL, so the 1-tautology problem for the logic SBL is in coNP.

### Other infinite sums

T-algebras which are infinite sums and generate a variety distinct from either **BL** or **SBL** have only finitely many (possibly no) Ł-segments.

We now explain how to encode the varieties (or sets of tautologies) generated by t-algebras which are infinite sums by use of finite strings. The idea that this can be done comes from [Han02] but here we use a simpler description developed in [EGM03], where the authors introduce the concept of *canonical* t-algebra and show that for each t-algebra there is a canonical one with the same set of tautologies. We adopt this concept for its simplicity (cf. [EGM03], Definition 4.6).

**Definition 4.2.7 canonical t-algebra)** *A t-algebra is canonical iff it is an infinite sum of Ł-segments only, or a sum of a* $\Pi$*-segment followed by infinite sum of Ł-segments only, or a finite sum of segments of type Ł, G,* $\Pi$ *and* $\infty\Pi$ *(standing for an infinite sum of* $\Pi$*-segments only), where each G-segment is not preceded or followed by another G, and each segment* $\infty\Pi$ *is not preceded or followed by G,* $\Pi$ *or another* $\infty\Pi$*.*

To see that canonical t-algebras generate all subvarieties of **BL** that are generated by a single t-algebra, consider Lemma 3.2.7 and the discussion following it. However, not only canonical algebras cover all possible subvarieties of **BL** generated by a single t-algebra, but, as shown in [EGM03], distinct canonical algebras generate distinct varieties. The strings in the alphabet {Ł, $\infty$Ł, G, $\Pi$, $\infty\Pi$} give a nice finite-string representation of each of the sets of propositional tautologies. Here we disregard the two canonical algebras represented by $\infty$Ł and $\Pi \oplus \infty$Ł.

For a t-algebra **A** which is an infinite sum with only finitely many Ł-segments, we show that the set of its 1-tautologies is in coNP. Again, the coNP-hardness was shown in Theorems 4.2.1 and 4.2.2.

**Theorem 4.2.8** *Let* **A** *be a t-algebra which is an infinite sum with only finitely many Ł-segments. Then* $\mathrm{TAUT}_1^{\mathbf{A}}$ *is in coNP.*

The proof forms the rest of this section.

Fix an arbitrary canonical t-algebra **A** which is an infinite sum with finitely many Ł-segments. Let us use $l(\mathbf{A})$ for the length of the finite string in the alphabet {Ł, G, Π, ∞Π} representing **A**.

`Preliminaries.` For a propositional formula $\varphi$, let $m$ denote the number of subformulas in $\varphi$. Further, $k$ denotes the number of propositional variables in $\varphi$.

`// algorithm INF for infinite sum` **A**

`input:` $\varphi$

`begin`

`guessCardinality()`

`guessLayout()` Assign to each $i = 1, \ldots, n$ one of the symbols Ł, G, Π, signifying the type of the $i$-th segment of the sum.

We use the term 'constructed sum' and the symbol **C** to denote this finite sum.

`checkEmbedding()` Check whether the constructed sum is $1-1$ embeddable into **A** (as a sequence of symbols into a sequence of symbols), in such a way that a potential initial Ł of the constructed sum is mapped to an initial Ł in **A**.

From now on the algorithm works with **C**.

`nameSubformulas()`

`cutpointVariables()`

`guessOrder()`

`checkOrder()`

`checkExternal()`

`checkInternal()`

`end`

We discuss the `checkEmbedding()` step, showing that it is an NP subroutine and explaining why it works the way it does and what its modifications could be.

**Lemma 4.2.9** *The embeddability of the constructed sum* $\mathbf{C}$ *into* $\mathbf{A}$ *can be checked by an NP algorithm (w. r. t. the length* $n$ *of* $\mathbf{C}$*).*

*Proof.* The algorithm works with the representation of $\mathbf{A}$, which is a string of length $l(\mathbf{A})$ in the alphabet {Ł, G, Π, ∞Π}, and the constructed sum $\mathbf{C}$, which presents itself as $n$ symbols from the alphabet {Ł,G,Π}. It works in two stages: first it guesses, for each symbol in $\mathbf{C}$, an index into the representation of $\mathbf{A}$, i. e., a natural number in $[0, l(\mathbf{A}) - 1]$. If the guess is sound, this should be a 1-1 embedding of $\mathbf{C}$ (as a sequence of segments) into $\mathbf{A}$ (as a sequence of segments). Note that the information guessed is polynomial since $l(\mathbf{A})$ is constant. Then it performs a verification of whether the guess was sound: an (initial) Ł-segment in $\mathbf{C}$ may only map to an (initial) Ł-segment in $\mathbf{A}$; a G-segment in $\mathbf{C}$ may only map to a G-segment in $\mathbf{A}$, and a Π-segment in $\mathbf{C}$ may map either to a Π segment or an ∞Π-segment in $\mathbf{A}$. The indices must be nondecreasing (w. r. t. the sequence $\mathbf{C}$), no two Ł-segments in $\mathbf{C}$ may have the same index, no two G-segments in $\mathbf{C}$ may have the same index, and two Π-segments may have the same index iff it specifies an ∞Π-segment of $\mathbf{A}$. Obviously this check is polynomial in $n$ since it is sufficient to consider the indices of each two neighbouring segments in turn. $\mathcal{QED}$

Soundness: if there is a counterexample evaluation, then there is a finite subsum of $\mathbf{A}$ harbouring it. We know by Lemma 3.2.5 it is enough to search all finite subsums up to length $k + 1$. The algorithm works with each such subsum as finite sum and works in exactly the same way as in the case for finite sums.

Again, it is clear that the output of the algorithm is 'yes' (on at least one branch) iff the formula $\varphi$ is not a 1-tautology of $\mathbf{A}$, thus 1-tautologies of $\mathbf{A}$ are in coNP.

It is perhaps worth pointing out that under this construction, the algorithm "throws away" some of the constructed sums which actually could supply a counterexample evaluation embeddable into $\mathbf{A}$; for example, if we permitted G-segments in $\mathbf{C}$ to map onto ∞Π segments, the algorithm would still be sound because, if later the algorithm puts the values of some variables into this G-segment, it would be easy to map these finitely many idempotents to some cutpoints in the ∞Π-segment. We prefer, however, to work solely with the string representations and not to go back to the structure of the t-algebras.

## 4.3 Logics with an involutive negation

For general definitions and results on logics with an involutive negation, see the paper [EGHN00] (basics reproduced here in Section 2.6.2).

We shall need a generalization of the $\sim$-standard completeness for $\Pi_\sim$, obtained in [EGHN00], to all calculi $\mathcal{C}_\sim$, where $\mathcal{C}$ is an extension of SBL given by some t-algebra.

**Theorem 4.3.1 ($\sim$-standard completeness for $\mathcal{C}_\sim$)** *Let $\mathcal{C}$ be the logic of some standard SBL-algebra $\mathbf{A}$. Then $\mathcal{C}_\sim$ is $\sim$-standard complete: provable formulas are exactly tautologies of all $\sim$-standard $\mathcal{C}_\sim$-algebras.*

*Proof.* Assume $\varphi$ is a formula in the language of $\mathcal{C}_\sim$, but not provable in $\mathcal{C}_\sim$. Then there is a $\mathcal{C}_\sim$-chain $\mathbf{L}$ in which $\varphi$ does not hold; fix an evaluation $e$ in $\mathbf{L}$ s. t. $e(\varphi) < 1$. Let $a_1 < \cdots < a_m \in L$ be a complete evaluation of $\varphi$ under $e$.

If $\sim a_i = a_j$ for some $1 \leq i, j \leq m$, then we say $(a_i, a_j)$ form a pair. Note that all pairs defined by $\varphi$ and $e$ in $\mathbf{L}$ are *nested*, i. e., for any two pairs $(a_i, a_j)$ and $(a_k, a_l)$ if $a_i < a_k$, then $a_l < a_j$.

We need to embed the partial subalgebra on $\{0, 1\} \cup \{a_1, \ldots, a_m\}$, with operations $*$, $\Rightarrow$ and 0 only, into $\mathbf{A}$. To this end, let $\mathbf{H}_0 \oplus \cdots \oplus \mathbf{H}_{m'}$, $m' \leq m$, be the subsum of hoops of $\mathbf{L}$ given by the evaluation $e(\varphi)$ (see Definition 3.2.3). By [EGM03], Lemma 3.7 (ii), we can find a corresponding subsum of hoops in $\mathbf{A}$ such that: if $\mathbf{H}_i$ is an MV-algebra, then $\mathbf{A}_i$ is Ł, if $\mathbf{H}_i$ is cancellative then $\mathbf{A}_i$ is either $C$ of Ł, and if $\mathbf{H}_i$ is 2 then $\mathbf{A}_i$ is 2 or Ł (see Section 3.2.2 for definitions of Ł, C, and 2). Thus $\mathbf{H}_0 \oplus \cdots \oplus \mathbf{H}_{m'}$ is partially embeddable into $\mathbf{A}$, and the partial isomorphism $f$ gives a sequence of values $0 \leq f(a_1) < \cdots < f(a_m) \leq 1$ in $\mathbf{A}$.

Thanks to Lemma 2.6.7 there is a decreasing involution in $\mathbf{A}$ which heeds the pairing of values $f(a_1) < \cdots < f(a_m)$ prescribed by $\varphi$. Thus $\mathbf{A}$ offers a complete evaluation of $\varphi$ (namely, the values $f(a_i)$) which gives value less than 1. $\mathcal{QED}$

We prove coNP-completeness of each calculus $\mathcal{C}_\sim$ (i. e., by Theorem 4.3.1, of the set of 1-tautologies of the class of $\sim$-standard $\mathcal{C}_\sim$-algebras). Recall that the coNP-hardness of any $\mathcal{C}_\sim$ follows from results in Section 4.2.1.

**Theorem 4.3.2** *Let $\mathcal{C}$ be the propositional logic given by some standard SBL-algebra. Then $\mathcal{C}_\sim$ is in coNP.*

*Proof.* In what follows we modify the algorithms `FIN`, `SBL`, `INF` to incorporate the involutive negation. We give the algorithm for finite sums, the modifications for infinite sums being obviously the same as in cases without the involutive negation.

Let $\mathbf{A}$ be an arbitrary standard SBL-algebra which is a finite sum and $\mathcal{C}$ be the propositional logic given by $\mathbf{A}$. Let $n$ denote the cardinality of the sum. For a propositional formula $\varphi$, the symbol $m = |\varphi|$ denotes the number of subformulas in $\varphi$.

```
// algorithm FIN∼
input:  φ //in the language expanded by ∼
begin
nameSubformulas()
cutpointVariables()
guessOrder()
checkOrder()
```

`checkInvolution()` for $\varphi_i$, $\varphi_j$ subformulas of $\varphi$, where $i, j \in \{0, \ldots, m-1\}$, if $\varphi_i$ is $\sim \varphi_j$, put down a pair $\{x_i, x_j\}$. Do this for every occurrence of $\sim$ in $\varphi$. Check that the pairs are mutually exclusive (up to $\approx$) and that they are nested w. r. t. $\preceq$.

```
checkExternal()
checkInternal()
end
```

Soundness: suppose the formula $\varphi$ has a counterexample evaluation in an algebra $\mathbf{A}_\sim$ obtained from $\mathbf{A}$ by adding (some) decreasing involution; this defines an assignment of values in $\mathbf{A}_\sim$ to subformulas of $\varphi$. Then the equations in the language $*$, $\Rightarrow$ will be solvable, as proved in Section 4.2. Moreover, the properties of the involution on $\mathbf{A}$ warrant that the conditions in the step `checkInvolution()` will be satisfied. The output then will be "yes". On the other hand, if the algorithm says "yes" on $\varphi$, then the equations in $*$, $\Rightarrow$ are solvable in $\mathbf{A}$ (as before), and by Lemma 2.6.7 there is a decreasing involution on $\mathbf{A}$ which satisfies all the equations prescribed by $\varphi$. Note that the check of soundness of the ordering w. r. t. involution is independent of the other steps and can be performed at any stage (after the ordering is established).

It is easy to see that the `checkInvolution()` step runs in time polynomial in $m$. $\mathcal{QED}$

# Chapter 5

# A set theory in fuzzy logic

This chapter is based on the paper [HH03].

## 5.1 Introduction

### 5.1.1 Some historical remarks

If anything comes to people's minds on the term 'fuzzy set theory' being used, it usually is a theory identifying fuzzy sets with real-valued functions ("generalized membership functions") on a fixed universe $\mathcal{U}$ within the classical set-theoretic universe, in the style of [Zad65]. The universe is *flat*, i. e., a set cannot be a member of another set. Various set-theoretic operations may be available, based on a set of propositional connectives with a fuzzy interpretation on the reals (e. g., an intersection of two sets $X$ and $Y$ is a set $Z$ such that $\forall u \in \mathcal{U}(u \in Z \equiv (u \in X \mathbin{\&} u \in Y))$. A theory of fuzzy sets in this sense is built over classical logic.

A few exceptions from this general expectation have, however, emerged: these are formal axiomatic theories *within* the respective many-valued logics. Moreover, the universe of sets generated by the theory is a *cumulative* one, which allows for development of non-trivial mathematics within the theory. Often the agenda of the papers is metamathematical and the development of, e. g., ordinal numbers is only a technical means of showing relative consistency of ZF w. r. t. the theory, or results of similar nature.

We remark on some works, not all of which are relevant to our present development of the topic, but they indicate various possible approaches. Further we rely especially on those works which develop a theory in the language and style of the classical Zermelo-Fraenkel set theory, modifying in some degree its axiomatic system.

Beginning with Skolem's work [Sko57], a number of works investigates consistency of the axiom of comprehension

$$\forall x_1, \ldots, x_n \exists z \forall u (u \in z \equiv \varphi(u, x_1, \ldots, x_n))$$

in predicate Łukasiewicz logic, under various assumptions about $\varphi$. In particular, [Sko57] shows that this principle is consistent for open formulas. [Cha63] shows consistency for formulas without parameters (but with arbitrary quantification) or formulas with parameters *and* quantification but with restrictive condition on the quantified variables. A further generalization is presented in the work [Fen64]. Finally, [Whi79] shows that the comprehension principle is consistent for arbitrary formulas. This work also shows that the axiom of extensionality cannot be consistently added to full comprehension, as the latter implies crispness of $=$ (under presence of congruence axioms, esp. (E5)—see Section 5.2.3).

Another early effort, completely different in approach, is presented in the works of Klaua ([Kla65], [Kla67], [Kla66]), who does not develop axiomatic theory but constructs cumulative hierarchies of sets, defining many-valued truth functions of $=$, $\subseteq$ and $\in$. The set of truth values is either the standard or a finite MV-algebra. In [Kla67] the author constructs a cumulative universe similar to ours in definition of its elements and the value of the membership function (cf. Section 5.3.5), but has also a non-crisp equality; his universe then validates extensionality and comprehension, but fails to validate the congruence axioms (see Section 5.3.2). The works of Klaua have been continued and made accessible to a wider audience in the works of Gottwald ([Got76a], [Got76b], [Got77], [Got81], [Got84]).

A fruitful material for study is a small selection of papers on ZF-like set theory in the intuitionistic logic.[I)] [Pow75] defines a ZF-like theory with an additional axiom of *double complement* $\exists y \forall z (\neg\neg(z \in x) \to z \in y)$, develops some technical means (like ordinals and rank) and defines a class of *stabilized sets* which it proves to be an inner model of classical ZF. [Gra79b] omits double complement but uses collection instead of replacement, and (within the theory) constructs a Heyting-valued universe over a complete Heyting algebra. Using a particular Boolean algebra which it constructs, it then obtains a "model" of

[I)] Gödel logic is a strengthening of the intuitionistic logic, hence these papers are directly relevant when developing set theory over Gödel logic. Their significance, esp. that of [Gra79b], is however wider.

ZF, i. e., shows relative consistency of ZF. This paper is an excellent material regarding some of the difficulties of developing set theory in a weak logic, above all, the danger of assembling an axiomatic system which strengthens the underlying logic to the classical one. For an important example, the axiom of foundation, together with a very weak fragment of ZF, implies the law of the excluded middle, which yields the full classical logic (both in intuitionistic logic and in the logic we use later in this chapter), and thus the theory becomes classical. It also shows (e. g., by using $\in$-induction instead of foundation) that some classically equivalent principles are no longer equivalent in a weaker logical setting.

[Gra79b] has inspired the paper [TT84], which develops a ZF-like set theory and a considerable portion of calculus in this set theory over Gödel logic (their axiomatization, obtained from Gentzen style axiomatization of the intuitionistic logic, includes an infinitary rule). More importantly perhaps, [TT92] sets off by developing a logical system combining Łukasiewicz connectives with the product conjunction, the strict negation and a constant denoting 1/2 on $[0, 1]$ (thus defining the well-known *logic of Takeuti and Titani*, having its followers in the logics ŁΠ and ŁΠ1/2—see [Háj98b], Section 9.1, [HC03], [EGM01]). This logic contains Gödel logic, and it is Gödel logic that is used in ZF-style axiomatics. Equality is many-valued, thus (due to the presence of Łukasiewicz connectives and product conjunction) for example the substitution theorem spells $\Delta(x = y) \to (\varphi(x) \equiv \varphi(y))$. The axiomatics includes the double complement axiom (cf. [Pow75]). In the paper [Tit99] the author repeats most of the constructions of [TT92] (including a "completeness" theorem) in a lattice-valued set theory. We remark that the author arrives, apparently unawares, at a crisp equality.

In coping with some of the difficulties one encounters when working with non-idempotent conjunction (see Section 5.3.2), we appreciated elegant solutions found in [Shi99]. The author works over the so-called phase spaces as algebras of truth values and builds, using an analogy of the construction of a Boolean-valued universe (over a phase-space instead of a Boolean algebra), a class, with class operations evaluating formulas in the language $\{\in, \subseteq, =\}$, in which he verifies the chosen axioms of his set theory. Having observed that the standard Łukasiewicz algebra enriched with the $\Delta$ operator is a particular phase space, we studied this paper thinking of a more general approach employing BL-chains with $\Delta$; this should form a common generalization of the approach of [TT84] and [TT92] and that of [Shi99].

A rather successful attempt at a close study of basic natural properties of a set theory over Gödel logic has emerged recently in [Běh02].

### 5.1.2 Position and aims

Since our logical system is weaker than the classical logic, an analogy of ZF (i. e., a theory whose axioms are the axioms or theorems of ZF) is trivially consistent relative to ZF. It is non-trivial to show that the constructed theory is distinct from ZF or its classical fragment (cf. [Gra79b] and Section 5.3.2).

We develop an axiomatic theory FST ('fuzzy set theory') and show its non-triviality by constructing a BL$\Delta$-valued universe over an arbitrary complete BL$\Delta$-chain in which all its axioms were valid. Here we propose a simpler definition of the universe and evaluation of the basic predicate symbols than we used in our paper [HH01]. We discuss the choices we have made when assembling the axioms. Finally, we show relative consistency of ZF w. r. t. FST by exhibiting its inner model, consisting of hereditarily crisp sets.

## 5.2 An overview of the calculus BL$\forall\Delta$

The aim of this section is to go over some important facts about the specific kind of predicate calculus we use in this chapter. We try to point out some distinctive properties, especially with respect to the calculus BL$\forall$ developed in [Háj98b], Chapter 5, which is a predicate calculus over BL. Our calculus BL$\forall\Delta$ has the following specific traits:

– its propositional fragment is (not BL but) the logic BL$\Delta$ developed in Section 2.4 of [Háj98b] (see also Section 2.6.1).

– it considers first-order languages with (predicate and) function symbols. We rely on the paper [Háj00].

– it contains equality.

We remark that, although the calculus BL$\forall\Delta$ as defined in this work has not been investigated before, many of its properties are implicit in [Háj98b].

We sometimes try to give a general formulation of our results, considering a *schematic extension* of BL$\forall\Delta$. We stress that here a schematic extension of BL$\forall\Delta$ is a calculus obtained by adding some propositional schemata of axioms in the language of BL to BL$\forall\Delta$; i. e., no new axioms for $\Delta$, $\forall$ or $\exists$ are added. A more pedantic denotation would be $\mathcal{C}\Delta\forall$. A $\mathcal{C}$-algebra (chain) is just a BL$\Delta$-algebra in which all the additional schemata are valid.

A complete and motivating treatise on predicate calculi over BL will be found in [Háj98b], Chapter 5.

### 5.2.1 Language and syntax

The logical symbols in the language of BL$\forall\Delta$ are the logical connectives of BL$\Delta$, object variables $\mathcal{V}$ (members usually denoted $x$, $y$, $z$, $w$ etc.) and quantifiers $\forall$ and $\exists$. We include $=$ among logical symbols.

A language of a particular theory has constants, predicate symbols *and* function symbols. As in classical logic, an *arity* is given for each predicate and function symbol. Generally in a theory with fuzzy equality, a *syntactic* or *congruence degree* should be specified for each predicate and function symbol (see [Háj98b], 5.6.7). In this case however equality is crisp (see Sections 5.2.3 and 5.3.2), thus all predicate an function symbols have a congruence degree 1.

Terms are defined by the usual inductive definition, starting with constants and object variables and employing function symbols. Atomic formulas have the form $P(t_1, \ldots, t_n)$, where $P$ is a predicate symbol of arity $n$ and $t_1, \ldots, t_n$ are terms. Formulas are defined as usual, i. e., inductively using &, $\rightarrow$, 0, $\Delta$, $\forall$ and $\exists$ for induction steps.

A *theory* in a language $\mathcal{L}$ is a set of formulas of $\mathcal{L}$.

The basic language of our theory FST contains a binary predicate symbol $\in$ (together with $=$ already included as a logical symbol). We define some other predicate and function symbols commonly used in set theories (see conservative extension theorems in Section 5.2.4).

**Definition 5.2.1 (Bounded quantifiers)**
(i) *The formula* $\forall x \in y(\varphi)$ *is an abbreviation for* $\forall x(x \in y \rightarrow \varphi)$.
(ii) *The formula* $\exists x \in y(\varphi)$ *is an abbreviation for* $\exists x(x \in y \,\&\, \varphi)$.

### 5.2.2 Structures

The semantics of a predicate language $\mathcal{L}$, over a calculus $\mathcal{C}$ which is a schematic extension of BL$\forall\Delta$, is given by an $\mathbf{L}$-structure $\mathbf{M}$, $\mathbf{L}$ being a $\mathcal{C}\Delta$-chain. $\mathbf{M} = (M, (r_P)_{P \text{ predicate symbol}}, (m_c)_{c \text{ constant}}, (f_F)_{F \text{ function symbol}})$ has a domain $M \neq \emptyset$, for each $n$-ary predicate symbol $P$ of $\mathcal{L}$ an $\mathbf{L}$-fuzzy $n$-ary relation $r_P : M^n \rightarrow L$, for each constant $c$ of $\mathcal{L}$ an element $m_c \in M$, and for each $n$-ary function symbol $F$ of $\mathcal{L}$ a function $f_F : M^n \longrightarrow M$ (see [Háj98b], [Háj00]).

An $\mathbf{M}$-evaluation of object variables is a mapping $v : \mathcal{V} \longrightarrow M$. For two evaluations $v$, $v'$, $v \equiv_x v'$ means $v(y) = v'(y)$ for each variable $y$ distinct from $x$.

The value $\|t\|_{\mathbf{M},v}$ of a term $t$ under evaluation $v$ is defined inductively: $\|x\|_{\mathbf{M},v} = v(x)$, $\|c\|_{\mathbf{M},v} = m_c$, for $n$-ary function symbol $F$ and terms $t_1,\ldots,t_n$, $\|F(t_1, \ldots, t_n)\|_{\mathbf{M},v} = f_F(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v})$.

The full Tarski-style definition of the value $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ of a formula $\varphi$ given by an **L**-structure **M** and evaluation $v$ in **M** is again inductive:
$\|P(t_1,\ldots,t_n)\|^{\mathbf{L}}_{\mathbf{M},v} = r_P(\|t_1\|_{\mathbf{M},v},\ldots,\|t_n\|_{\mathbf{M},v})$
$\|\varphi \,\&\, \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} * \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$
$\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$
$\|0\|^{\mathbf{L}}_{\mathbf{M},v} = 0$
$\|\Delta\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \Delta\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$
$\|\forall x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{v\equiv_x v'} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}$
$\|\exists x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{v\equiv_x v'} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}$

Since **L** need not be complete lattice, this value may be undefined; the **L**-structure **M** is *safe* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for each $\varphi$ and $v$.

The truth value of a formula $\varphi$ of a predicate language $\mathcal{L}$ in a safe **L**-structure **M** for $\mathcal{L}$ is

$$\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \bigwedge_{v \text{ an } \mathbf{M}-\text{evaluation}} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$$

We call an interpretation **M** of $\mathcal{L}$ *admissible* if all the axioms for $=$ (see Section 5.2.3) are 1-true in **M**.

Let $\mathcal{C}$ be a schematic extension of BL$\forall\Delta$, $T$ a theory over $\mathcal{C}$, **L** a $\mathcal{C}$-chain and **M** a safe admissible **L**-structure for the language of $T$. **M** is an **L**-model of $T$ iff all axioms of $T$ are 1-true in **M**, i. e., $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$ for each $\varphi \in T$.

A formula $\varphi$ of a predicate language $\mathcal{L}$ is an **L**-tautology iff $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ for each safe admissible **L**-structure **M**.

In Section 5.3.5 we use *complete* BL$\Delta$-chains. Particular examples are the algebras on the real unit interval given by a continuous t-norm (these have the standard order of reals), among them the three ones given by Łukasiewicz, Gödel, and product t-norms. Each complete BL-chain is an ordered sum of copies of complete MV-chains, G-chains, and product chains, and these are as follows (the degenerate one-element algebra and any isomorphic copies disregarded):
– complete MV-chains: $[0,1]_{\text{Ł}}$ and its finite $n$-element subalgebras, $n \geq 1$;
– complete G-chains: any complete linear order determines a G-algebra;
– complete product chains: $[0,1]_{\Pi}$, the two-element Boolean algebra, and the restriction of the standard product algebra to $\{0\} \cup \{(1/2)^n; n \geq 0\}$.

### 5.2.3 Equality

For a thorough discussion of equality and its interpretation in various BL-chains see [Háj98b], Section 5.6. Some relevant passages (as to congruence degrees and crisp equality) can be found in [Háj00]. Regarding equality in set theory in the ZF-style, see Section 5.3.2.

Having function symbols in the language leads to considering equality as a logical predicate symbol with the usual axioms, i. e., reflexivity, symmetry, transitivity, and congruence w. r. t. other basic predicate and function symbols in the language (in case of FST, the only basic symbol is $\in$):

(E1) (reflexivity) $\forall x(x = x)$
(E2) (symmetry) $\forall x, y(x = y \to y = x)$
(E3) (transitivity) $\forall x, y, z(x = y \,\&\, y = z \to x = z)$
(E4) (congruence) $\forall x, y, z(x = y \,\&\, z \in x \to z \in y)$
(E5) (congruence) $\forall x, y, z(x = y \,\&\, y \in z \to x \in z)$

We remark that in the present theory, (E1)–(E4) are a consequence of the axiom of extensionality (cf. Section 5.3.4).

In general, having non-idempotent conjunction *and* a fuzzy equality, one cannot prove the usual congruence axioms for new predicates and functions defined by formulas. This leads to the notion of *syntactic degrees* of formulas (see [Háj98b], Definition 5.6.7 and Lemma 5.6.8 for the definition and how to compute them).

Fortunately these awkward computations are not our destiny since there are good reasons for having a *crisp* equality (see Section 5.3.2). Crispness of $=$ is a consequence of the extensionality axiom as formulated in 5.3.4. But let us assume an additional "crispness" axiom for $=$ in the logic:
(EC) (crispness of $=$) $\forall x, y(x = y \vee \neg x = y)$
This will spare us the necessity of considering syntactic degrees of formulas when formulating theorems on conservativity of introducing function symbols etc. Note that with the axiom EC (due to Lemmas 5.3.2 and 5.3.4) $\forall x, y(x = y \to \Delta(x = y))$. Thus $=$ is an idempotent relation.

### 5.2.4 Axiomatization and some theorems

In this section we give axiomatization of our logic BL$\forall\Delta$. All theorems implicitly assume the calculus in question is BL$\forall\Delta$ or its schematic extension.

The notions of *free variable* and *substitutability* in the following definition are the same as in classical logic.

**Definition 5.2.2** *Let $\mathcal{C}$ be a schematic extension of BL. The predicate logic $\mathcal{C}\forall$ is obtained by adding to the axioms of $\mathcal{C}$ the following axioms for quantifiers:*
($\forall$1) $\forall x\varphi(x) \to \varphi(t)$ *(t substitutable for x in $\varphi$)*
($\exists$1) $\varphi(t) \to \exists x\varphi(x)$ *(t substitutable for x in $\varphi$)*
($\forall$2) $\forall x(\chi \to \varphi) \to (\chi \to \forall x\varphi)$ *(x not free in $\chi$)*
($\exists$2) $\forall x(\varphi \to \chi) \to (\exists x\varphi \to \chi)$ *(x not free in $\chi$)*
($\forall$3) $\forall x(\varphi \vee \chi) \to (\forall x\varphi \vee \chi)$ *(x not free in $\chi$).*
*and the rule of generalization: $\varphi/\forall x\varphi$.*

BL$\forall\Delta$ is a formal logical system with the following axioms: (A1)–(A7), ($\Delta$1)–($\Delta$5), ($\forall$1), ($\exists$1), ($\forall$2), ($\exists$2), ($\forall$3), (E1)–(E3) and congruence axioms for all predicate and function symbols in the language (here (E4), (E5)). We assume in addition (EC) for crispness of $=$. The deduction rules of BL$\forall\Delta$ are modus ponens, generalization, and $\Delta$-generalization ($\varphi/\Delta\varphi$).

**Theorem 5.2.3 (Deduction theorem)** *Let $T$ be a theory and let $\varphi$ be a closed formula of the language of $T$. Then $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash \Delta\varphi \to \psi$.*

The following completeness theorem for BL$\forall\Delta$ is obtained by inspection of the proof of Theorem 5. 2. 9 of [Háj98b], considering the presence of equality and function symbols (and $\Delta$).

**Theorem 5.2.4 (Completeness)** *Let $\mathcal{C}$ be a schematic extension of* BL$\forall\Delta$*, let $T$ be a theory over $\mathcal{C}$, and let $\varphi$ be a formula of the language of $T$. Then $T$ proves $\varphi$ iff for each $\mathcal{C}$-chain $\mathbf{L}$ and each safe admissible $\mathbf{L}$-model $\mathbf{M}$ of $T$, $\varphi$ holds in $\mathbf{M}$.*

As announced, we define constants, predicate and function symbols in our theory. Moreover, to advance our proof techniques, we sometimes use analogies on theorems about the classical predicate logic. These concern conservativity of some extensions of theories. We give a brief overview of the statements we rely on.

**Theorem 5.2.5** *Let $T$ be a theory in a language $\mathcal{L}$, let $\varphi(x_1, \ldots, x_n)$ be a formula of $\mathcal{L}$ and $c_1, \ldots, c_n$ be new constants (not in $\mathcal{L}$). Let $\mathcal{L}' = \mathcal{L} \cup \{c_1, \ldots, c_n\}$. Then*

$$T \vdash_{\mathcal{L}} \varphi \text{ iff } T \vdash_{\mathcal{L}'} \varphi(x_i/c_i)$$

where the subscript of $\vdash$ indicates language in which the proof is carried out.

*Proof.* Analogous to the proof for classical logic. $\mathcal{QED}$

**Definition 5.2.6 (Conservative extension)** *Let $T_1$ be a theory in a language $\mathcal{L}_1$ and $T_2 \supseteq T_1$ a theory in a language $\mathcal{L}_2 \supseteq \mathcal{L}_1$. We say $T_2$ is a* conservative extension *of $T_1$ iff for each formula $\varphi$ of $\mathcal{L}_1$, if $T_2 \vdash \varphi$ then $T_1 \vdash \varphi$.*

**Theorem 5.2.7** *Let $T$ be a theory in a language $\mathcal{L}$.*
(i) *Let $P \notin \mathcal{L}$ be an $n$-ary predicate symbol and $\varphi(x_1, \ldots, x_n)$ a formula of $\mathcal{L}$. If $T'$ results from $T$ by adding the formula*

$$\forall x_1, \ldots, x_n (P(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n)),$$

*then $T'$ is a conservative extension of $T$.*
(ii) *Let $c \notin \mathcal{L}$ be a constant, $\exists x \Delta\varphi(x)$ a closed formula of $\mathcal{L}$ and provable in $T$. If $T'$ results from $T$ by adding the formula $\varphi(c)$, then $T'$ is a conservative extension of $T$.*
(iii) *Let $F \notin \mathcal{L}$ be an $n$-ary function symbol and $\varphi(x_1, \ldots, x_n, y)$ a formula of $\mathcal{L}$, such that $T \vdash \forall x_1, \ldots, x_n \exists y \Delta\varphi(x_1, \ldots, x_n, y)$. If $T'$ results from $T$ by adding the formula*

$$\forall x_1, \ldots, x_n \, \varphi(x_1, \ldots, x_n, F(x_1, \ldots, x_n))$$

*and the congruence axiom*

$$x_1 = z_1 \,\&\, \ldots \,\&\, x_n = z_n \to F(x_1, \ldots, x_n) = F(z_1, \ldots, z_n),$$

*then $T'$ is a conservative extension of $T$.*
*If, in addition, $T \vdash \forall x_1, \ldots, x_n \exists! y \varphi(x_1, \ldots, x_n, y)$, then each formula of $T'$ is $T'$-equivalent to a $T$-formula.*

For proofs, see 5.2.15 of [Háj98b] and Theorem 3 of [Háj00], taking into account our reformulation of the Deduction theorem.

## 5.3 The Theory FST

We introduce the theory FST with a basic predicate symbol $\in$ (having $=$ as a logical symbol). The underlying logic of FST is a schematic extension $\mathcal{C}$ (possibly void) of BL$\forall\Delta$. When proving theorems within FST, we only rely on those logical axioms which form BL$\forall\Delta$. On the other hand, for a given extension $\mathcal{C}$, the universe $V^{\mathbf{L}}$—for $\mathbf{L}$ a $\mathcal{C}$-chain—will yield an interpretation of FST over $\mathcal{C}$ in ZF over classical logic.

We start this section with developing technical means, then point out some problems and pitfalls one may run into when using a weak logic with non-idempotent conjunction. Then we give the axiomatics of FST and construct the BL$\Delta$-valued universe. Finally we define a class of $H$ of hereditarily crisp sets, which we show to be an inner model of classical ZF in FST.

### 5.3.1 Technicalities

We use as needed theorems of BL$\forall\Delta$ which are proved in or a direct consequence of [Háj98b] We prove a few additional theorems of BL$\forall\Delta$. This section is intended mainly for reference.

**Lemma 5.3.1** BL$\forall\Delta$ *proves the following:*
(i) $\forall x\varphi \,\&\, \psi \to \forall x(\varphi \,\&\, \psi)$ *($x$ not free in $\psi$)*
(ii) $\forall x\varphi \& \forall x\psi \to \forall x(\varphi \& \psi)$
(iii) $\exists x\Delta\varphi \to \Delta\exists x\varphi$

*Proof.* (i) BL$\forall \vdash \psi \to (\varphi \to (\varphi \,\&\, \psi))$ (2.2.8 (5) of [Háj98b])
BL$\forall \vdash \psi \to \forall x(\varphi \to (\varphi \,\&\, \psi))$ (generalization and ($\forall$2))
BL$\forall \vdash \psi \to (\forall x\varphi \to \forall x(\varphi \,\&\, \psi))$ (5.1.16 (5) of [Háj98b])
BL$\forall \vdash \forall x\varphi \,\&\, \psi \to \forall x(\varphi \,\&\, \psi)$
(ii) BL$\forall \vdash \varphi \to (\psi \to \varphi \,\&\, \psi)$
BL$\forall \vdash \forall x(\varphi \to (\psi \to \varphi \,\&\, \psi))$
BL$\forall \vdash \forall x\varphi \to (\forall x\psi \to \forall x(\varphi \,\&\, \psi))$ (repeated 5.1.16 (5) of [Háj98b])
(iii) BL$\forall\Delta \vdash \varphi \to \exists\varphi$ ($\exists$1)
BL$\forall\Delta \vdash \Delta\varphi \to \Delta\exists x\varphi$ ($\Delta$-gen. and ($\Delta$5))
BL$\forall\Delta \vdash \exists x\Delta\varphi \to \Delta\exists x\varphi$ (generalization and ($\exists$2))

$\mathcal{QED}$

**Lemma 5.3.2** BL$\forall\Delta \vdash \forall x\Delta\varphi \equiv \Delta\forall x\varphi$.

*Proof.* The right-to-left implication is easy. We give a BL$\forall\Delta$-proof of the converse one (an analogy to the proof of the Barcan formula in S5) in several steps. Let $\Diamond\varphi$ stand for $\neg\Delta\neg\varphi$.
(i) $\varphi \to \neg\Delta\neg\varphi$.
In BL$\Delta$, $\Delta\neg\varphi \to \neg\varphi$, thus $(\Delta\neg\varphi\&\varphi) \to 0$, thus $\varphi \to (\Delta\neg\varphi \to 0)$.
(ii) $\neg\Delta\varphi \to \Delta\neg\Delta\varphi$.
In BL$\Delta$, $\Delta\varphi \vee \neg\Delta\varphi$, thus $\Delta\Delta\varphi \vee \Delta\neg\Delta\varphi$. This gives $\neg\Delta\varphi \to \Delta\neg\Delta\varphi$ and

$\neg\Delta\neg\Delta\varphi \to \Delta\varphi$ (as $\varphi \vee \psi \to (\neg\varphi \to \psi)$ is provable in BL).
Thus the following two are provable in BL$\Delta$:
(iii) $\varphi \to \Delta\Diamond\varphi$
(iv) $\Diamond\Delta\varphi \to \Delta\varphi$.
Next, (v) $\Delta(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$,
since $\Delta(\varphi \to \psi) \to \Delta(\neg\psi \to \neg\varphi) \to (\Delta\neg\psi \to \Delta\neg\varphi) \to (\Diamond\varphi \to \Diamond\psi)$.
(vi) $\Diamond\forall x\varphi \to \forall x\Diamond\varphi$ is a consequence of (v), obtaining $\Delta(\forall x\varphi \to \varphi)$ from ($\forall$1), then generalize.
Finally, $\forall x\Delta\varphi \to \Delta\Diamond\forall x\Delta\varphi \to \Delta\forall x\Diamond\Delta\varphi \to \Delta\forall x\Delta\varphi \to \Delta\forall x\varphi$. $\mathcal{QED}$

**Corollary 5.3.3** BL$\forall\Delta \vdash \forall x\Delta(\varphi \,\&\, \psi) \to \forall x\Delta\varphi \,\&\, \forall x\Delta\psi$

*Proof.* BL$\forall\Delta \vdash \forall x\Delta(\varphi \,\&\, \psi) \to \forall x\Delta\varphi$, and
BL$\forall\Delta \vdash \forall x\Delta(\varphi \,\&\, \psi) \to \forall x\Delta\psi$, so by 2.2.8 (7) of [Háj98b],
BL$\forall\Delta \vdash (\forall x\Delta(\varphi \,\&\, \psi))^2 \to \forall x\Delta\varphi \,\&\, \forall x\Delta\psi$; now use Lemma 5.3.2 and 2.4.11 (1) of [Háj98b]. $\mathcal{QED}$

**Lemma 5.3.4** BL$\Delta \vdash \Delta(\varphi \vee \neg\varphi) \equiv \Delta(\varphi \to \Delta\varphi)$.

*Proof.* Left to right: $\Delta(\varphi \vee \neg\varphi) \to (\Delta\varphi \vee \Delta\neg\varphi) \to [\varphi \to (\varphi\&(\Delta\varphi \vee \Delta\neg\varphi))] \to [\varphi \to (\Delta\varphi \vee (\varphi\&\Delta\neg\varphi))]$. In the last formula in the chain, $\varphi\&\Delta\neg\varphi \to 0$, thus the last formula implies $\varphi \to \Delta\varphi$; we get $\Delta(\varphi \vee \neg\varphi) \to (\varphi \to \Delta\varphi)$, which $\Delta$-generalizes to the desired implication.
Conversely, $\Delta\varphi \to \Delta(\varphi \vee \neg\varphi)$, thus also
(i) $\Delta\varphi \to [\Delta(\varphi \to \Delta\varphi) \to \Delta(\varphi \vee \neg\varphi)]$.
Moreover $\Delta(\varphi \to \Delta\varphi) \to \Delta(\neg\Delta\varphi \to \neg\varphi) \to (\Delta\neg\Delta\varphi \to \Delta\neg\varphi) \to [\Delta\neg\Delta\varphi \to \Delta(\varphi \vee \neg\varphi)]$, thus
(ii) $\Delta\neg\Delta\varphi \to [\Delta(\varphi \to \Delta\varphi) \to \Delta(\varphi \vee \neg\varphi)]$.
Since BL$\forall\Delta$ proves $\Delta\varphi \vee \Delta\neg\Delta\varphi$, we get the right-to-left implication by putting together (i) and (ii). $\mathcal{QED}$

We write $\bowtie \varphi$ for $\Delta(\varphi \vee \neg\varphi)$.

**Definition 5.3.5** (i) *In a theory $T$, we say that a formula $\varphi(x_1, \ldots, x_n)$ in the language of $T$ is* crisp *iff $T \vdash \forall x_1, \ldots, x_n \bowtie \varphi(x_1, \ldots, x_n)$.*
(ii) *In a (set) theory with language containing $\in$ we define* $\mathrm{Crisp}(x) \equiv \forall u \bowtie (u \in x)$.

Note that (i) is equivalent to $T \vdash (\varphi \vee \neg\varphi)$. If $\varphi$ is an atomic formula $P(x_1, \ldots, x_n)$, we sometimes say that $P$ (or the corresponding property) is crisp instead of saying that $P(x_1, \ldots, x_n)$ is crisp. Especially, we say $\in$ or $=$ is crisp (in a given theory) iff the above is the case.

Using Lemma 5.3.4 we get

**Corollary 5.3.6** $\forall x(\mathrm{Crisp}(x) \equiv \forall u \Delta(u \in x \to \Delta(u \in x))$.

Note that $\mathrm{Crisp}(x) \equiv \Delta \forall u(u \in x \to \Delta(u \in x)) \to \Delta\Delta\forall u(u \in x \to \Delta(u \in x))$, so crispness is a crisp property:

**Lemma 5.3.7** $\mathrm{Crisp}(x) \to \Delta\mathrm{Crisp}(x)$.

Thus $\mathrm{Crisp}(x) \to \Delta\mathrm{Crisp}(x) \to (\Delta\mathrm{Crisp}(x))^2 \to (\mathrm{Crisp}(x))^2$.

**Lemma 5.3.8** $\mathrm{BL}\Delta \vdash (\bowtie \varphi \& (\varphi \to \Delta\psi)) \to \Delta(\varphi \to \psi)$.

*Proof.* $[\Delta\varphi \& (\varphi \to \Delta\psi)] \to \Delta\psi \to \Delta(\varphi \to \psi)$, and $\Delta\neg\varphi \to \Delta(\varphi \to \psi)$, so $(\Delta\varphi \vee \neg\Delta\varphi) \to ((\varphi \to \Delta\psi) \to \Delta(\varphi \to \psi))$. $\mathcal{QED}$

**Lemma 5.3.9** $\mathrm{BL}\forall\Delta \vdash \forall x, y((\mathrm{Crisp}(x) \,\&\, \mathrm{Crisp}(y) \& x \subseteq y) \to \Delta(x \subseteq y))$.[II)]

*Proof.* $[\mathrm{Crisp}(x) \,\&\, \mathrm{Crisp}(y) \,\&\, (u \in x \to u \in y)] \to$
$[\bowtie (u \in x) \& (u \in y \to \Delta(u \in y)) \& (u \in x \to u \in y)] \to$
$[\bowtie (u \in x) \& (u \in x \to \Delta u \in y)] \to$
$[\Delta(u \in x \to u \in y)]$. $\mathcal{QED}$

### 5.3.2 What we cannot have

This section should clarify our choice of axioms. We in fact reproduce three results from the unpublished note [Háj01]. The underlying logic considered in the three lemmas is $\mathrm{BL}\forall$.

The danger, well known from Grayson's paper ([Gra79b]) on set theory in the intuitionistic logic and referred to by Hájek as 'horror vacui', is that some axioms of ZF or their combinations may strengthen the underlying logic so that it becomes the classical predicate logic.

It is well known that BL together with the propositional schema $\varphi \vee \neg\varphi$ (the law of the excluded middle, LEM) yields the classical logic; so if a combination of axioms allows us to derive LEM for an arbitrary $\varphi$, then our underlying logic $\mathrm{BL}\forall\Delta$ is strengthened to the classical predicate logic and thus we get the classical ZF theory (see axioms in Section 5.3.4).

[II)] We define $\forall x, y(x \subseteq y \equiv \forall z \in x(z \in y))$ (cf. Section 5.3.4).

**Lemma 5.3.10** *A theory with separation (for open formulas), pairing or singletons, and congruence for $\in$ over a logic which proves the propositional formula $(\varphi \to \varphi \,\&\, \varphi) \to (\varphi \vee \neg\varphi)$ proves $\forall x, y(x = y \vee \neg(x = y))$.*

*Proof.* Given $x$, $y$, let $z$ be such that $u \in z \equiv (u \in \{x\} \& u = x)$, i. e., $u \in z \equiv (u = x)^2$. Since $(x = x)^2$, we have $x \in z$. If $y = x$ then $y \in z$ by congruence (E5), but then $(y = x)^2$; thus we have proved $y = x \to (y = x)^2$, thus (by assumption on the logic) $(x = y \vee \neg(x = y))$. $\mathcal{QED}$

Unless we wish to give up some of the (basic) principles like separation or congruence, in Łukasiewicz logic and in product logic we get a crisp $=$. For this reason we make crisp equality a universal decision in FST.

Moreover, under the usual formulation of extensionality (with no $\Delta$'s), crispness of $=$ implies crispness of $\in$ (cf. [Gri99]):

**Lemma 5.3.11** *In a theory with extensionality, successors*[III] *and congruence, crispness of $=$ implies crispness of $\in$.*

*Proof.* Given $x$, $y$, take $y \cup \{x\}$. From crispness of $=$ we get $y = y \cup \{x\} \vee \neg(y = y \cup \{x\})$. The first case implies $x \in y$ by congruence. On the other hand, extensionality gives $\neg(y = y \cup \{x\}) \to$
$\neg\forall u((u \in y \vee u = x) \to u \in y) \to$
$\neg\forall u((u \in y \to u \in y) \wedge (u = x \to u \in y)) \to$
$\neg\forall u(u = x \to u \in y) \to \neg(x \in y)$. $\mathcal{QED}$

Naturally, crispness of both our basic predicate symbols means that our theory proves the crispness of all formulas and thus becomes classical (since we can prove LEM for any formula).

To avoid this trivialization we borrow the solution from [Shi99]: a modification of extensionality using the $\Delta$ connective, which invalidates the proof of Lemma 5.3.11 (then we get $x \in y \vee \neg\Delta(x \in y)$, which is provable in BL$\Delta$) and at the same time defines the desired relationship between a crisp $=$ and a generally non-crisp $\in$.

Further, similarly as in set theory over the intuitionistic logic (see [Gra79b]), the axiom of regularity in a very weak setting (with separation, empty set, and pairing) implies LEM.

[III] i. e., $\forall x, y \exists z \forall u(u \in z \equiv u \in x \vee u = y)$, introducing the corresponding operation $x \cup \{y\}$; alternatively we might request binary unions and singletons, etc.

**Lemma 5.3.12** *In a theory with separation (in a weak axiomatic setting), the axiom of foundation*

$$\forall x(\neg(x = 0) \to \exists y \in x \neg\exists w(w \in x \,\&\, w \in y))$$

*implies LEM.*

*Proof.* Let $\varphi$ be an arbitrary formula. Denote $1 = \{\emptyset\}$. By separation, define the set $\{\emptyset|\varphi\}$, so that $x \in \{\emptyset|\varphi\} \equiv x \in \{\emptyset\} \,\&\, \varphi \equiv (x = \emptyset) \,\&\, \varphi$.

Put $z = \{1\} \cup \{\emptyset|\varphi\}$. Then by definition $u \in z \equiv ((u = \emptyset \,\&\, \varphi) \vee u = 1)$, and $\neg(z = \emptyset)$ since $1 \in z$. By definition also $\emptyset \in 1$.

Apply foundation to $z$. Then $\exists y(((y = \emptyset \,\&\, \varphi) \vee y = 1) \,\&\, \neg\exists w(w \in z \,\&\, w \in y))$. If $\exists y(y = \emptyset \,\&\, \varphi \,\&\, \neg\exists w(\dots))$, then $\varphi$ holds. On the other hand if $\exists y(y = 1 \,\&\, \neg\exists w(w \in z \,\&\, w \in y))$, then $\neg\exists w(w \in z \,\&\, w \in 1)$, so $\neg(\emptyset \in z \,\&\, \emptyset \in 1)$, and so $\neg(\emptyset \in z)$ (since $\neg(\varphi \,\&\, \psi) \,\&\, \varphi \to \neg\psi$ is provable in BL), which by definition of $z$ is equivalent to $\neg\varphi$. We obtained $\varphi \vee \neg\varphi$. $\mathcal{QED}$

### 5.3.3 The role of $\Delta$

As explained in Section 5.3.2, we use $\Delta$ in the axiom of extensionality to prevent $\in$ from becoming crisp.

We also consistently use $\Delta$ after the existential quantifiers in axioms allowing us to define some of the standard set-theoretic operations like the empty set, a pair, a union, the set $\omega$ etc. Note the semantics of the existential quantifier: mere validity of a formula $\exists x\varphi(x)$ in a model $\mathbf{M}$ does *not* guarantee that there is an object $m$ for which $\|\varphi(m)\|_{\mathbf{M}} = 1$. Note also Theorem 5.2.7 on conservativity of introducing constants and function symbols in a theory with $\Delta$.

The $\Delta$ in the schema of $\in$-induction is introduced to weaken the statement, see Lemma 5.3.19 which shows that the standard form of $\Delta$-induction is not valid in $V^{\mathbf{L}}$ (in particular, if $\mathbf{L}$ is $[0,1]_{\text{Ł}}$).

Similarly, in the weak power set axiom $\Delta$ weakens the statement (see the footnote in the verification of weak power in $V^{\mathbf{L}}$, Section 5.3.5).

### 5.3.4 Axiomatization of FST

FST is a theory over BL$\forall\Delta$ with basic predicate symbols $=$ and $\in$. We list the axioms, defining new predicate and function symbols in the progress.

We define $\forall x, y(x \subseteq y \equiv \forall z \in x(z \in y))$.

(i) (extensionality) $\forall x, y(x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$

Note the condition is BL$\forall\Delta$-equivalent to $\forall z\Delta(z \in x \equiv z \in y)$ (use Corollary 5.3.3).

(ii) (empty set) $\exists x\Delta\forall y\neg(y \in x)$

We introduce a new constant $\emptyset$.[IV] Note Crisp($\emptyset$). Moreover, if for some $x$ we can prove $\forall y\neg(y \in x)$, then we can prove $x = \emptyset$.

(iii) (pair) $\forall x\forall y\exists z\Delta\forall u(u \in z \equiv (u = x \vee u = y))$

We introduce a binary function symbol $\{x, y\}$. We also introduce a unary function $\{x\}$; this set is called the *singleton* of $x$.

(iv) (union) $\forall x\exists z\Delta\forall u(u \in z \equiv \exists y(u \in y \,\&\, y \in x))$

We introduce a unary function symbol $\bigcup x$. Sometimes we also use a binary function $x \cup y$, which is $\bigcup\{x, y\}$.

(v) (weak power) $\forall x\exists z\Delta\forall u(u \in z \equiv \Delta(u \subseteq x))$

We introduce a unary function symbol WP$(x)$. Moreover, $WPC(x) = \{u \in WP(x) \mid \mathrm{Crisp}(u)\}$.

(vi) (infinity) $\exists z\Delta(\emptyset \in z \,\&\, \forall x \in z(x \cup \{x\} \in z))$

(vii) (separation) $\forall x\exists z\Delta\forall u(u \in z \equiv (u \in x \,\&\, \varphi(u, x)))$, for any formula not containing $z$ as a free variable

(viii) (collection) $\forall x\exists z\Delta[\forall u \in x\exists v\, \varphi(u, v) \to \forall u \in x\exists v \in z\varphi(u, v)]$ for any formula not containing $z$ as a free variable

(ix) ($\in$-induction) $\Delta\forall x(\forall y \in x\varphi(y) \to \varphi(x)) \to \Delta\forall x\varphi(x)$ for any formula $\varphi$

(x) (support) $\forall x\exists z(\mathrm{Crisp}(z)\&\Delta(x \subseteq z)))$

We introduce a unary function symbol Supp$(x)$. Note that Supp$(x)$ is always a crisp set.

### 5.3.5 BL$\Delta$-valued universe

We define a universe $V^{\mathbf{L}}$ over a complete BL$\Delta$-chain $\mathbf{L}$, define interpretation of formulas of FST in $V^{\mathbf{L}}$, verify validity of axioms and deduction rules of FST in $V^{\mathbf{L}}$, and thus show that FST does not collapse to (classical) ZF.

Consider the classical ZFC. Fix $\mathcal{C}$ as a schematic extension of BL$\forall\Delta$, and fix a constant $\mathbf{L}$ for an arbitrary linearly ordered complete $\mathcal{C}$-algebra; write $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1, \Delta)$. In ZFC let us make the following construction: in analogy to the construction of a Boolean-valued universe over a complete

[IV] According to Theorem 5.2.7, we at the same time add an axiom $\forall y\neg(y \in \emptyset)$ to the theory. Further we add axioms for new constants and function symbols, as specified in Theorem 5.2.7, without explicit notice.

Boolean algebra, we build the class $V^{\mathbf{L}}$ by ordinal induction. Define $L^+ = L - \{0\}$.

$$V_0^{\mathbf{L}} = \{\emptyset\}$$

$$V_{\alpha+1}^{\mathbf{L}} = \{f : \mathrm{Fnc}(f) \,\&\, D(f) \subseteq V_\alpha^{\mathbf{L}} \,\&\, R(f) \subseteq L^+\}$$

for any ordinal $\alpha$, and for limit ordinals $\lambda$

$$V_\lambda^{\mathbf{L}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbf{L}}$$

Here $\mathrm{Fnc}(x)$ is a unary predicate stating that $x$ is a function, and $D(x)$ and $R(x)$ are unary functions assigning to $x$ its domain and range, respectively.

Note that functions taking the value 0 on any element of their domain are not considered as elements of the universe.

Finally we put

$$V^{\mathbf{L}} = \bigcup_{\alpha\in\mathrm{On}} V_\alpha^{\mathbf{L}}$$

Observe that for $\alpha \leq \beta$, $V_\alpha^{\mathbf{L}} \subseteq V_\beta^{\mathbf{L}}$.

For any $a \in V^{\mathbf{L}}$ we define $\rho(a) = \min\{\alpha \,|\, a \in V_\alpha^{\mathbf{L}}\}$.

We define two binary functions from $V^{\mathbf{L}}$ into $L$, assigning to any $u, v \in V^{\mathbf{L}}$ the values $\|u \in v\|$ and $\|u = v\|$ (representing the "truth values" of the two predicates $\in$ and $=$)[V]:

$$\|u \in v\| = v(u) \text{ if } u \in D(v), \text{ otherwise } 0$$
$$\|u = v\| = 1 \text{ if } u = v, \text{ otherwise } 0$$

We now use induction on the complexity of formulas to define for any formula $\varphi(x_1, \ldots, x_n)$—the free variables in $\varphi$ being $x_1, \ldots, x_n$—a corresponding $n$-ary function from $(V^{\mathbf{L}})^n$ into $L$, assigning to an $n$-tuple $u_1, \ldots, u_n$ the value $\|\varphi(u_1, \ldots, u_n)\|$. The induction steps admit the following cases (we omit parameters for short):

$\varphi$ is 0: then $\|\varphi\| = 0$;
$\varphi$ is $\psi \,\&\, \chi$; then $\|\varphi\| = \|\psi\| * \|\chi\|$;
$\varphi$ is $\psi \to \chi$: then $\|\varphi\| = \|\psi\| \Rightarrow \|\chi\|$;

[V] We depart from the usual definition of these functions, since it turns out that this much simpler approach works due to crispness of our equality predicate. Cf. our paper [HH01], where the standard definition is used.

$\varphi$ is $\psi \wedge \chi$: then $\|\varphi\| = \|\psi\| \wedge \|\chi\|$;
$\varphi$ is $\psi \vee \chi$: then $\|\varphi\| = \|\psi\| \vee \|\chi\|$;
$\varphi$ is $\Delta\psi$; then $\|\varphi\| = \Delta\|\psi\|$:
$\varphi$ is $\forall x\psi$: then $\|\varphi\| = \bigwedge_{u \in V^{\mathbf{L}}} \|\psi(u)\|$;
$\varphi$ is $\exists x\psi$: then $\|\varphi\| = \bigvee_{u \in V^{\mathbf{L}}} \|\psi(u)\|$.

**Definition 5.3.13** *Let $\varphi$ be a closed formula. We say that $\varphi$ is valid in $V^{\mathbf{L}}$ iff $\|\varphi\| = 1$ is provable in ZFC.*

**Lemma 5.3.14** *(1) ZFC proves for $u \in V^{\mathbf{L}}$:*
*$u(x) = \|x \in u\|$ for $x \in D(u)$, $\|u = u\| = 1$*
*(2) ZFC proves for $u, v, w \in V^{\mathbf{L}}$:*
(i)$\|u = v\| * \|v = w\| \leq \|u = w\|$
(ii)$\|u \in v\| * \|v = w\| \leq \|u \in w\|$
(iii)$\|u = v\| * \|v \in w\| \leq \|u \in w\|$

*Proof.* (1) immediate,
(2) (i) immediate,
(ii) if $v = w$, then $\|v = w\| = 1$, so $\|u \in v\| = \|u \in v\| * \|v = w\| = \|u \in w\| * \|v = w\| = \|u \in w\|$; otherwise $\|v = w\| = 0$, in which case the statement holds trivially.
(iii) analogously. $\mathcal{QED}$

In the following two lemmas we omit parameters in the formulas involved for the sake of readability.

**Lemma 5.3.15 (Substitution)** *For any formula $\varphi(x)$, ZFC proves:*
*$\forall u, v \in V^{\mathbf{L}}(\|u = v\| * \|\varphi(u)\| \leq \|\varphi(v)\|)$.*

**Lemma 5.3.16 (Bounded quantifiers)** *For any formula $\varphi(y)$, ZFC proves $\forall u \in V^{\mathbf{L}}$:*
(i) $\|\exists y \in u\, \varphi(y)\| = \|\exists y(y \in u \,\&\, \varphi(y))\| = \bigvee_{v \in D(u)}(u(v) * \|\varphi(v)\|)$
(ii) $\|\forall y \in u\, \varphi(y)\| = \|\forall y(y \in u \to \varphi(y))\| = \bigwedge_{v \in D(u)}(u(v) \Rightarrow \|\varphi(v)\|)$

*Proof.* (i) $\|\exists y(y \in u \,\&\, \varphi(y))\| = \bigvee_{y \in V^{\mathbf{L}}}(\|y \in u\| * \|\varphi(y)\|) = \bigvee_{v \in D(u)}(u(v) * \|\varphi(v)\|)$ since $\|v \in u\|$ is nonzero only if $v \in D(u)$, and in that case it is $u(v)$.
(ii) analogously. $\mathcal{QED}$

For $u, v \in V^{\mathbf{L}}$, the value of $\|u \subseteq v\|$ is $\bigwedge_{z \in D(u)}(u(z) \Rightarrow \|z \in v\|)$ (i. e., the value of the defining formula).

**Theorem 5.3.17** *Let $\varphi$ be a closed formula provable in FST. Then $\varphi$ is valid in $V^{\mathbf{L}}$.*

Proof by induction on length of proofs. We omit validity proofs for logical axioms and verification that inference rules preserve validity. The validity of congruence axioms for $\in$ has been verified in Lemma 5.3.14. It remains to verify the set-theoretic axioms.

**Lemma 5.3.18** *The set-theoretic axioms (i)-(x) hold in $V^{\mathbf{L}}$.*

*Proof.* (extensionality) For fixed $x, y \in V^{\mathbf{L}}$, either $x = y$ and then $\|\Delta(x \subseteq y)\| = \|\Delta(y \subseteq x)\| = 1$, or $x \neq y$ and $\|x = y\| = 0$; then either $D(x) = D(y)$ and w. l. o. g. there is a $z \in D(x)$ s. t. $x(z) < y(z)$, so $\|\Delta y \subseteq x\| = 0$, or w. l. o. g. there is a $z \in D(x)$ s. t. $z \notin D(y)$, and then $\|z \in x\|$ is nonzero while $\|z \in y\|$ is zero thus $\|\Delta x \subseteq y\| = 0$.

(empty set) There is only one candidate for the role of an "empty set in $V^{\mathbf{L}}$", and this is the $\emptyset$ in $V_0^{\mathbf{L}}$. Indeed, for an arbitrary $x$ we get $\|x \in \emptyset\| = 0$ since no $x$ can be in the domain of $\emptyset$ (taken as a function).

(pair) For fixed $x, y \in V^{\mathbf{L}}$, there is a $z \in V^{\mathbf{L}}$ such that $D(z) = \{x, y\}$ and $z(x) = z(y) = 1$. The set $z$ has the desired properties: for arbitrary $u \in V^{\mathbf{L}}$, either $u \in D(z)$, then either $u = x$ or $u = y$ and $\|u \in z\| = z(u) = 1$, or $u \notin D(z)$, and then $\|u \in z\| = \|u = x\| = \|u = y\| = 0$.

(union) For a fixed $x \in V^{\mathbf{L}}$, define (auxiliary) $D^2(x) = \bigcup\{D(v) : v \in D(x)\}$. Define $z$ s. t. $D(z) = \{u \in D^2(x) : \bigvee_{v \in D(x)}(v(u) * x(v)) > 0\}$ (with a nilpotent t-norm, the union of a nonempty set may well be empty), and for $u \in D(z)$ set $z(u) = \bigvee_{v \in D(x)}(v(u) * x(v))$. Then for an arbitrary $u \in V^{\mathbf{L}}$, if $u \in D(z)$ then $\|u \in z\| = z(u) = \bigvee_{v \in D(x)}(v(u) * x(v)) = \|\exists y \in x(u \in y)\|$. If $u \notin D(z)$, then $\|u \in z\| = 0$, and also $\|\exists y \in x(u \in y)\| = 0$ by definition of $D(z)$.

(weak power) For a fixed $x \in V^{\mathbf{L}}$, define $z$ s. t. $D(z) = \{u \in V^{\mathbf{L}} : D(u) \subseteq D(x) \,\&\, u(v) \leq x(v) \text{ for } v \in D(u)\}$, and $z(u) = 1$ for $u \in D(z)$. For $u \in V^{\mathbf{L}}$, either $u \in D(z)$ and then $\|u \in z\| = z(u) = 1$, and also (by definition of $D(z)$) $\|u \subseteq x\| = 1 = \|\Delta u \subseteq x\|$, or $u \notin D(z)$, thus $\|u \in z\| = 0$, and (by definition of $D(z)$) either $D(u) \not\subseteq D(x)$, or for some $v \in D(u)$, $u(v) > x(v)$, and in either case $\|\Delta \forall v \in u(v \in x)\| = \Delta \bigwedge_{v \in D(u)}(u(v) \Rightarrow \|v \in x\|) = 0$.[VI)]

[VI)] Note, however, that the "standard" power set axiom is *not* valid in $V^{\mathbf{L}}$: consider $\mathbf{L}$ to be $[0,1]_{\mathrm{L}}$, and fix $x$ as the set $\emptyset$. Then the axiom would require existence of $z \in V^{\mathbf{L}}$ such that $\|u \in z\| = \bigwedge_{v \in D(u)} -u(v)$. This obviously requires $D(z)$ to be a proper class (which contains, e. g., all "subnormal singletons" on each $V_\alpha$).

(infinity) Define a function $z$ with $D(z) = V^{\mathbf{L}}_{\omega}$ and $z(u) = 1$ for $u \in D(z)$. Then $\|\emptyset \in z\| = 1$ and, since for $x \in V^{\mathbf{L}}_{\alpha}$ the function representing $x \cup \{x\}$ is in $V^{\mathbf{L}}_{\alpha+1}$, also $\|\forall x \in z(x \cup \{x\} \in z)\| = 1$.

(separation) For a fixed $x \in V^{\mathbf{L}}$, and a given $\varphi$, define $z$ s. t. $D(z) = \{u \in D(x) : x(u) * \|\varphi(u)\| > 0\}$ and for $u \in D(z)$ set $z(u) = x(u) * \|\varphi(u)\|$. Obviously this definition of $z$ demonstrates the validity of separation in $V^{\mathbf{L}}$.

(collection)[VII] Work with a fixed $x \in V^{\mathbf{L}}$ and a given $\varphi$. For each $u \in V^{\mathbf{L}}$, define $F_u = \{w \in [0,1] \mid \exists v \in V^{\mathbf{L}} \|\varphi(u,v)\| = w\}$. Then $\forall w \in F_u \exists \alpha \exists v (\|\varphi(u,v)\| = w \;\&\; \rho(v) = \alpha)$. Hence by collection in ZF, $\exists y \forall w \in F_u \exists \alpha \in y \exists v (\|\varphi(u,v)\| = w \;\&\; \rho(v) = \alpha)$.

Let $\alpha_u = \{\alpha \in y \mid \alpha \in \mathrm{On}\}$. Then $F_u = \{w \in [0,1] \mid \exists v \in V^{\mathbf{L}}_{\alpha_u} (\|\varphi(u,v)\| = w)\}$. Hence $\bigvee_{v \in V^{\mathbf{L}}} \|\varphi(u,v)\| = \bigvee_{v \in V^{\mathbf{L}}_{\alpha_u}} \|\varphi(u,v)\|$.

Let $\beta = \bigcup\{\alpha_u \mid u \in D(x)\}$. Then

$\|\forall u \in x \exists v \varphi(u,v)\| =$

$= \bigwedge_{u \in D(x)} (x(u) \Rightarrow \bigvee_{v \in V^{\mathbf{L}}} \|\varphi(u,v)\|) =$

$= \bigwedge_{u \in D(x)} (x(u) \Rightarrow \bigvee_{v \in V^{\mathbf{L}}_{\alpha_u}} \|\varphi(u,v)\|) \leq$

$\leq \bigwedge_{u \in D(x)} (x(u) \Rightarrow \bigvee_{v \in V^{\mathbf{L}}_{\beta}} \|\varphi(u,v)\|)$.

Define $z \in V^{\mathbf{L}}$ so that $D(z) = V^{\mathbf{L}}_{\beta}$ and $z(a) = 1$ for $a \in D(z)$. Then $\bigvee_{v \in V^{\mathbf{L}}_{\beta}} \|\varphi(u,v)\| = \bigvee_{v \in D(z)} (z(v) * \|\varphi(u,v)\|) = \|\exists v \in z \varphi(u,v)\|$.

Then $\|\forall u \in x \exists v \varphi(u,v)\| \leq \bigwedge_{u \in D(x)} (x(u) \Rightarrow \|\exists v \in z\, \varphi(u,v)\|)$. In other words, we have managed to construct a $z$ for which the desired inequality holds in $V^{\mathbf{L}}$.

($\in$-induction) Fix a formula $\varphi$ and suppose the axiom does not hold in $V^{\mathbf{L}}$. Then (since $\Delta$ is two-valued) it must be the case that $\|\Delta \forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x))\| = 1$ and $\|\Delta \forall x \varphi(x)\| = 0$, thus there is a successor ordinal $\alpha$ s. t. $\exists x \in V^{\mathbf{L}}_{\alpha} (\|\varphi(x)\| < 1) \,\&\, \forall \beta < \alpha \forall y \in V^{\mathbf{L}}_{\beta} (\|\varphi(y)\| = 1)$. Suppose first $\alpha = 0$; but $\|\forall y \in \emptyset \varphi(y) \rightarrow \varphi(\emptyset)\| = (1 \Rightarrow \|\varphi(\emptyset)\|) = \|\varphi(\emptyset)\| < 1$, thus the antecedent would be 0. Suppose now $\alpha > 0$, and $x \in V^{\mathbf{L}}_{\alpha}$ is s. t. $\|\varphi(x)\| < 1$. From the condition that $\|\forall y \in x \varphi(y) \rightarrow \varphi(x)\| = 1$, and since $\|\forall y \in x \varphi(y)\| = 1$, we get $\|\varphi(x)\| = 1$, a contradiction.

(support) For a fixed $x \in V^{\mathbf{L}}$ take $z$ such that $D(z) = D(x)$ and $\forall u \in D(z)$ $z(u) = 1$. Then $\|\mathrm{Crisp}(z)\| = 1$ and $\bigwedge_{u \in D(x)} (x(u) \Rightarrow \|u \in z\|) = 1$. $\mathcal{QED}$

**Lemma 5.3.19** *The $\in$-induction schema $\forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$ is* not *valid in $V^{\mathbf{L}}$ over* $[0,1]_{\mathrm{L}}$.

[VII] Proof taken from [Shi99].

*Proof.*[VIII)] We shall need Definition 5.3.23 of a hereditarily crisp set (the unary predicate symbol is $H(x)$). Let $\varphi(x)$ be the formula $H(x) \to (x \in p)$, where $p$ is a free variable. Let $\mathrm{IND}(x)$ stand for $\forall y \in x\varphi(y) \to \varphi(x)$.

Fix $\mathbf{L}$ as $[0,1]_{\mathrm{L}}$. Observe that for $a \in V^{\mathbf{L}}$, $\|H(a)\| = 1$ iff $a$ is hereditarily crisp, i. e., hereditarily built of functions taking only the value 1; otherwise $\|H(a)\| = 0$ (cf. Lemma 5.3.24).

Let $p^{\mathbf{L}}$ be an element of $V^{\mathbf{L}}$ such that $D(p^{\mathbf{L}}) = \{\emptyset, \langle\emptyset, 1\rangle\}$ (we denote the latter element 1) and $p^{\mathbf{L}}(\emptyset) = 1$, $p^{\mathbf{L}}(1) = 1/2$.

(i) For any element $a$ of $V^{\mathbf{L}}$ which is hereditarily crisp and distinct from $\emptyset$ or 1, we get $\|\varphi(a)\| = 0$.

(ii) $\|\varphi(\emptyset)\| = 1$, $\|\varphi(1)\| = 1/2$.

(iii) $\|\mathrm{IND}(\emptyset)\| = 1$, $\|\mathrm{IND}(\emptyset)\| = 1/2$.

(iv) For $a \in V^{\mathbf{L}}$ which is not hereditarily crisp, $\|\mathrm{IND}(a)\| = 1$ since $\|\varphi(a)\| = 1$.

(v) If $a \in V^{\mathbf{L}}$ is hereditarily crisp then $\|\mathrm{IND}(a)\| \geq 1/2$. Consider $a$ distinct from $\emptyset$ or 1. Then there is some $b \neq \emptyset$ such that $a(b) = 1$. If $b = 1$ then $\|\varphi(b)\| = 1/2$, otherwise $\|\varphi(b)\| = 0$. Thus $\|\forall y \in a\varphi(y)\| \leq 1/2$, so $\|\mathrm{IND}(a)\| \geq 1/2$.

For the formula $\varphi$ therefore, the value of the induction schema in $V^{\mathbf{L}}$ is at most 1/2. $\mathcal{QED}$

We know, however, that "standard" forms of power set axiom and the schema of $\in$-induction will result in a consistent theory (relative to ZF).

### 5.3.6 An Interpretation of ZF in FST

Within FST, we shall define a class (i. e., we shall give a formula with one free variable, in the language of FST) of hereditarily crisp sets, which will be proved an inner model of ZF in FST.

**Definition 5.3.20 (Hereditarily crisp transitive set)**
$HCT(x) \equiv \mathrm{Crisp}(x) \,\&\, \forall u \in x(\mathrm{Crisp}(u) \& u \subseteq x)$

The formula $HCT(x)$ defines a class, and we adopt the habit of writing $x \in HCT$ instead of $HCT(x)$ and approach classes in a similar way we approach sets. (However, we prefer to treat $\mathrm{Crisp}(x)$ as a predicate). A "crisp class" $C$ is a class for which $\forall x\Delta(x \in C \vee \neg(x \in C))$, or equivalently, $\forall x\Delta(x \in C \to \Delta(x \in C))$.

VIII) Due to Hájek, unpublished note.

**Lemma 5.3.21** $x \in HCT \equiv \mathrm{Crisp}(x) \,\&\, \forall u \in x(\mathrm{Crisp}(u)) \,\&\, \forall u \in x(u \subseteq x)$

Before starting a proof, observe $\bowtie \alpha \to (\alpha \to \alpha^2)$ (a consequence of Lemma 5.3.4).

*Proof.* Right to left: $\mathrm{Crisp}(x) \,\&\, \forall u \in x\, \mathrm{Crisp}(u) \,\&\, \forall u \in x(u \subseteq x) \to$
$\mathrm{Crisp}(x) \,\&\, \forall u((u \in x)^2 \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)) \to$
$\mathrm{Crisp}(x) \,\&\, \mathrm{Crisp}(x) \,\&\, \forall u((u \in x)^2 \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)) \to$
$\mathrm{Crisp}(x) \,\&\, \forall u(\bowtie (u \in x) \,\&\, ((u \in x)^2 \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x))) \to$
$\mathrm{Crisp}(x) \,\&\, \forall u(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x))$, using the above observation.
Left to right: first, $HCT(x) \equiv$
$\mathrm{Crisp}(x) \,\&\, \mathrm{Crisp}(x) \,\&\, \forall u(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)) \equiv$
$\mathrm{Crisp}(x) \,\&\, \forall u(u \in x \to (\mathrm{Crisp}(x) \,\&\, \mathrm{Crisp}(u) \,\&\, u \subseteq x) \to$
$\mathrm{Crisp}(x) \,\&\, \forall u(u \in x \to (\Delta\mathrm{Crisp}(u) \,\&\, \Delta(u \subseteq x))$, the last implication due to Lemma 5.3.9 and idempotence of $\mathrm{Crisp}(u)$ (Lemma 5.3.7). Further,
$\mathrm{Crisp}(x) \,\&\, \forall u(u \in x \to (\Delta\mathrm{Crisp}(u) \,\&\, \Delta(u \subseteq x)) \to$
$\forall u(\bowtie (u \in x) \,\&\, (u \in x \to (\Delta\mathrm{Crisp}(u) \,\&\, \Delta(u \subseteq x))) \to$
$\forall u(\Delta(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)))$, using Lemma 5.3.8. The latter implies
$\forall u(\Delta(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)))^2$, thus also
$(\forall u(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)))^2$ (since $\forall u(\Delta\alpha)^2 \to (\forall u \Delta\alpha)^2$). Further $\forall u(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)) \to \forall u(u \in x \to \mathrm{Crisp}(u))$ (and similarly for $u \subseteq x$). This concludes the proof. $\mathcal{QED}$

**Lemma 5.3.22 (Crispness of $HCT$)** $x \in HCT \to \Delta(x \in HCT)$.

*Proof.* See the proof of Lemma 5.3.21, left to right, for proof that $x \in HCT \to \forall u \Delta(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x)))$. By Lemma 5.3.2, the latter is equivalent to $\Delta \forall u(u \in x \to (\mathrm{Crisp}(u) \,\&\, u \subseteq x))$. Finally, observe $x \in HCT \to \Delta\mathrm{Crisp}(x) \,\&\, x \in HCT$, using Lemma 5.3.7. $\mathcal{QED}$

**Definition 5.3.23 (Hereditarily crisp set)**
$\mathrm{H}(x) \equiv \mathrm{Crisp}(x) \,\&\, \exists x' \in \mathrm{HCT}(x \subseteq x')$.

**Lemma 5.3.24 (Crispness of $H$)** $x \in H \to \Delta(x \in H)$.

*Proof.* We prove
$(\mathrm{Crisp}(x) \,\&\, \exists y \in \mathrm{HCT}(x \subseteq y)) \to \Delta(\mathrm{Crisp}(x) \,\&\, \exists y \in \mathrm{HCT}(x \subseteq y))$.
Since $\mathrm{Crisp}(x) \to \Delta\mathrm{Crisp}(x)$, it suffices to prove
$(\mathrm{Crisp}(x) \,\&\, \exists y \in \mathrm{HCT}(x \subseteq y)) \to \Delta(\exists y \in \mathrm{HCT}(x \subseteq y))$.
First, $(\mathrm{Crisp}(x) \,\&\, y \in \mathrm{HCT} \,\&\, x \subseteq y) \to \Delta(y \in \mathrm{HCT} \,\&\, x \subseteq y)$ by Lemma

5.3.22 and Lemma 5.3.9. Now generalize:
$\forall y((\mathrm{Crisp}(x) \,\&\, y \in \mathrm{HCT} \,\&\, x \subseteq y) \to \Delta(y \in \mathrm{HCT} \,\&\, x \subseteq y))$; hence
$(\mathrm{Crisp}(x) \,\&\, \exists y(y \in \mathrm{HCT} \,\&\, x \subseteq y)) \to \exists y \Delta(y \in \mathrm{HCT} \,\&\, x \subseteq y)$; and the succedent implies
$\Delta \exists y(y \in \mathrm{HCT} \,\&\, x \subseteq y)$ (see Lemma 5.3.1 (iii)). $\mathcal{QED}$

**Lemma 5.3.25 (Transitivity of $H$)** *$y \in x \,\&\, x \in H \to y \in H$.*

*Proof.* First, note $y \in x \,\&\, x \in H \to (y \in x)^2 \,\&\, (x \in H)^2$ since $H$ is crisp and so is $x$. Now
$[(y \in x) \,\&\, \exists x' \in HCT(x \subseteq x')] \to [\exists x' \in HCT(y \in x \,\&\, x \subseteq x')] \to \mathrm{Crisp}(y)$,
and $[(y \in x) \,\&\, \exists x' \in HCT(x \subseteq x')] \to [\exists x' \in HCT(y \in x')] \to$
$\exists x' \in HCT(y \subseteq x')$. $\mathcal{QED}$

We show that FST proves $H$ to be an inner model of ZF. In more detail, we consider the classical language with connectives $\&$, $\to$ and 0 and the universal quantifier $\forall$, and for $\varphi$ a formula in the language of ZF, define $\varphi^H$ inductively as follows:
$\varphi^H = \varphi$ for $\varphi$ atomic;
$\varphi^H = \varphi$ for $\varphi = 0$;
$\varphi^H = \psi^H \,\&\, \chi^H$ for $\varphi = \psi \,\&\, \chi$;
$\varphi^H = \psi^H \to \chi^H$ for $\varphi = \psi \to \chi$;
$\varphi^H = (\forall x \in H)\psi^H$ for $\varphi = (\forall x)\psi$.

Note that for introduced symbols of classical logic, we would obtain $\varphi^H = \neg\psi^H$ for $\varphi = \neg\psi$; $\varphi^H = \psi^H \vee \chi^H$ for $\varphi = \psi \vee \chi$, and $\varphi^H = (\exists x \in H)\psi^H$ for $\varphi = (\exists x)\psi$; the latter is since $(\neg\forall x \neg\varphi)^H = (\forall x \neg\varphi \to 0)^H = (\forall x \neg\varphi)^H \to 0) = (\forall x \in H \neg\varphi^H \to 0)$ which is BL$\forall$ equivalent to
$\neg(\forall x(x \in H \to (\varphi^H \to 0)))$;
$\neg\forall x(\neg(x \in H \,\&\, \varphi^H)$;
$\neg\neg\exists x(x \in H \& \varphi^H)$. The latter is BL$\forall\Delta$-equivalent to $(\exists x \in H)\varphi^H$ since in BL$\Delta$ we can prove $\bowtie \varphi \to (\neg\neg\varphi \to \varphi)$.

**Theorem 5.3.26** *Let $\varphi$ be a theorem of ZF. Then* $\mathrm{FST} \vdash \varphi^{\mathrm{H}}$.

The remaining part of this section gives a proof of this theorem. We first show that the law of the excluded middle (LEM) holds in H; since BL$\forall$ together with LEM yield classical logic, we will have proved that all axioms of classical logic are provable when relativized to H. Then we prove the H-relativized versions of all the axioms of the ZF set theory. (Note that all deduction rules of classical logic are available in FST).

**Lemma 5.3.27** *Let $\varphi(x_1, \dots, x_n)$ be a ZF-formula whose free variables are among $x_1, \dots, x_n$. Then FST proves*

$$\forall x_1 \in H \dots \forall x_n \in H(\varphi^H(x_1, \dots, x_n) \vee \neg\varphi^H(x_1, \dots, x_n))$$

*Proof.* The proof proceeds by induction on the complexity of $\varphi$. An $n$-tuple of variables $x_1, \dots, x_n$ is denoted $\overline{x}$. Note that for any $\varphi(\overline{x})$, the following formulas are equiprovable in FST:
$\forall\overline{x}(\overline{x} \in H \to (\varphi(\overline{x}) \vee \neg\varphi(\overline{x})))$;
$\forall\overline{x}(\overline{x} \in H \to \Delta(\varphi(\overline{x}) \vee \neg\varphi(\overline{x})))$;
$\forall\overline{x}(\overline{x} \in H \to (\varphi(\overline{x}) \to \neg\varphi(\overline{x})))$;
$\forall\overline{x}(\overline{x} \in H \to \Delta(\varphi(\overline{x}) \to \Delta\varphi(\overline{x})))$.

Atomic subformulas: $=$ is crisp, for $\in$ we have to prove $x \in y \to \Delta x \in y$ assuming $x, y \in H$. In fact $y \in H$ implies $\mathrm{Crisp}(y)$, which entails $\forall x(x \in y \to \Delta x \in y)$.

Conjunction: for a subformula $\psi_1(\overline{x}, \overline{y}) \,\&\, \psi_2(\overline{x}, \overline{z})$ of $\varphi$ assume (we omit listings of free variables) $\overline{x}, \overline{y} \in H \to (\psi_1^H \to \Delta\psi_1^H)$ and $\overline{x}, \overline{z} \in H \to (\psi_2^H \to \Delta\psi_2^H)$. Then $((\overline{x} \in H)^2 \,\&\, \overline{y}, \overline{z} \in H) \to ((\psi_1^H \,\&\, \psi_2^H) \to \Delta(\psi_1^H \,\&\, \psi_2^H))$, and since membership in $H$ is idempotent, this completes the induction step for conjunction.

Implication: for a subformula $\psi_1(\overline{x}, \overline{y}) \to \psi_2(\overline{x}, \overline{z})$ of $\varphi$, assume $\overline{x}, \overline{y} \in H \to (\psi_1^H \to \Delta\psi_1^H)$ and $\overline{x}, \overline{z} \in H \to (\psi_2^H \to \Delta\psi_2^H)$. Thus $(\psi_1^H \to \psi_2^H) \,\&\, (\overline{x}, \overline{z} \in H) \to (\psi_1^H \to \Delta\psi_2^H)$, and since $\overline{x}, \overline{y} \in H$ implies crispness of $\psi_1^H$, using Lemma 5.3.8 we get $\overline{x}, \overline{y} \in H \to ((\psi_1^H \to \Delta\psi_2^H) \to \Delta(\psi_1^H \to \psi_2^H))$. Thus $\overline{x}, \overline{y}, \overline{z} \in H \to ((\psi_1^H \to \psi_2^H) \to \Delta(\psi_1^H \to \psi_2^H))$.

The universal quantifier: for a subformula $\forall y \psi(\overline{x}, y)$ of $\varphi$, the induction hypothesis is $\overline{x}, y \in H \to (\psi^H(\overline{x}, y) \to \Delta\psi^H(\overline{x}, y))$. Generalize in $y$: $\overline{x} \in H \to \forall y(y \in H \to (\psi^H(\overline{x}, y) \to \Delta\psi^H(\overline{x}, y))$; now since for a crisp $\alpha$, BL proves $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$, we may modify the succedent to $\forall y((y \in H \to \psi^H(\overline{x}, y)) \to (y \in H \to \Delta\psi^H(\overline{x}, y))$, and distributing $\forall y$, we have proved: $\overline{x} \in H \to (\forall y \in H\psi^H(\overline{x}, y) \to \forall y \in H\Delta\psi^H(\overline{x}, y))$. To flip the $\Delta$ and the $\forall y \in H$ in the succedent, use Lemma 5.3.8 and Lemma 5.3.2. $\mathcal{QED}$

**Lemma 5.3.28** *$\forall x(x \subseteq H \,\&\, \mathrm{Crisp}(x) \to x \in H)$.*

*Proof.* Prove for $x$ a (new) constant, then apply Theorem 5.2.5. We show $FST, x \subseteq H, \mathrm{Crisp}(x) \vdash x \in H$ and to get the statement we apply the Deduction theorem; note that under the assumption $\mathrm{Crisp}(x)$, $x \subseteq H$ is crisp as well, due to Lemma 5.3.9. Henceforth we prove in the enriched theory.

$x \subseteq H$ is by definition $\forall u \in x(\mathrm{Crisp}(u) \,\&\, \exists u'(u' \in HCT \,\&\, u \subseteq u'))$. By Lemma 5.3.9, this entails $\forall u \in x \exists u'(u' \in HCT \,\&\, \Delta(u \subseteq u'))$. Thus by collection, $\exists v_0 \Delta \forall u \in x \exists u' \in v_0(u' \in HCT \,\&\, \Delta(u \subseteq u'))$. Fix $v_0$ using Theorem 5.2.7.

Thanks to axiom of support, we have also
$\forall u \in x \exists u'(\Delta(u' \in \mathrm{Supp}(v_0)) \,\&\, u' \in HCT \,\&\, \Delta(u \subseteq u'))$. Note that $\mathrm{Supp}(v_0)$ is crisp. Separate: $v = \{u' \in \mathrm{Supp}(v_0) \mid u' \in HCT\}$. Fix $v$, using again Theorem 5.2.7. Observe that $\forall u \in x \exists u' \in v(u' \in HCT \,\&\, \Delta(u \subseteq u'))$.

Now we argue (in the enriched theory):
(i) $\mathrm{Crisp}(v)$. $v$ is separated from a crisp set by a crisp formula.
(ii) $(\forall u \in v)\mathrm{Crisp}(u)$, since $u \in v \to u \in HCT$[IX].
(iii) $\mathrm{Crisp}(\bigcup v)$ as a consequence of (i), (ii): we have $u \in \bigcup v \equiv \exists b(u \in b \,\&\, b \in v) \to \exists b(\Delta(u \in b) \,\&\, \Delta(b \in v)) \to \Delta \exists b(u \in b \,\&\, b \in v)$, the last implication by Lemma 5.3.1 (iii).
(iv) $(\forall a \in \bigcup v)\mathrm{Crisp}(a)$, because $a \in \bigcup v \equiv \exists b \in HCT(a \in b \in v)$ by (ii), hence $a$ is crisp. Also $\forall a \in \bigcup v(a \subseteq \bigcup v)$: we have $b \in a \in \bigcup v \to \exists y(b \in a \in y \in v)$, and since $y \in HCT$ is transitive, $b \in y \in v$ and $b \in \bigcup v$.
(v) $\bigcup v \in HCT$ as a consequence of (iv).

Now consider $(\bigcup v) \cup x$ as a possible "witness" for $x \in H$, i. e., such that $(\bigcup v) \cup x \in HCT$. Obviously the set is crisp. Moreover, $(y \in (\bigcup v) \cup x) \equiv (y \in \bigcup v \vee y \in x) \to \mathrm{Crisp}(y)$. Finally, $y \in (\bigcup v) \cup x \to (y \in \bigcup v \vee y \in x) \to (y \subseteq \bigcup v \vee \exists y' \in v(y \subseteq y'))$, and the latter disjunct implies $\exists y' \subseteq \bigcup v(y \subseteq y')$, thus $y \subseteq \bigcup v$. We have shown that $(\bigcup v) \cup x$ satisfies the conditions of Lemma 5.3.21 and thus is in $HCT$. $\mathcal{QED}$

We consider ZF with the following axioms: empty set, pair, union, power set, infinity, separation, collection, extensionality, $\in$-induction.

**Theorem 5.3.29** *For $\varphi$ being any of the abovementioned axioms of ZF, FST proves $\varphi^H$.*

*Proof.* **(empty set)** $\exists z \forall u \neg(u \in z)$.
The H-translation, which reads $\exists z \in H \forall u \in H \neg u \in z$, is provable since $0 \in H$.

**(pair)** $\forall x, y \exists z \forall u(u \in z \equiv (u = x \vee u = y))$.
The H-translation is $\forall x, y \in H \exists z \in H \forall u \in H(u \in z \equiv (u = x \vee u = y))$. Assuming for the while that $x, y$ are new constants, we prove $x, y \in H \vdash \exists z \in$

[IX] In fact $u \in v \to (u \in v \,\&\, u \in HCT)$.

$H\forall u \in H(u \in z \equiv (u = x) \vee (u = y))$, then we use Deduction theorem (note that $x, y \in H$ is a crisp formula) and to obtain the desired statement we apply Theorem 5.2.5.

We argue the operation is absolute in $H$ by showing that $\{x, y\} \in H$. Obviously it is a crisp set; we need to find a "witness", i. e., to show
(i) $\exists z' \in HCT(\{x, y\} \subseteq z')$.

Observe that (in the enriched theory), from the assumption $x \in H$ we get) $\exists x'\Delta(x' \in HCT \& x \subseteq x')$ and $x' \in HCT \& x \subseteq x' \to \Delta(x' \in HCT \& x \subseteq x')$ by Lemma 5.3.9, so we may introduce new constants $x'$ and $y'$. It remains to show $\{x, y\} \cup x' \cup y' \in HCT$, which is obvious. This gives (i) using dual form of specification.

**(union)** $\forall x \exists z \forall u(u \in z \equiv \exists y(u \in y \in x))$.
The H-translation is $\forall x \in H \exists z \in H \forall u \in H(u \in z \equiv \exists y \in H(u \in y \in x))$. We show the operation is absolute in $H$. Fix an $x$, and let $x' \in HCT$ witness $x \in H$. Then $\bigcup x$ is a crisp set with crisp elements (since $x \subseteq x'$), and $\bigcup x \subseteq \bigcup x' \in HCT$, which witnesses $\bigcup x \in H$.

**(power set)** $\forall x \exists z \forall u(u \in z \equiv u \subseteq x)$
The $H$-translation reads $\forall x \in H \exists z \in H \forall u \in H(u \in z \equiv \forall y \in H(y \in u \to y \in x))$. Let $x'$ be a witness for $x \in H$. Then $\forall u \in H(u \in WPC(x) \equiv \Delta(u \subseteq x \,\&\, \mathrm{Crisp}(u)) \equiv \Delta(u \subseteq x) \equiv \Delta(u \subseteq x)^H$, the last equivalence due to transitivity of $H$. $WPC(x)$ is crisp, and $WPC(x) \cup x'$ is a transitive crisp set with crisp elements, thus in $HCT$ and a witness for $WPC(x) \in H$.

**(separation)** $\forall x \exists z \forall u(u \in z \equiv (u \in x \,\&\, \varphi(u)))$ for a ZF-formula not containing $z$ freely.
Let $x' \in HCT$ witness $x \in H$ and set $z = \{u \in x; \varphi^H(u)\}$, then $z$ is a crisp set and $z \subseteq x \subseteq x' \in HCT$ (i. e., $x'$ is a witness for $z \in H$).

**(infinity)** $\exists z(0 \in z \,\&\, \forall u \in z(u \cup \{u\} \in z))$.
It suffices to prove that there is a set $z \in H$ s. t. $0 \in z \,\&\, \forall u \in z(u \cup \{u\} \in z)$ (as $H$ is transitive). We in fact find a $z \in HCT$ satisfying the condition. Let $z_0$ be any set satisfying the axiom of infinity in FST and define $z_1 = \{x \in z_0 : \Delta(x \in z_0) \,\&\, \mathrm{Crisp}(x)\}$. Then $z_1$ is a subset of $z_0$ and $0 \in z_1$ and $u \in z_1 \to u \cup \{u\} \in z_1$. Now let $z = \{x \in z_1 : x \subseteq z_1\}$, i. e., a transitive subset of $z_1$. Obviously $0 \in z$, let us prove $x \in z \to (x \cup \{x\}) \in z$, that is by definition of $z$, $[x \in z_1 \,\&\, \forall y \in x(y \in z_1)] \to [x \cup \{x\} \in z_1 \,\&\, \forall a(a \in x \vee a = x \to a \in z_1)]$. We know $x \in z_1 \to (x \cup \{x\} \in z_1)$. Also, $x \in z_1 \,\&\, (y \in x \to y \in z_1) \to (y \in x \vee y = x \to y \in z_1)$. (Note the presumption $x \in z_1$ is crisp). Finally, $z$ is a crisp transitive set with crisp elements, so $z \in HCT$.

**(extensionality)** $\forall x, y(x = y \equiv \forall z(z \in x \equiv z \in y))$.

The H-translation $\forall x, y \in H(x = y \equiv \forall z \in H(z \in x \equiv z \in y))$ follows from extensionality in FST by H being transitive and by the crispness of its elements (the $\Delta$'s may be left out).

**(collection)** $\forall u \exists v(\forall x \in u \exists y \varphi(x, y) \rightarrow \forall x \in u \exists y \in v \varphi(x, y))$ for $\varphi$ not containing $v$ freely.
The H-translation reads $\forall u \in H \exists v \in H(\forall x \in H(x \in u \rightarrow \exists y \in H \varphi^H(x, y)) \rightarrow \forall x \in H(x \in u \rightarrow \exists y \in H(y \in v \,\&\, \varphi^H(x, y))))$. Fix $u \in H$ and a ZF-formula $\varphi$; we want to find a corresponding $v \in H$ s. t. the above is true. Define $v_0$ by collection in FST for $u$ and the formula $y \in H \,\&\, \varphi^H(x, y)$; let $v_1$ be the support of $v_0$. Separate $v = \{x \in v_1 : \Delta(x \in v_1) \,\&\, x \in H\}$. Then $v \subseteq H$ is a crisp set and satisfies the collection axiom for $u$ and $y \in H \,\&\, \varphi^H(x, y)$. By Lemma 5.3.28, $v \in H$.

**($\in$-induction)** $\forall x(\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$ for any ZF-formula $\varphi$.
The H-translation is $\forall x \in H(\forall y \in H(y \in x \rightarrow \varphi^H(y)) \rightarrow \varphi^H(x)) \rightarrow \forall x \in H \varphi^H(x)$. Fix a ZF-formula $\varphi$, and consider the instance of $\in$-induction in FST for the formula $x \in H \rightarrow \varphi^H(x)$:
$\Delta \forall x(\forall y \in x(y \in H \rightarrow \varphi^H(y)) \rightarrow (x \in H \rightarrow \varphi^H(x))) \rightarrow \Delta \forall x(x \in H \rightarrow \varphi^H(x))$. This formula is our aim except for the $\Delta$'s. Let us denote $A$ the antecedent and $S$ the succedent in the implication, with the $\Delta$'s omitted. Since $\Delta S \rightarrow S$, it remains to be proved that $A \rightarrow \Delta A$. Let us further denote $\forall y \in H(y \in x \rightarrow \varphi^H(y))$ with $I$. First, it is obvious that
$x \in H \rightarrow \forall y(y \in x \rightarrow \bowtie \varphi^H(y))$, and
$x \in H \rightarrow \forall y \bowtie (y \in x)$. Thus,
$x \in H \rightarrow \forall y(\bowtie (y \in x) \,\&\, (y \in x \rightarrow \bowtie \varphi^H(y)))$, hence
$x \in H \rightarrow \forall y(\bowtie (y \in x \rightarrow \varphi^H(y)))$, and
$x \in H \rightarrow \bowtie \forall y(y \in x \rightarrow \varphi^H(y))$, thus $x \in H \rightarrow \bowtie I$. Since $x \in H \equiv \bowtie (x \in H)$, we get
$A \rightarrow \forall x \in H(\bowtie I \,\&\, (I \rightarrow \Delta \varphi^H(x)))$, hence by use of Lemma 5.3.8
$A \rightarrow \forall x \in H \Delta(I \rightarrow \varphi^H(x))$, so
$A \rightarrow \Delta \forall x \in H(I \rightarrow \varphi^H(x))$, which is $A \rightarrow \Delta A$. $\mathcal{QED}$

## 5.4 Concluding remarks

We have proposed an axiomatic set theory in a fuzzy predicate logic. The theory is sufficiently strong and at the same time admits many-valued interpretations. Our future aim is to refine the axiomatic system and develop it mathematically.

The mathematical development should cover some basic and interesting concepts like orderings, similarities, mappings, the notion of infinite set and cardinalities, etc. These are well worth investigating in themselves. Moreover, such an attempt might hopefully lead to further and more specific demands on the axiomatic system itself and its various strengthenings and/or modifications, motivated by specific needs.

Having the class of hereditarily crisp set in FST, which is an inner model of ZF, we may rely on some of the classical technical concepts, like ordinal numbers, and try to form analogies of some more sophisticated constructions analogous to those of ZF (e. g., showing that each set has a *rank* in the ordinals given by the crisp core). This might be another potential source of inspiration for additional axioms.

One might also continue the work in some schematic extensions of BL$\forall\Delta$; for example, Łukasiewicz logic with $\Delta$. On a broader scope, in view of the benefits the additional connective $\Delta$ has brought into our predicate logic, and regarding also the possibilities of having multiple conjunctions with a fixed "arithmetical" semantics, as in the logic of Takeuti and Titani, we might consider enriching our language with additional propositional connectives.

It is also desirable to perform a more detailed comparison of our theory FST with other systems, in order to find out what we are missing. This has already been started by Hájek, who compared FST with the system of Titani.

# Bibliography


[AFM03] P. Aglianò, I. M. A. Ferreirim, and F. Montagna. Basic hoops: an algebraic study of continuous t-norms. Studia Logica, to appear, 2003.

[AM03] P. Aglianò and F. Montagna. Varieties of BL-algebras I: general properties. *J. Pure Appl. Algebra*, 181:105–129, 2003.

[Bar77] Jon Barwise, editor. *Handbook of Mathematical Logic.* North Holland, 1977.

[Běh02] Libor Běhounek. Teorie množin v Gödelově logice. Master's thesis, Filozofická fakulta Univerzity Karlovy v Praze, 2002.

[BF00] W. J. Blok and I. M. A. Ferreirim. On the structure of hoops. *Algebra univers.*, 43:233–257, 2000.

[BHKŠ98] Matthias Baaz, Petr Hájek, Jan Krajíček, and David Švejda. Embedding logics into product logic. *Studia Logica*, 61:35–47, 1998.

[BHMV02] Matthias Baaz, Petr Hájek, Franco Montagna, and Helmut Veith. Complexity of t-tautologies. *Ann. Pure Appl. Logic*, 113(1-3):2–11, 2002.

[BP94] W. J. Blok and D. Pigozzi. On the structure of varieties with equationally definable principal congruences III. *Algebra Univers.*, 32:545–608, 1994.

[BŠ86] Bohuslav Balcar and Petr Štěpánek. *Teorie množin.* Academia, 1986.

[CDM00] Roberto L.O. Cignoli, Itala M. L. D'Ottaviano, and Daniele Mundici. *Algebraic Foundations of Many-valued Reasoning.* Kluwer Academic Publishers, 2000.

[CEGT00] Roberto Cignoli, Francesc Esteva, Lluís Godo, and Antoni Torrens. Basic Fuzzy Logic is the logic of continuous t-norms and their residua. *Soft Comput.*, 4:106–112, 2000.

[Cha58] C. C. Chang. Algebraic analysis of many-valued logics. *Trans. AMS*, 88:467–490, 1958.

[Cha59] C. C. Chang. A new proof ot the completeness of the Łukasiewicz axioms. *Trans. AMS*, 93:74–90, 1959.

[Cha63] C. C. Chang. The axiom of comprehension in infinite valued logic. *Math. Scand.*, 13:9–30, 1963.

[Cin01a] Petr Cintula. About axiomatic systems of product fuzzy logic. *Soft Comput.*, 5:243–244, 2001.

[Cin01b] Petr Cintula. An alternative approach to ŁΠ logic. *Neural Network World*, 11:561–572, 2001.

[dN91] Antonio di Nola. MV-algebras in the treatment of uncertainty. In *Fuzzy Logic: Proceedings of the International Congress IFSA*, pages 123–131, Bruxelles, 1991. Kluwer Academic Publishers.

[dNL99] Antonio di Nola and Anna Lettieri. Equational characterization of all varieties of MV algebras. *J. of Algebra*, 22:463–474, 1999.

[EGH96] Francesc Esteva, Lluís Godo, and Petr Hájek. A complete many-valued logic with product conjunction. *Arch. Math. Logic*, 35:191–208, 1996.

[EGHM03] Francesc Esteva, Lluís Godo, Petr Hájek, and Franco Montagna. Hoops and fuzzy logic. Journal of Logic and Computation, to appear, 2003.

[EGHN00] Francesc Esteva, Lluís Godo, Petr Hájek, and Mirko Navara. Residuated fuzzy logic with an involutive negation. *Arch. Math. Logic*, 39:103–124, 2000.

[EGM01] Francesc Esteva, Lluís Godo, and Franco Montagna. The ŁΠ and ŁΠ1/2 logics: two complete fuzzy systems joining Łukasiewicz and product logics. *Arch. Math. Logic*, 40:39–67, 2001.

[EGM03] Francesc Esteva, Lluís Godo, and Franco Montagna. Equational characterization of the subvarieties of BL generated by t-norm algebras. preprint, 2003.

[Fen64] Jens Erik Fenstad. On the consistency of the axiom of comprehension in the Łukasiewicz infinite valued logic. *Math. Scand.*, 14:65–74, 1964.

[Fer92] Isabel Maria André Ferreirim. *On varieties and quasivarieties of hoops and their reducts*. PhD thesis, University of Illinois at Chicago, 1992.

[GG94] Dov M. Gabbay and F. Guenthner, editors. *Handbook of Philosophical Logic*. Kluwer Academic Publishers, 1994.

[Göd32] Kurt Gödel. Zum Intuizionistischen Aussagenkalkül. *Anzeiger Akademie der Wissenschaften Wien*, 69:65–66, 1932.

[Got76a] Siegfried Gottwald. Untersuchungen zur mehrwertigen Mengenlehre I. *Math. Nachrichten*, 72:297–303, 1976.

[Got76b] Siegfried Gottwald. Untersuchungen zur mehrwertigen Mengenlehre II. *Math. Nachrichten*, 74:329–336, 1976.

[Got77] Siegfried Gottwald. Untersuchungen zur mehrwertigen Mengenlehre III. *Math. Nachrichten*, 79:207–217, 1977.

[Got81] Siegfried Gottwald. Fuzzy Mengen und ihre Anwendungen. Ein Überblick. *Elektronische Informationsverarbeitung und Kybernetik*, 17:207–235, 1981.

[Got84] Siegfried Gottwald. Fuzzy set theory: some aspects of the early development. In H. J. Skala, S. Termini, and E. Trillas, editors, *Aspect of Vagueness*, pages 13–29. D. Reidel Publishing Company, 1984.

[Got88] Siegfried Gottwald. *Mehrwertige Logik*. Akademie-Verlag, Berlin, 1988.

[Grä79a] G. Grätzer. *Universal Algebra*. Springer, Berlin, Heidelberg, New York, 1979.

[Gra79b] R. J. Grayson. Heyting-valued models for intuitionistic set theory. *Lecture Notes Math.*, 753:402–414, 1979.

[Gri99] V. N. Grishin. O vese aksiomy svertyvania v teorii, osnovannoj na logike bez sokrashtchenij. *Matematicheskije zamjetki*, 66(5):643–652, 1999.

[Háj98a] Petr Hájek. Basic fuzzy logic and BL-algebras. *Soft Comput.*, 2:124–128, 1998.

[Háj98b] Petr Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 1998.

[Háj00] Petr Hájek. Function symbols in fuzzy logic. In *Proceedings of the East-West Fuzzy Colloquium*, pages 2–8, Zittau/Görlitz, 2000. IPM.

[Háj01] Petr Hájek. What we cannot have in fuzzy set theory. unpublished manuscript, 2001.

[Han01] Zuzana Haniková. Standard algebras for fuzzy propositional calculi. *Fuzzy Sets and Systems*, 124(3):309–320, 2001.

[Han02] Zuzana Haniková. A note on the complexity of propositional tautologies of individual t-algebras. *Neural Network World*, 12(5):453–460, 2002.

[Han03] Zuzana Haniková. On the complexity of propositional logics with an involutive negation. In *Proceedings of Eusflat 2003*, pages 636–639, Zittau/Görlitz, 2003. EUSFLAT.

[HC03] Rostislav Horčík and Petr Cintula. Product Łukasiewicz logic. to appear in Arch. Math. Logic, 2003.

[HH01] Petr Hájek and Zuzana Haniková. A set theory within fuzzy logic. In *Proceedings of the ISMVL 2001*, pages 319–323, Warsaw, 2001. IEEE Computer Society.

[HH03] Petr Hájek and Zuzana Haniková. A Development of Set Theory in Fuzzy Logic. In Melvin Fitting and Ewa Orłowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, Studies in Fuzziness and Soft Computing, pages 273–285. Physica-Verlag, 2003.

[Hor95] A. S. Hornby. *Oxford Advanced Learner's Dictionary of Current English*. Oxford University Press, 1995.

[Jec02] Thomas Jech. *Set Theory*. Springer-Verlag, 2002.

[Kla65] D. Klaua. Über einen Ansatz zur mehrwertigen Mengenlehre. *Monatsb. Deutsch. Akad. Wiss. Berlin*, 7:859–867, 1965.

[Kla66] D. Klaua. Über einen zweiten Ansatz zur mehrwertigen Mengenlehre. *Monatsb. Deutsch. Akad. Wiss. Berlin*, 8:782–802, 1966.

[Kla67] D. Klaua. Ein Ansatz zur mehrwertigen Mengenlehre. *Math. Nachr.*, 33:273–296, 1967.

[KMP00] Erich Peter Klement, Radko Mesiar, and Endre Pap. *Triangular norms*. Kluwer Academic Publishers, 2000.

[MNP00] Jiří Močkoř, Vilém Novák, and Irina Perfilieva. *Mathematical Principles of Fuzzy Logic*. Kluwer Academic Publishers, 2000.

[MS57] P. S. Mostert and A. L. Shields. On the structure of semigroups on a compact manifold with boundary. *Annals of Math.*, 65:117–143, 1957.

[Mun87] D. Mundici. Satisfiability in many-valued sentential logic is NP-complete. *Theor. Comput. Sci.*, 52:145–153, 1987.

[Pap94] Christos H. Papadimitriou. *Computational Complexity*. Addison-Wesley Publishing Company, 1994.

[Pow75] William C. Powell. Extending Gödel's negative interpretation to ZF. *J. Symb. Logic*, 40:221–229, 1975.

[Shi99] M. Shirahata. Phase-valued models of linear set theory. preprint, 1999.

[Ske10] Walter W. Skeat. *An Etymological Dictionary of the English Language*. Clarendon Press, Oxford, 1910.

[Sko57] Thoralf Skolem. Bemerkungen zum Komprehensionsaxiom. *Z. Math. Logik Grundlagen Math.*, 3:1–17, 1957.

[Soc01] Antonín Sochor. *Klasická matematická logika*. Karolinum, 2001.

[Šve02] Vítězslav Švejdar. *Logika: neúplnost, složitost a nutnost.* Academia, 2002.

[Tit99] Satoko Titani. A lattice-valued set theory. *Arch. Math. Logic*, 38:395–421, 1999.

[TT84] Gaisi Takeuti and Satoko Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *J. Symb. Logic*, 49:851–866, 1984.

[TT92] Gaisi Takeuti and Satoko Titani. Fuzzy logic and fuzzy set theory. *Arch. Math. Logic*, 32:1–32, 1992.

[Whi79] Richard B. White. The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz. *J. Philosophical Logic*, 8:509–534, 1979.

[Zad65] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.