Axiomatizations of standard algebras for fuzzy PC by use of truth constants

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Abstract

This paper deals with fuzzy propositional calculi (PC) given by continuous t-norms. The logical approach is that of [3]; enlarging upon that, we try to axiomatize a logic given by a particular t-norm, namely, an ordered sum of a copy of Lukasiewicz's t-norm and a copy of product t-norm. We enrich the propositional language by a truth constant denoting the one delimiting idempotent; then the axiomatization can be obtained in a uniform way.

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Notation and conventions

Closed intervals (mostly of reals) are denoted by square brackets, like $[0,1]$; open intervals by round brackets, like $(0,1)$.

By a 'formula' we mean a propositional formula in the corresponding language. Formulas are always denoted with Greek letters.

Precedence of connectives in decreasing order: negation, conjunctions, disjunction, implication, equivalence. We sometimes omit parentheses for readability's sake. Thus $\neg\varphi \,\&\, \psi \to \chi$ should be parsed as $((\neg\varphi) \,\&\, \psi) \to \chi$.

We say at times that 'intervals are isomorphic', which should be read as 'isomorphic w.r.t. all operations defined'; or that 't-norms are isomorphic on $[x,y]$', which in turn means that the underlying interval(s) are isomorphic w.r.t. the t-norms; we refrain from such abbreviations where confusion could arise.

1. A Reader's Digest of Fuzzy PC

This section gives a very brief overview of basic notions and statements of fuzzy propositional calculus, following the approach of [3]. It is intended as a referential background to the results in the following section; those who wish to pursue the matter in depth are referred to [3] as a good starting point.

1.1. t-norms and their residua

We define truth functions for the basic propositional connectives of fuzzy PC, namely &, $\rightarrow$, and the constant $\overline{0}$. We start with the strong conjunction &; note that the choice of the truth function for & uniquely determines the whole algebra of truth values.

1.1.1 Definition. *A t-norm is a binary operation $*$ on $[0,1]$, satisfying the following conditions:*

(i) *$*$ is commutative and associative*
(ii) *$*$ is non-decreasing in both arguments*
(iii) *$1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.*

We shall only be interested in *continuous* t-norms[1] (i.e., continuous mappings of $[0,1]^2$ onto $[0,1]$) as possible truth functions for &.

For each continuous t-norm $*$ there is a unique operation $\Rightarrow$, defined (for $x, y \in [0,1]$) as $x \Rightarrow y = \max\{z; x * z \leq y\}$; this operation is called the *residuum* of the t-norm $*$, and is used as the truth function for the implication $\rightarrow$. Note that by the above definition, $x \Rightarrow y = 1$ iff $x \leq y$.

There are three outstanding examples of continuous t-norms (their importance is justified by a theorem included further on): Lukasiewicz's t-norm $*_{\mathrm{L}}$, Gödel's t-norm $*_{\mathrm{G}}$, and product t-norm $*_{\Pi}$; their definitions, including the respective residua for $x > y$, are listed below:

$$
\begin{array}{lll}
\mathrm{L}: & x * y = \max(0, x + y - 1) & x \Rightarrow y = 1 - x + y \\
\mathrm{G}: & x * y = \min(x, y) & x \Rightarrow y = y \\
\Pi: & x * y = x.y & x \Rightarrow y = \dfrac{y}{x}
\end{array}
$$

A continuous t-norm $*$ determines the algebra $[0,1]_* = ([0,1], 0, *, \Rightarrow)$; these structures are called *standard algebras* for fuzzy PC.

We introduce a theorem which characterizes all continuous t-norms and explains the importance of the three examples above. Let $*$ be a continuous t-norm. An element $x \in [0,1]_*$ is *idempotent* (w.r.t. $*$) iff $x * x = x$. The set of all idempotents of $*$ is a closed subset of $[0,1]$. Its complement is a union of countably many non-overlapping open intervals; denote this set of intervals $\mathcal{I}_\circ$. Let $\mathcal{I}$ be the set of closures of intervals in $\mathcal{I}_\circ$.

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[1] Should t-norms appear in this paper without the attribute 'continuous', we mean continuous t-norms.

1.1.2 Representation theorem.

(i) *For each* $[a,b] \in \mathcal{I}$, $*$ *on* $[a,b]$ *is isomorphic either to the product t-norm (on* $[0,1]$*) or to Łukasiewicz t-norm (on* $[0,1]$*).*

(ii) *If for* $x, y \in [0,1]$ *there is no* $[a,b] \in \mathcal{I}$ *such that* $x, y \in [a,b]$*, then* $x * y = \min(x,y)$*.*

To close this section we include a useful lemma.

1.1.3 Lemma. *For* $x, u, y \in [0,1]$*, for any t-norm* $*$*:*

(i) *if* $y \leq x$*, then* $\exists z \in [0,1]\,(y = z * x)$

(ii) $1 \Rightarrow y = y$

1.2. Basic logic and its extensions

The language and syntax of fuzzy PC are almost the same as in classical propositional logic: the language has in addition a truth constant $\overline{0}$ (which is *not* definable here) and the connective $\wedge$ (or &). Formulas, proofs etc. are defined as usual. Propositional connectives of fuzzy PC are defined from &, $\rightarrow$, and $\overline{0}$ as follows:

$$
\begin{array}{rcl}
\varphi \wedge \psi & \text{is} & \varphi \,\&\, (\varphi \rightarrow \psi) \\
\varphi \vee \psi & \text{is} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\
\neg\varphi & \text{is} & \varphi \rightarrow \overline{0} \\
\varphi \equiv \psi & \text{is} & (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)
\end{array}
$$

Fix a continuous t-norm $*$; this determines uniquely a standard algebra $[0,1]_*$ and the corresponding propositional calculus PC($*$), where evaluations of propositional variables extend to formulas as follows:

$$
\begin{aligned}
e(\overline{0}) &= 0 \\
e(\varphi \,\&\, \psi) &= e(\varphi) * e(\psi) \\
e(\varphi \rightarrow \psi) &= e(\varphi) \Rightarrow e(\psi)
\end{aligned}
$$

The shortcuts for evaluating other propositional connectives are the *precomplement* (the truth function of negation $\neg$), defined as $-x = x \Rightarrow 0$, and the operations $\min(x,y)$ and $\max(x,y)$ (truth functions of $\wedge$ and $\vee$).

A formula φ is a 1-tautology of a standard algebra $[0,1]_*$ iff it evaluates to 1 under any evaluation in $[0,1]_*$. φ is a *t-tautology* iff it is a 1-tautology of $[0,1]_*$ for any continuous t-norm $*$. Note that formulas translate to terms, i.e., for each formula φ there is a term τ in the language of standard algebras, such that for every standard algebra $[0,1]_*$ and every evaluation e in this algebra, τ evaluates to 1 under e iff $[0,1]_* \models e(\varphi)$.

Now we define the *basic logic*; its axioms are t-tautologies.

1.2.1 Definition. *The following formulas are the axioms of the basic logic (denoted* BL*).*

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

(A2) $(\varphi \,\&\, \psi) \rightarrow \varphi$

(A3) $(\varphi \,\&\, \psi) \rightarrow (\psi \,\&\, \varphi)$

(A4) $(\varphi \,\&\, (\varphi \rightarrow \psi)) \rightarrow (\psi \,\&\, (\psi \rightarrow \varphi))$

(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \,\&\, \psi) \rightarrow \chi)$

(A5b) $((\varphi \,\&\, \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\overline{0} \to \varphi$

The inference rule of BL *is modus ponens.*

For example and for convenience, we list formulas provable in BL that will be used further in this work; for proofs and/or more examples of BL-provable formulas, see [3].

1.2.2 Lemma. BL *proves these formulas:*

(i) $\varphi \to (\psi \to \varphi)$
(ii) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(iii) $\varphi \to (\psi \to (\varphi \,\&\, \psi))$
(iv) $(\varphi \,\&\, (\varphi \to \psi)) \to \psi$
(v) $((\alpha \to \beta) \,\&\, (\gamma \to \delta)) \to ((\alpha \,\&\, \gamma) \to (\beta \,\&\, \delta))$
(vi) $\varphi \wedge \psi \to \varphi$, $\varphi \wedge \psi \to \psi$, $\varphi \,\&\, \psi \to \varphi \wedge \psi$
(vii) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
(viii) $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$, $(\varphi \vee \psi) \to (\psi \vee \varphi)$
(ix) $(\varphi \,\&\, (\psi \wedge \chi)) \equiv ((\varphi \,\&\, \psi) \wedge (\varphi \,\&\, \chi))$

BL may be extended by additional axioms (or, schemas of axioms); below are listed those that yield the calculi given by Łukasiewicz's, Gödel's, and product t-norm, respectively:

Lukasiewicz's logic: $$\neg\neg\varphi \to \varphi \tag{L}$$

Gödel's logic: $$\varphi \to (\varphi \,\&\, \varphi) \tag{G}$$

Product logic: $$\neg\neg\chi \to [((\varphi \,\&\, \chi) \to (\psi \,\&\, \chi)) \to (\varphi \to \psi)] \tag{Π1}$$
and $$\varphi \wedge \neg\varphi \to \overline{0} \tag{Π2}$$

1.3. BL-algebras and completeness theorems

In this section we define a general version of algebras of truth values for fuzzy PC (i.e., BL and its extensions). Standard algebras are a subclass of these.

1.3.1 Definition. *A residuated lattice is an algebra* $\mathbf{L} = (L, \cup, \cap, *, \Rightarrow, 0, 1)$ *with four binary operations and two constants such that*

(i) $(L, \cup, \cap, 0, 1)$ *is a lattice with largest element* 1 *and least element* 0 *(with respect to the lattice ordering* $\leq$*)*
(ii) $(L, *, 1)$ *is a commutative semigroup with unit element* 1*, i.e.,* $*$ *is commutative, associative,* $1 * x = x$ *for all* x
(iii) $*$ *and* $\Rightarrow$ *form an adjoint pair, i.e.,* $z \leq (x \Rightarrow y)$ *iff* $x * z \leq y$ *for all* $x, y, z \in L$.

A residuated lattice is a BL-algebra iff the following identities hold for all $x, y \in L$*:*

(i) $x \cap y = x * (x \Rightarrow y)$
(ii) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$

A BL-algebra is linearly ordered iff its lattice ordering is linear, i.e., for any x,$y \in L$, $x \cap y = x$ *or* $x \cap y = y$*. Linearly ordered BL-algebras are also called BL-chains.*

Note that the class of BL-algebras is a variety.

1.3.2 Lemma. *Let* $\mathbf{L} = (L, \cup, \cap, *, \Rightarrow, 0, 1)$ *be a BL-algebra and* $x, y, u \in L$. *Then*
(i) *if* $x \leq u \leq y$ *and* u *is idempotent then* $x * y = x$
(ii) *if* $x < u \leq y$ *and* u *is idempotent then* $y \Rightarrow x = x$

For proof, see [2].

Let $\mathbf{L}$ be a BL-algebra. An $\mathbf{L}$-evaluation of propositional variables is a mapping e, assigning to each propositional variable an element of L; evaluation of formulas is obtained using 0, $*$, and $\Rightarrow$ (in $\mathbf{L}$) to evaluate $\overline{0}$, &, and $\rightarrow$. A formula φ is an $\mathbf{L}$-tautology iff $e(\varphi) = 1$ for all $\mathbf{L}$-evaluations e. A formula is a BL-tautology iff it is an $\mathbf{L}$-tautology for all BL-algebras $\mathbf{L}$.

Let $\mathcal{C}$ be a schematic extension of BL (i.e., a set of propositional formulas, including the axioms of BL). A BL-algebra $\mathbf{L}$ is a $\mathcal{C}$-algebra iff any $\varphi \in \mathcal{C}$ is an $\mathbf{L}$-tautology, i.e., $e(\varphi) = 1$ for all $\mathbf{L}$-evaluations e. A formula is a $\mathcal{C}$-tautology iff it is an $\mathbf{L}$-tautology for all $\mathcal{C}$-algebras $\mathbf{L}$. A $\mathcal{C}$-chain is a linearly ordered $\mathcal{C}$-algebra.

1.3.3 Theorem. BL *is sound w.r.t. BL-algebras, i.e., if* φ *is provable in* BL, *it is an* $\mathbf{L}$*-tautology for each BL-algebra. Generally, if a schematic extension* $\mathcal{C}$ *proves* φ, *then* φ *is an* $\mathbf{L}$*-tautology for each* $\mathcal{C}$*-algebra* $\mathbf{L}$.

1.3.4 Theorem (Completeness). *The following three conditions are equivalent:*
(i) φ *is provable in* BL
(ii) *for each BL-algebra* $\mathbf{L}$, φ *is an* $\mathbf{L}$*-tautology*
(iii) *for each linearly ordered BL-algebra* $\mathbf{L}$, φ *is an* $\mathbf{L}$*-tautology.*

The proof of this statement also shows that a schematic extension $\mathcal{C}$ of BL proves φ iff φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-algebra $\mathbf{L}$ iff φ is an $\mathbf{L}$-tautology for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$.

This means that the three calculi Ł, G, and Π are complete w.r.t. the respective classes of algebras. In particular, BL-algebras $\mathbf{L}$ for which $\neg\neg\varphi \rightarrow \varphi$ is an $\mathbf{L}$-tautology are called MV-algebras (MV standing for "many-valued"). BL-algebras satisfying the axiom G are called regular Heyting algebras or simply G-algebras; BL-algebras satisfying the two axioms of Π are called product algebras or Π-algebras.

In the following we shall also need the notion of an ordered (or ordinal) sum of BL-chains.

1.3.5 Definition. *Let* $\mathcal{C}$ *be a linearly ordered set with least element* 0 *and largest element* 1. *For* $c \in \mathcal{C}$, *let* c^+ *be the upper neighbour of* c *in* $\mathcal{C}$ *if it exists, otherwise* $c^+ = c$. *For each* $c \in \mathcal{C}$, *let* A_c *be an isomorphic copy of a BL-chain with the least element* c *and the largest element* c^+. *Assume that the non-extremal elements of* $A_c; c \in \mathcal{C}$ *are not elements of any* $A_{c'}, c' \in \mathcal{C}, c \neq c'$.
The ordered sum $\bigoplus_{c\in\mathcal{C}} A_c$ *is defined as follows:*
(i) *the domain is* $\bigcup_{c\in\mathcal{C}} A_c$
(ii) *for* $x \in A_{c_1}$, $y \in A_{c_2}$ *define* $x \leq y$ *iff either* $c_1 = c_2$ *and* $x \leq_{c_1} y$, *or* $c_1 < c_2$, $c_1, c_2 \in \mathcal{C}$
(iii) $x * y = x *_c y$ *for* $x, y \in A_c, c \in \mathcal{C}$, *otherwise* $x * y = \min(x, y)$
(iv) $x \Rightarrow y = 1$ *iff* $x \leq y$
(v) $x \Rightarrow y = x \Rightarrow_c y$ *for* $x, y \in A_c$, $x > y$, $c \in \mathcal{C}$
(vi) $x \Rightarrow y = y$ *for* $x \in A_{c_1}$, $y \in A_{c_2}$, $c_1, c_2 \in \mathcal{C}$, *and* $c_1 > c_2$.

1.3.6 Lemma. *In the above notation,* $\mathbf{A} = \bigoplus_{c \in \mathcal{C}} A_c$ *is a BL-chain.*

For the following statement we need the notion of saturation. A precise definition would demand more terminology and will be found in [2]. Saturation means that there is an idempotent element delimiting (in the sense of ordering) each two consecutive copies of BL-chains in the sum (i.e., some of the elements of $\mathcal{C}$ may be dropped and the result is still a BL-chain). For each BL-chain there is a (unique) saturation with the same set of tautologies. We shall only deal with saturated BL-chains.

1.3.7 Theorem. *Each saturated BL-chain is an ordered sum of isomorphic copies of MV-chains, G-chains, and Π-chains.*

For proof, see [1].

1.4. Standard completeness

It has long been an open problem whether BL is complete w.r.t. the class of standard algebras. In [2], this problem is reduced to the question of whether two additional axioms B1 and B2 are redundant (provable in BL). An affirmative answer to this question was recently presented in [1].

1.4.1 Theorem. BL *is complete w.r.t. standard algebras; i.e., a formula* φ *is a theorem of* BL *iff it is a t-tautology.*

1.4.2 Theorem. Ł *is complete w.r.t.* $[0,1]_{Ł}$*,* G *is complete w.r.t.* $[0,1]_{G}$*,* Π *is complete w.r.t.* $[0,1]_{\Pi}$.

The standard completeness proofs are carried out in slightly different fashion in each case. In all three cases the crucial point is the statement that each finite subset of a linearly ordered MV-algebra (G-algebra, Π-algebra respectively) can be locally embedded into $[0,1]_{Ł}$ ($[0,1]_{G}$, $[0,1]_{\Pi}$ respectively). For example, let $\mathbf{L}$ be a linearly ordered MV-algebra, and S be a finite subset of L. Then there is a partial isomorphism of S into $[0,1]_{Ł}$, i.e., a 1-1 mapping f s.t. for all $x, y, z \in S$,

$$\begin{aligned} z = x * y \quad &\text{iff} \quad f(z) = f(x) * f(y) \\ z = x \Rightarrow y \quad &\text{iff} \quad f(z) = f(x) \Rightarrow f(y) \\ x \leq y \quad &\text{iff} \quad f(x) \leq f(y) \end{aligned}$$

The same result can be obtained (through a different construction) for G and Π.

Therefore, if a formula φ is not provable in Ł and consequently is not an $\mathbf{L}$-tautology of an MV-chain $\mathbf{L}$, then it is not a tautology of $[0,1]_{Ł}$; similarly for G and Π.

2. Ł⊕Π-norms

This chapter investigates the logic given by one particular example of a continuous t-norm. A truth constant has been added to the propositional language to ease the task. Axiomatization of this logic without the additional constant is yet to be found; BL is not complete w.r.t. the standard algebra given by this t-norm, i.e., some schematic extension is needed.

2.1. Ł⊕Π-norms

Informally, an Ł⊕Π-norm is obtained by sticking together an isomorphic copy of the Łukasiewicz's t-norm and an isomorphic copy of the product t-norm on $[0,1]$, in the sense of the representation theorem for continuous t-norms, i.e., delimited with one idempotent element named h (for 'half'). We require that $h \neq 0$ and $h \neq 1$; it does not matter which element plays the role of h, as long as it is neither 0 nor 1.

2.1.1 Definition. *A continuous t-norm* $*_{\text{Ł}\oplus\Pi}$ *is an* Ł⊕Π*-norm iff there is an element* $h \in (0,1)$ *such that* $h *_{\text{Ł}\oplus\Pi} h = h$*, and bijective mappings* f *and* g*,*

$$f : [0,1] \longrightarrow [0,h],\ g : [0,1] \longrightarrow [h,1],$$

such that for $a, b \in [0,1]$*,*

$$f(a *_{\text{Ł}} b) = f(a) *_{\text{Ł}\oplus\Pi} f(b),\ g(a *_{\Pi} b) = g(a) *_{\text{Ł}\oplus\Pi} g(b)$$

2.1.2 Lemma. *The mapping* f *from the previous definition is increasing and continuous. Moreover, if* $0 \leq x < y \leq h$*, then also* $f(y \Rightarrow x) = f(y) \Rightarrow f(x)$*. Similarly for* g*.*

We prove a more general version of this statement:

2.1.3 Theorem. *Let* $*_a$*,* $*_b$ *be two continuous t-norms, let* $0 \leq a_1 < a_2 \leq 1$*, where* a_1 *and* a_2 *are idempotents of* $*_a$*, let* $0 \leq b_1 < b_2 \leq 1$ *and let* f *be a bijective mapping of* $[a_1, a_2]$ *onto* $[b_1, b_2]$*, such that* $(\forall x, y \in [a_1, a_2])\, f(x *_a y) = f(x) *_b f(y)$*. Then*

(i) *f is increasing (and hence continuous) on $[a_1, a_2]$, and*
(ii) *$(\forall x, y \in [a_1, a_2])$, $x > y$ implies $f(x \Rightarrow_a y) = f(x) \Rightarrow_b f(y)$.*

Proof.
(i) Suppose $x < y$, $x, y \in [a_1, a_2]$. Then, by 1.1.3 (i), $\exists z(z *_a y = x)$. In this case $a_1 \leq z \leq a_2$, because $a_1 *_a y = a_1$ and $a_2 *_a y = y$. Thus $f(z) *_b f(y) = f(x)$ and therefore $f(x) \leq f(y)$. Because $f(x) = f(y)$ is impossible, $f(x) < f(y)$.
(ii) We want to prove $f(x \Rightarrow_a y) = f(x) \Rightarrow_b f(y)$ for $x > y$, $x, y \in [a_1, a_2]$.
By definition, $f(x) \Rightarrow_b f(y) = \max\{z; z *_b f(x) \leq f(y)\}$. Since f is increasing, this is equal to $f(\max\{u; u *_a x \leq y\})$, i.e., $f(x \Rightarrow_a y)$. $\mathcal{QED}$

2.1.4 Corollary. *Let* $*_1$*,* $*_2$ *be two* Ł⊕Π*-norms. Then the two standard algebras* $[0,1]_{*_1}$ *and* $[0,1]_{*_2}$ *are isomorphic.*

In a fixed language, all isomorphic structures have the same set of tautologies, therefore in the following we pick any representative from the (uncountable) class of Ł⊕Π-norms.

2.2. Axioms of Ł⊕Π

We assemble propositional formulas which determine, as closely as possible, the particulars of the Ł⊕Π-norm defined above. For a revision, the formulas should guarantee the existence of a non-extremal idempotent, should specify that $*$ on the "bottom" behaves like Łukasiewicz's t-norm, and on the "top" like product t-norm. In the first attempt, we include all promising formulas, disregarding independence claims. Refinements will follow in 2.5.

We introduce a new truth constant $\overline{h}$, and add an axiom

$$\overline{h} \,\&\, \overline{h} \equiv \overline{h} \tag{Id}$$

This states the existence of an idempotent element h. Adding another axiom $\neg\neg\overline{h}$ ascertains that $h \neq 0$. Whenever h is idempotent (as secured by (Id) in this case), $e(\neg\neg\overline{h}) = 1$ iff $e(\overline{h}) \neq 0$ for any t-norm $*$ and any evaluation e. There is however, as shown in 2.3.2, no formula expressing $h \neq 1$.

The rest of intended axioms will be "translated" axioms of Łukasiewicz and product logics (including the axioms of BL). We introduce two functions $\flat$ and $\sharp$, operating on formulas, defined as follows[1]:

$$\begin{aligned}
\overline{0}^\flat &= \overline{0} & \overline{0}^\sharp &= \overline{h} \\
\overline{1}^\flat &= \overline{h} & \overline{1}^\sharp &= \overline{1} \\
p^\flat &= p \wedge \overline{h} & p^\sharp &= p \vee \overline{h} \\
(\varphi \& \psi)^\flat &= \varphi^\flat \& \psi^\flat & (\varphi \& \psi)^\sharp &= \varphi^\sharp \& \psi^\sharp \\
(\varphi \to \psi)^\flat &= (\varphi^\flat \to \psi^\flat) \wedge \overline{h} & (\varphi \to \psi)^\sharp &= \varphi^\sharp \to \psi^\sharp
\end{aligned}$$

2.2.1 Theorem. *If $\flat$ and $\sharp$ are as above, φ and ψ are formulas, then:*

(i) *$\varphi \equiv \psi$ is a tautology of $[0,1]_{\text{Ł}}$ iff $\varphi^\flat \equiv \psi^\flat$ is a tautology of $[0,1]_{\text{Ł}\oplus\Pi}$*

(ii) *$\varphi \equiv \psi$ is a tautology of $[0,1]_\Pi$ iff $\varphi^\sharp \equiv \psi^\sharp$ is a tautology of $[0,1]_{\text{Ł}\oplus\Pi}$*

Proof. In $[0,1]_{\text{Ł}\oplus\Pi}$, define new operations $1^h = h$ and $a \Rightarrow^h b = \min\{a \Rightarrow_{\text{Ł}\oplus\Pi} b, h\}$. Then $[0,h]$ is closed with respect to $*_{\text{Ł}\oplus\Pi}$ and $\Rightarrow^h$, and the structure

$$\text{Ł}^\flat = ([0,h], 0, 1^h, *_{\text{Ł}\oplus\Pi}, \Rightarrow^h)$$

is isomorphic to $[0,1]_{\text{Ł}}$ using lemma 2.1.2; denote this isomorphism f. Therefore, if φ and ψ are equivalent under an evaluation e in $[0,1]_{\text{Ł}}$, they are equivalent under an evaluation e' in $\text{Ł}^\flat$, where $e'(p_i) = f(e(p_i))$ for each propositional variable in φ or ψ, and vice versa.

By induction on φ one proves that $\varphi^\flat(a_1, \ldots, a_n)$ evaluated in $[0,1]_{\text{Ł}\oplus\Pi}$ yields the same element as $\varphi(\min(a_1,h), \ldots, \min(a_n,h))$ in $\text{Ł}^\flat$.

The case of $\sharp$ is analogous; define $0^h = h$, then the structure

$$\Pi^\sharp = ([h,1], 0^h, 1, *_{\text{Ł}\oplus\Pi}, \Rightarrow_{\text{Ł}\oplus\Pi})$$

is isomorphic to $[0,1]_\Pi$. $\mathcal{QED}$

[1] though $\overline{1}$ is definable in BL, we include it for clarity's sake

In particular, if φ is a tautology of $[0,1]_{\text{Ł}}$, then $\varphi^\flat \equiv \overline{h}$ is a tautology of $[0,1]_{\text{Ł}\oplus\Pi}$. Let $\text{BL}^\flat$ be the set of formulas $\{\varphi^\flat \equiv \overline{h};\ \varphi$ is an axiom of BL$\}$. Applying $\flat$ to (Ł), one gets

$$\left(\left[\left(\left[\left(\varphi^\flat \to \overline{0}\right) \wedge \overline{h}\,\right] \to \overline{0}\right) \wedge \overline{h}\,\right] \to \varphi^\flat\right) \wedge \overline{h}$$

which can be simplified using negation; the desired axiom is

$$\left[\,\left(\left[\neg\left[\neg\varphi^\flat \wedge \overline{h}\,\right] \wedge \overline{h}\,\right] \to \varphi^\flat\right) \wedge \overline{h}\,\right] \equiv \overline{h} \qquad (\text{Ł}^\flat)$$

Similarly, $\text{BL}^\sharp$ are the axioms of BL translated using $\sharp$, and $\sharp$ applied to (Π1) and (Π2) in turn yields

$$((\chi^\sharp \to \overline{h}) \to \overline{h}) \to ((\varphi^\sharp \mathbin{\&} \chi^\sharp \to \psi^\sharp \mathbin{\&} \chi^\sharp) \to (\varphi^\sharp \to \psi^\sharp))$$
$$(\varphi^\sharp \wedge (\varphi^\sharp \to \overline{h})) \to \overline{h}$$

respectively; these formulas will be denoted $\Pi 1^\sharp$ and $\Pi 2^\sharp$ respectively.

2.2.2 Definition. *The set of axioms of the logic* Ł$\oplus\Pi$ *is as follows:*

$$\text{Ł} \oplus \Pi = \text{BL} \cup \{\overline{h} \mathbin{\&} \overline{h} \equiv \overline{h}\} \cup \{\neg\neg\overline{h}\} \cup \text{BL}^\flat \cup \text{BL}^\sharp \cup \{\text{Ł}^\flat\} \cup \{\Pi 1^\sharp\} \cup \{\Pi 2^\sharp\}$$

From theorem 2.2.1, and considering that h is fixed as a nonzero element of $[0,1]$, we immediately obtain:

2.2.3 Lemma. *The axioms of* Ł$\oplus\Pi$ *are sound with respect to any* $[0,1]_{\text{Ł}\oplus\Pi}$.

2.3. Ł$\oplus\Pi$-algebras

2.3.1 Definition. *An* Ł$\oplus\Pi$*-algebra is a BL-algebra with an additional constant element h, satisfying all the axioms of* Ł$\oplus\Pi$.

Ł$\oplus\Pi$-algebras are defined by a set of propositional formulas and consequently form a variety.

By checking the proof of completeness theorem for BL and its schematic extensions, we obtain completeness of Ł$\oplus\Pi$ w.r.t. Ł$\oplus\Pi$-algebras: a propositional formula is provable in Ł$\oplus\Pi$ iff it is a tautology of all Ł$\oplus\Pi$-algebras iff it is a tautology of all linearly ordered Ł$\oplus\Pi$-algebras.

The aim of the following section will be to prove completeness with respect to $[0,1]_{\text{Ł}\oplus\Pi}$. Note that any *MV*-algebra is an Ł$\oplus\Pi$-algebra with $h = 1$. (Due to the axiom $\neg\neg\overline{h}$, the only Ł$\oplus\Pi$-algebra satisfying $h = 0$ is the trivial one-element algebra.) We cannot, however, prevent h from being equal to 1:

2.3.2 Lemma. *The condition $h \neq 1$ is not expressible by a propositional formula (in a system extending* $\text{BL} \cup \{\text{Id}\}$*).*

Proof. The proof is based on the observation that for any linearly ordered algebra (in our language), satisfying BL and (Id), there is a homomorphism sending all elements of $[h,1]$ to 1.
Add a constant h to the language of standard algebras and suppose there is a formula φ that holds in a standard algebra iff $h * h = h$ and $h \neq 1$; let $\mathcal{A}$ be the class of all

standard algebras where φ holds. Then any $[0,1]_{\text{Ł}\oplus\Pi}$ is in $\mathcal{A}$; pick one standard Ł$\oplus\Pi$-algebra A. In $[0,1]_{\text{Ł}}$, set $h = 1$. Define a mapping $f : A \longrightarrow [0,1]_{\text{Ł}}$, sending $[0,h]$ in A isomorphically to $[0,1]$ (by definition of Ł$\oplus\Pi$-norms such isomorphism always exists), and all elements of $[h,1]$ to 1. Then f is a homomorphism: if $x \leq h \leq y$, we get $f(x * y) = f(\min(x,y)) = f(x) = f(x) *_{\text{Ł}} 1 = f(x) *_{\text{Ł}} f(y)$; as to $\Rightarrow$, if $x < h < y$, then $f(y \Rightarrow x) = f(x) = 1 \Rightarrow_{\text{Ł}} f(x) = f(y) \Rightarrow_{\text{Ł}} f(x)$; if $h \leq x \leq y$, then $f(y \Rightarrow x) \geq f(x) = 1$, and $f(y) \Rightarrow_{\text{Ł}} f(x) = 1 \Rightarrow_{\text{Ł}} 1 = 1$. Since homomorphisms preserve validity of formulas, φ must hold in $[0,1]_{\text{Ł}}$. But in $[0,1]_{\text{Ł}}$, $h = 1$; this is contradictory, and therefore, no such φ exists. $\mathcal{QED}$

This means that all Ł$\oplus\Pi$-tautologies not containing $\overline{h}$ are also MV-tautologies; moreover, all Ł$\oplus\Pi$-tautologies without $\overline{h}$ are also Π-tautologies; this simple case is easily verified by observing that in any (linearly ordered) Ł$\oplus\Pi$-algebra, the subalgebra $\{0\} \cup (h,1]$ is a linearly ordered Π-algebra). By the completeness theorem therefore, for any formula φ not containing $\overline{h}$, if Ł$\oplus\Pi\vdash \varphi$ then Ł $\vdash \varphi$ and $\Pi \vdash \varphi$. The same is true for standard algebras: any tautology of $[0,1]_{\text{Ł}\oplus\Pi}$ not containing $\overline{h}$ is a tautology of $[0,1]_{\text{Ł}}$ and of $[0,1]_{\Pi}$.

2.3.3 Lemma. *A BL-chain with a nonzero idempotent h is an* Ł$\oplus\Pi$*-algebra iff it is either trivial, or it is an ordered sum of a non-trivial MV-chain and a (possibly trivial) Π-chain.*

Proof. It is easy to check that in a sum of a non-trivial MV-chain and a Π-chain, denoting the delimiting idempotent h, all the axioms of Ł$\oplus\Pi$ hold.
To prove the left-to-right implication, let A be a non-trivial BL-chain with a nonzero idempotent h, such that the axioms of Ł$\oplus\Pi$ hold in it. We know that the operation $-$, corresponding to negation, applied to any nonzero element of A except the elements of a potential initial MV-segment, yields 0. Suppose there is an element $a \in A; a < h$ such that $-a = 0$. Consider the axiom Ł$^\flat$ for a propositional variable p, i.e., $\big[\,([\neg[\neg(p \wedge \overline{h}) \wedge \overline{h}\,] \wedge \overline{h}\,] \to (p \wedge \overline{h})) \wedge \overline{h}\,\big] \equiv \overline{h}$. If $e(p) = a$, then all the left-hand side of the equivalence is p, while the right side is $h > p$, and the axiom does not hold; therefore no such element can exist. Consequently, $[0,h]$ in A is a copy of an MV-chain.
By an equally easy consideration, the translation $\Pi 1^\sharp$ secures that the segment $[h,1]$ is either an MV-chain or a Π-chain (it is violated by any idempotent element in $(h,1)$). The translation $\Pi 2^\sharp$ excludes the MV-chain, because if the (only) variable in it is evaluated by an element a sufficiently close to h in such an MV-chain, then both a and $a \Rightarrow h$ are greater than h. $\mathcal{QED}$

2.4. Standard completeness for Ł$\oplus\Pi$

We aim now at proving the set of axioms Ł$\oplus\Pi$ to be complete with respect to any $[0,1]_{\text{Ł}\oplus\Pi}$. Our definition of an Ł$\oplus\Pi$-norm does not allow h to be 0 or 1; this is necessary, since, if $h = 1$, we get Łukasiewicz t-norm—but Ł$\oplus\Pi$ is obviously not complete with respect to the standard algebra given by Łukasiewicz t-norm ($\neg\neg\varphi \to \varphi$ is not provable in Ł$\oplus\Pi$).

Thanks to the completeness theorem (for Ł$\oplus\Pi$-algebras), the question whether Ł$\oplus\Pi$ is complete with $[0,1]_{\text{Ł}\oplus\Pi}$ can be reformulated thus: if a formula is a tautology for all $[0,1]_{\text{Ł}\oplus\Pi}$, is it also an **L**-tautology for any Ł$\oplus\Pi$-chain **L**? (If so, then it is provable in Ł$\oplus\Pi$.) To be able to answer this question in the affirmative, we follow in the footsteps of [3], namely, we use local embeddings.

2.4.1 Theorem (Standard completeness for Ł⊕Π). *A formula φ is provable within Ł⊕Π iff it is a 1-tautology of any $[0,1]_{Ł\oplus\Pi}$.*

Proof. We have observed already that all algebras given by Ł⊕Π-norms are isomorphic, and hence have the same sets of tautologies. It is therefore sufficient to prove the theorem for any $[0,1]_{Ł\oplus\Pi}$.

Suppose there is a formula φ (in the language of Ł⊕Π) and a linearly ordered Ł⊕Π-algebra $\mathbf{L}$ such that $\mathbf{L} \not\models \varphi$, i.e., there is an $\mathbf{L}$-evaluation e such that $e(\varphi) \neq 1$ (hence φ is not provable in Ł⊕Π). Fix the algebra $\mathbf{L} = (L, 0, 1, h, \cup, \cap, *, \Rightarrow)$, the evaluation e in $\mathbf{L}$, and the formula φ (note that $\mathbf{L}$ is non-trivial). We shall define an evaluation e_s in $[0,1]_{Ł\oplus\Pi}$ such that $e_s(\varphi) \neq 1$, and hence $[0,1]_{Ł\oplus\Pi} \not\models \varphi$ either. Therefore, no formula which is not provable in Ł⊕Π can be valid in $[0,1]_{Ł\oplus\Pi}$.

Formulas translate to terms, i.e., for each formula φ there is a term σ in the language of Ł⊕Π-algebras such that, for any Ł⊕Π-algebra $\mathbf{L}$, $\mathbf{L} \models \varphi$ iff $\sigma = 1$ holds in $\mathbf{L}$.

Let $\sigma = 1$ be the identity corresponding to φ, and let $\{x_1, \ldots, x_n\}$ be all the variables in σ. Since the evaluation e is fixed, let $a_i = e(x_i)$ and $A = \{a_1, \ldots, a_n\}$ (in $\mathbf{L}$). Let $\overline{A} = \{a \in L; a = e(\tau), \tau \text{ subterm of } \sigma\}$ (i.e., $\overline{A}$ is the set of evaluations of all subterms of σ under e in $\mathbf{L}$).

We are looking for an isomorphic embedding α of $\overline{A}$ into $[0,1]_{Ł\oplus\Pi}$ such that $\alpha(\tau(x_1, \ldots, x_n)) = \tau(\alpha(x_1), \ldots, \alpha(x_n))$ for any subterm τ of σ. Once this is accomplished, we put $e_s(x_i) = \alpha(a_i)$; this defines the desired evaluation e_s in $[0,1]_{Ł\oplus\Pi}$.

It is shown in detail in [3] that any finite part of an MV-algebra (Π-algebra) can be isomorphically embedded in $[0,1]_Ł$ ($[0,1]_\Pi$). Since the Ł⊕Π-algebra is an ordered sum of an MV-algebra and a Π-algebra, the embedding may, with slight modifications, be performed separately for each part. The modifications are as follows: define $L^\flat = \{x \in L : x \leq h\}$ and $L^\sharp = \{x \in L : x \geq h\}$, $1^h = h$, $0^h = h$, and an operation $x \Rightarrow^h y = \min(x \Rightarrow y, h)$. Note that in $x, y \in L^\flat$ we have $x \Rightarrow y = 1$ iff $x \Rightarrow^h y = h$ (and in $x \in L^\sharp$, $-x = 0$ iff $-^h x = h$). Define $\mathbf{L}^\flat = (L^\flat, 0, 1^h, \cup, \cap, *, \Rightarrow^h)$ and $\mathbf{L}^\sharp = (L^\sharp, 0^h, 1, \cup, \cap, *, \Rightarrow)$. Then $\mathbf{L}^\flat$ is an MV-algebra and $\mathbf{L}^\sharp$ is a Π-algebra.

Note that if $h = 1$ in $\mathbf{L}$, we may trivially use the embedding of MV into $[0,1]_Ł$, and then send $[0,1]_Ł$ isomorphically to $[0,h) \cup 1$—a subalgebra in $[0,1]_{Ł\oplus\Pi}$, and composing these two mappings yields α. In the following therefore suppose that $h \neq 1$ in $\mathbf{L}$.

Let $\overline{A_1} = \{a \in \overline{A}; a < h\}$ and $\overline{A_2} = \{a \in \overline{A}; a \geq h\}$. Then $\overline{A_1}$ is a finite subset of $L^\flat$; by [3], an embedding $f_1 : \overline{A_1} \longrightarrow [0,1]_Ł$ exists so that for $a, b, c \in \overline{A_1}$, $a * b = c$ iff $f_1(a) *_Ł f_1(b) = f_1(c)$ and $a \Rightarrow^h b = c$ iff $f_1(a) \Rightarrow_Ł f_1(b) = f_1(c)$; similarly f_2 for $\overline{A_2}$. By fixing $[0,1]_{Ł\oplus\Pi}$, we obtain isomorphisms $g_1 : [0,1]_Ł \longrightarrow [0,h]_{Ł\oplus\Pi}$ and $g_1 : [0,1]_\Pi \longrightarrow [h,1]_{Ł\oplus\Pi}$. Define $\alpha(0) = 0$, $\alpha(h) = h$, $\alpha(1) = 1$, for $x \in \overline{A_1}$ put $\alpha(x) = g_1(f_1(x))$ and for $x \in \overline{A_2}$ put $\alpha(x) = g_2(f_2(x))$. For $x, y, z \in \overline{A}$ we claim $x * y = z$ iff $\alpha(x) *_{Ł\oplus\Pi} \alpha(y) = \alpha(z)$ and $x \Rightarrow y = z$ iff $\alpha(x) \Rightarrow_{Ł\oplus\Pi} \alpha(y) = \alpha(z)$.

First observe that for any $x < h \leq y$, $x, y \in \overline{A}$ we have $\alpha(x * y) = \alpha(x) = \min(\alpha(x), \alpha(y)) = \alpha(x) *_{Ł\oplus\Pi} \alpha(y)$ and $\alpha(y \Rightarrow x) = \alpha(x) = \alpha(y) \Rightarrow_{Ł\oplus\Pi} \alpha(x)$ (and of course $\alpha(x \Rightarrow y) = \alpha(1) = 1 = \alpha(x) \Rightarrow_{Ł\oplus\Pi} \alpha(y)$). By definition, α satisfies the above conditions if the arguments x, y, z are in $[0,h) \cap \overline{A}_1$, or if x, y, z are in $[h,1] \cap \overline{A}_2$. Other possibilities follow from basic statements about BL-algebras. $\mathcal{QED}$

2.5. Reducing the axioms of Ł⊕Π

Suppose we extend BL by the axiom of idempotence for $\overline{h}$ only; this extension will be denoted BL$^+$.

2.5.1 Definition. BL^+ *is the logic* $\mathrm{BL} \cup \{\overline{h} \,\&\, \overline{h} \equiv \overline{h}\}$.

We show that the whole of $(\mathrm{BL})^\flat$ and $(\mathrm{BL})^\sharp$ are provable in BL^+. This reduces the axiomatics of the logic $\text{Ł}\oplus\Pi$ so that it only contains $\flat$- and $\sharp$-translations of the axioms actually specifying the behaviour of the t-norm on the intervals $[0, h]$ and $[h, 1]$.

Again, by an instance of the completeness theorem (for schematic extensions), any BL^+-tautology is provable in BL^+. We give detailed syntactic proofs of the translated axioms; note, however, that we need not do so—it is in some cases easier to verify that the formula is a BL^+-tautology and contend with the fact that the proof exists.

We are going to make use of these simple auxiliary statements:

2.5.2 Lemma. *Let $\mathcal{C}$ be a schematic extension of* BL. *If $\mathcal{C} \vdash \varphi$ and if ψ is any formula (in the language of $\mathcal{C}$), then $\mathcal{C} \vdash \psi \to \varphi$.*

2.5.3 Lemma. *Let $\mathcal{C}$ be a schematic extension of* BL. *If $\mathcal{C} \vdash \varphi$ and $\mathcal{C} \vdash \psi$, then $\mathcal{C} \vdash \varphi \wedge \psi$.*

Proof. Combine 1.2.2 (iii) and 1.2.2 (vi) using the transitivity of implication (A1). $\mathcal{QED}$

2.5.4 Lemma. $\mathrm{BL} \vdash (\alpha \to \beta) \to ((\alpha \wedge \gamma) \to (\beta \wedge \gamma))$.

Proof. By (A5), the desired statement is BL-equivalent to $[(\alpha \to \beta) \,\&\, (\alpha \wedge \gamma)] \to (\beta \wedge \gamma)$. By 1.2.2 (vii), it is enough to prove the two implications

$\mathrm{BL} \vdash ((\alpha \to \beta) \,\&\, (\alpha \wedge \gamma)) \to \beta$ and
$\mathrm{BL} \vdash ((\alpha \to \beta) \,\&\, (\alpha \wedge \gamma)) \to \gamma$.

Take the first one first:

$\mathrm{BL} \vdash ((\alpha \to \beta) \,\&\, (\alpha \wedge \gamma)) \to ((\alpha \to \beta) \,\&\, \alpha) \to \alpha$ using again 1.2.2 (iv).

The second case is equally easy:

$\mathrm{BL} \vdash ((\alpha \to \beta) \,\&\, (\alpha \wedge \gamma)) \to (\alpha \wedge \gamma)$
$\mathrm{BL} \vdash ((\alpha \to \beta) \,\&\, (\alpha \wedge \gamma)) \to \gamma$ by transitivity and 1.2.2 (vi). $\mathcal{QED}$

Now let us prove the translated axioms. We shall start with $\sharp$ since it is the easier part. Note that $(\mathrm{A1})^\sharp$–$(\mathrm{A6})^\sharp$ are obtained from (A1)–(A6) by substitution. It only remains to prove (A7).

2.5.5 Theorem. $\mathrm{BL}^+ \vdash \overline{h} \to \varphi^\sharp$ *for any formula* φ.

Proof. By induction on the structure of φ.

Let φ be atomic: then $\varphi^\sharp = (\varphi \vee \overline{h})$, and

$\mathrm{BL}^+ \vdash \overline{h} \to (\varphi \vee \overline{h})$ by 1.2.2 (viii).

From now on suppose $\mathrm{BL}^+ \vdash \overline{h} \to \varphi^\sharp$, $\mathrm{BL}^+ \vdash \overline{h} \to \psi^\sharp$. Then by 1.2.2 (v),

$\mathrm{BL}^+ \vdash \overline{h} \,\&\, \overline{h} \to \varphi^\sharp \,\&\, \psi^\sharp$, and using (Id),
$\mathrm{BL}^+ \vdash \overline{h} \to \varphi^\sharp \,\&\, \psi^\sharp$.

As to the implication,

$\mathrm{BL}^+ \vdash (\overline{h} \to \psi^\sharp) \to (\varphi^\sharp \to (\overline{h} \to \psi^\sharp))$ by 1.2.2 (i)
$\mathrm{BL}^+ \vdash \varphi^\sharp \to (\overline{h} \to \psi^\sharp)$ by the induction hypothesis

and, using 2.2.2 (ii),

$\mathrm{BL}^+ \vdash \overline{h} \to (\varphi^\sharp \to \psi^\sharp)$ $\mathcal{QED}$

Now we are going to prove the $\flat$-translations.

2.5.6 Theorem. BL$^+$ *proves these formulas:*

(i) $\big[\big([(\varphi^\flat \to \psi^\flat) \wedge \overline{h}] \to [\big([(\psi^\flat \to \chi^\flat) \wedge \overline{h}] \to [(\varphi^\flat \to \chi^\flat) \wedge \overline{h}]\big) \wedge \overline{h}]\big) \wedge \overline{h}\big] \equiv \overline{h}$

(ii) $\big[((\varphi^\flat \,\&\, \psi^\flat) \to \varphi^\flat) \wedge \overline{h}\big] \equiv \overline{h}$

(iii) $\big[((\varphi^\flat \,\&\, \psi^\flat) \to (\psi^\flat \,\&\, \varphi^\flat)) \wedge \overline{h}\big] \equiv \overline{h}$

(iv) $\big[\big((\varphi^\flat \,\&\, [(\varphi^\flat \to \psi^\flat) \wedge \overline{h}]) \to (\psi^\flat \,\&\, [(\psi^\flat \to \varphi^\flat) \wedge \overline{h}])\big) \wedge \overline{h}\big] \equiv \overline{h}$

(va) $\big[[(\varphi^\flat \to [(\psi^\flat \to \chi^\flat) \wedge \overline{h}]) \wedge \overline{h}] \to [((\varphi^\flat \,\&\, \psi^\flat) \to \chi^\flat) \wedge \overline{h}] \wedge \overline{h}\big] \equiv \overline{h}$

(vb) $\big[[((\varphi^\flat \,\&\, \psi^\flat) \to \chi^\flat) \wedge \overline{h}] \to [(\varphi^\flat \to [(\psi^\flat \to \chi^\flat) \wedge \overline{h}]) \wedge \overline{h}] \wedge \overline{h}\big] \equiv \overline{h}$

(vi) $\big[\big([([(\varphi^\flat \to \psi^\flat) \wedge \overline{h}] \to \chi^\flat) \wedge \overline{h}] \to [(([(\psi^\flat \to \varphi^\flat) \wedge \overline{h}] \to \chi^\flat) \wedge \overline{h}] \to \chi^\flat) \wedge \overline{h}]\big) \wedge \overline{h}\big] \equiv \overline{h}$

(vii) $\big[(\overline{0} \to \varphi^\flat) \wedge \overline{h}\big] \equiv \overline{h}$

Proof. All the formulas have the form $[\alpha \wedge \overline{h}] \equiv \overline{h}$. It is sufficient, of course, to prove $\overline{h} \to [\alpha \wedge \overline{h}]$ (in BL$^+$); then by 2.2.2 (vii), it is sufficient to prove $\overline{h} \to \alpha$ (for the given α), or just to prove α and then use 2.5.2. In all the proofs, we are going to leave the $\flat$'s out in superscripts for readability's sake.

(i)

BL$^+ \vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ (A1)

BL$^+ \vdash ((\psi \to \chi) \to (\varphi \to \chi)) \to \big([(\psi \to \chi) \wedge \overline{h}] \to [(\varphi \to \chi) \wedge \overline{h}]\big)$ by 2.5.4; then, using transitivity of the implication (A1),

BL$^+ \vdash (\varphi \to \psi) \to \big([(\psi \to \chi) \wedge \overline{h}] \to [(\varphi \to \chi) \wedge \overline{h}]\big)$. Using 2.5.4 again, we get

BL$^+ \vdash [(\varphi \to \psi) \wedge \overline{h}] \to [\big([(\psi \to \chi) \wedge \overline{h}] \to [(\varphi \to \chi) \wedge \overline{h}]\big) \wedge \overline{h}]$.

(ii), (iii), (vii)

Take (ii) for example. As discussed above, by 2.5.2 if $(\varphi \,\&\, \psi) \to \varphi$ is provable in BL$^+$, then BL$^+$ also proves $\overline{h} \to ((\varphi \,\&\, \psi) \to \varphi)$. Now $(\varphi \,\&\, \psi) \to \varphi$ is the axiom (A2), therefore provable. Cases (iii) and (vii) are analogous.

(iv) It is sufficient to show

BL$^+ \vdash \varphi \,\&\, [(\varphi \to \psi) \wedge \overline{h}] \to \psi \,\&\, [(\psi \to \varphi) \wedge \overline{h}]$; by 1.2.2 (ix), this is the same as
BL$^+ \vdash \big((\varphi \,\&\, (\varphi \to \psi)) \wedge (\varphi \,\&\, \overline{h})\big) \to \big((\psi \,\&\, (\psi \to \varphi)) \wedge (\psi \,\&\, \overline{h})\big)$.

We use 1.2.2 (vii) again;

BL$^+ \vdash (\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$ holds since it is the axiom (A4), while
BL$^+ \vdash ((\varphi \,\&\, (\varphi \to \psi)) \wedge (\varphi \,\&\, \overline{h}) \to (\psi \,\&\, \overline{h})$ follows from 1.2.2 (iv).

(va)

By 2.5.4, it is sufficient to prove $(\varphi \to [(\psi \to \chi) \wedge \overline{h}]) \to ((\varphi \,\&\, \psi) \to \chi)$ in BL$^+$. We know that

BL$^+ \vdash [(\psi \to \chi) \wedge \overline{h}] \to (\psi \to \chi)$; therefore, using (A1) and 2.2.2 (i),
BL$^+ \vdash (\varphi \to [(\psi \to \chi) \wedge \overline{h}]) \to (\varphi \to (\psi \to \chi))$ and, using (A5a),
BL$^+ \vdash (\varphi \to [(\psi \to \chi) \wedge \overline{h}]) \to ((\varphi \,\&\, \psi) \to \chi)$.

(vb)

We are going to adjust the antecedent of the desired implication first:

BL$^+ \vdash [((\varphi \,\&\, \psi) \to \chi) \wedge \overline{h}] \to [((\varphi \,\&\, \psi) \to \chi) \wedge (\overline{h} \wedge \overline{h})]$, using the fact that $\overline{h} \to \overline{h} \wedge \overline{h}$ and 2.5.4;

BL$^+ \vdash [((\varphi \,\&\, \psi) \to \chi) \wedge (\overline{h} \wedge \overline{h})] \to [(((\varphi \,\&\, \psi) \to \chi) \wedge \overline{h}) \wedge \overline{h}]$, since $\wedge$ is associative.

By 2.5.4, it is now sufficient to show that

$\mathrm{BL}^+ \vdash [((\varphi \,\&\, \psi) \to \chi) \wedge \overline{h}] \to (\varphi \to [(\psi \to \chi) \wedge \overline{h}])$; or, using (A5),

$\mathrm{BL}^+ \vdash [(\varphi \to (\psi \to \chi)) \wedge \overline{h}] \to (\varphi \to [(\psi \to \chi) \wedge \overline{h}])$; or, using (A5a),

$\mathrm{BL}^+ \vdash ([(\varphi \to (\psi \to \chi)) \wedge \overline{h}] \,\&\, \varphi) \to [(\psi \to \chi) \wedge \overline{h}]$.

Using 1.2.2 (vii), we will get this statement from the two implications

$\mathrm{BL}^+ \vdash ([(\varphi \to (\psi \to \chi)) \wedge \overline{h}] \,\&\, \varphi) \to \overline{h}$, which is quite clear, and

$\mathrm{BL}^+ \vdash ([(\varphi \to (\psi \to \chi)) \wedge \overline{h}] \,\&\, \varphi) \to [\psi \to \chi]$, which follows from 1.2.2 (x) and (iv).

(vi)

We are going to show that

$\mathrm{BL}^+ \vdash [([[(\varphi \to \psi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to [(([[(\psi \to \varphi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to \chi) \wedge \overline{h}]$.

By 2.2.2 (vii), it is sufficient to prove in BL^+ the two implications

$[([[(\varphi \to \psi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to \overline{h}$, which is obvious, and

$[([[(\varphi \to \psi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to ([([[(\psi \to \varphi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to \chi)$, which is, by (A5), the same as

$[([[(\varphi \to \psi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \,\&\, [([[(\psi \to \varphi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to \chi$

We will use (A6) now: it says (with (A5)) that

$\mathrm{BL}^+ \vdash ((\varphi \to \psi) \to \chi) \,\&\, ((\psi \to \varphi) \to \chi) \to \chi$; by transitivity of the implication and 2.2.2 (v), it is enough to show that

$\mathrm{BL}^+ \vdash [([[(\varphi \to \psi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to ((\varphi \to \psi) \to \chi)$ and

$\mathrm{BL}^+ \vdash [([[(\psi \to \varphi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \to ((\psi \to \varphi) \to \chi)$.

Take the first case, rewrite it as:

$[([[(\varphi \to \psi) \wedge \overline{h}] \to \chi) \wedge \overline{h}] \,\&\, (\varphi \to \psi) \to \chi$; for clarity, substitute α for $(\varphi \to \psi)$:

$[([\alpha \wedge \overline{h}] \to \chi) \wedge \overline{h}] \,\&\, \alpha \to \chi$; now use the definition of $\wedge$:

$(\alpha \,\&\, \overline{h} \,\&\, (\overline{h} \to ([\alpha \wedge \overline{h}] \to \chi))) \to \chi$; this is, using (Id),

$(\alpha \,\&\, \overline{h} \,\&\, \overline{h} \,\&\, (\overline{h} \to ([\alpha \wedge \overline{h}] \to \chi))) \to \chi$. By 1.2.2 (iv), it is now sufficient to show

$\mathrm{BL}^+ \vdash (\alpha \,\&\, \overline{h} \,\&\, ([\alpha \wedge \overline{h}] \to \chi)) \to \chi$; if we replace the first $\&$ by $\wedge$ (which is acceptable by 1.2.2 (vi)), we get

$([\alpha \wedge \overline{h}] \,\&\, ([\alpha \wedge \overline{h}] \to \chi)) \to \chi$, which is exactly 1.2.2 (iv), and hence provable in BL^+. The second case is analogous. $\mathcal{QED}$

2.6. Generalizations

We sketch briefly a possible generalization of the developed method, for countably many truth constants (which is just as much as is needed, for in any standard algebra there are at most countably many idempotents delimiting the isomorphic copies of the three algebras $[0,1]_{\mathrm{L}}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_{\Pi}$). We do not give proofs for the suggested generalization.

Take a linearly ordered countable set I of indexes, and introduce a set of truth constants $\mathcal{H} = \{h_i\}, i \in I$. We assume for convenience that for i_0 the minimum of I we have $h_{i_0} = \overline{0}$, and for i_1 the maximum of I $h_{i_1} = \overline{1}$. To indicate idempotence, add an axiom $h_i \,\&\, h_i \equiv h_i$ for all $i \in I$, and to indicate that each constant except h_{i_0} is distinct from $\overline{0}$, add $\neg\neg h_i$ for all $i \in I, i \neq 0$. Moreover, for $i < j, i, j \in I$ add $h_i \to h_j$.

For $h_i \in \mathcal{H}$, we write h_i^+ for h_{i^+}, i.e. the constant denoted by the successor of i in I if it exists, otherwise $h_i^+ = h_i$.

For each $i \in I$ except the maximum i_1, we define a translation function f_i working analogically to $\flat$ as follows:

$$
\begin{aligned}
\overline{0}^{f_i} &= h_i \\
\overline{1}^{f_i} &= h_i^+ \\
p^{f_i} &= (p \vee h_i) \wedge h_i^+ \\
(\varphi \& \psi)^{f_i} &= \varphi^{f_i} \& \psi^{f_i} \\
(\varphi \to \psi)^{f_i} &= (\varphi^{f_i} \to \psi^{f_i}) \wedge h_i^+
\end{aligned}
$$

Now we choose a standard algebra we want to axiomatize. Suppose that it is represented as an ordered sum of isomorphic copies $[c_i, c_i^+]$ of $[0,1]_{\text{Ł}}$, $[0,1]_{\text{G}}$, or $[0,1]_\Pi$, and the sum is indexed by a set I. We take this set (which is countable) as the index set of constants, and assign to each $i \in I$ a constant h_i as described above. For each i we add a translation of the axiom of the corresponding schematic extension Ł, G, or Π (depending on which standard algebra the interval $[c_i, c_i^+]$ is a copy of).

The claim is that BL plus this set of formulas is an axiomatics sound and complete w.r.t. the tautologies of the chosen standard algebra (in a language enriched with the added constants). The completeness proof for the corresponding subvariety of *BL*-algebras should present no difficulties. A standard completeness proof will then depend on a general version of lemma 2.3.3, asserting that each linearly ordered algebra satisfying all the axioms can be decomposed as an ordered sum of *BL*-chains along the same index set and with the same layout of Ł's, G's, and Π's as the original standard algebra. The local embedding argument will then be used. Of course this axiomatics is infinite iff the ordered sum is infinite.

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