Interpreting Lattice-Valued Set Theory in Fuzzy Set Theory

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Abstract

An interpretation of lattice-valued logic, defined by Titani, in basic fuzzy logic, defined by Hájek, is presented. Moreover, Titani's axioms of lattice-valued set theory are interpreted in fuzzy set theory, under slight modifications of the fuzzy set theory axiomatics.

1 Introduction

This paper presents a comparison of two axiomatic set theories over two non-classical logics. In particular, it suggests an interpretation of *lattice-valued set theory* as defined in [16] by S. Titani in *fuzzy set theory* as defined in [11] by authors of this paper.

There are many different conceptions of set theories in non-classical logics, or even in the much narrower territory of fuzzy logics; cf. [5], [6], [10] for overviews. From the viewpoint of this paper, a milestone was laid by G. Takeuti and S. Titani in their paper [15]. Their logic is Gödel fuzzy logic expanded with Łukasiewicz connectives and product conjunction, and their set theory is a variant of ZFC in the given logic. Their paper, as well as its predecessor [14], builds upon results of set theory in intuitionistic logic, as given by W. C. Powell [12] and R. J. Grayson [7], which is apparent, among other things, in its spelling of axioms—in a weak setting (such as that of an intermediate logic), different but classically equivalent versions of axioms are no longer equivalent; some are too strong, so that they strengthen the logic, while others appear to be too weak to prove desirable statements, and choosing suitable axiomatics involves avoiding both extremes. The paper [15] not only establishes strong results for the set theory it defines, but also contributes to the development of mathematical fuzzy logics, in that it meaningfully employs a richer-than-usual set of logical connectives; a bit later, this kind of logics re-emerged in P. Hájek's book [8] and in the papers of F. Esteva, L. Godo, F. Montagna, P. Cintula and others (cf. [3],[4],[1]).

S. Titani's paper [16] is similar to [15] in its development of set theory, but the logic is significantly different. Titani defines and uses the logic of complete lattices, taking

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the lattice operations as truth functions of the conjunction and the disjunction, and introducing the *basic implication* representing the lattice ordering, and a corresponding negation. Strong completeness of this logic w. r. t. models over complete lattices was proved in [13]. In this logic, she develops a first-order theory LZFZ which is an analogue of Zermelo-Fraenkel set theory with the axiom of choice. Note that lattice-valued logic, albeit many-valued, is not a fuzzy logic in the sense of [2]; in particular, it is not complete with respect to chains among its algebras of truth values, and the conjunction and the implication do not form a residuated pair: For example, $\varphi \rightarrow (\psi \rightarrow \varphi)$ is not a logically valid formula.

The theory FST (Fuzzy Set Theory) was defined and developed by P. Hájek and Z. Haniková in [10], [11]. Its logic is BL$\forall\Delta$, which is the first-order Basic Fuzzy Logic BL$\forall$, defined by Hájek in [8], expanded with the Δ connective. FST is a rendering of the classical Zermelo-Fraenkel set theory in a logic which is a common fragment of, e. g., Łukasiewicz logic and Gödel logic, and therefore significantly weaker than either of the two.

The above implies that the two set theories in question might be rather similar, which is indeed the case. The interpretation presented in this paper is in fact an interpretation of Titani's lattice-valued logic in the logic BL$\forall\Delta$, while the set-theoretic language is left intact and (the translations of) most LZFZ axioms are provable in FST straightforwardly. Among the similarities on might mention the use of the *globalization* connective in LZFZ and the Δ-connective in FST, which serve similar purposes—namely, suppressing many-valued semantics in certain statements, or the fact that equality is a two-valued predicate in both theories. There are two major differences inbetween the set of axioms of lattice-valued set theory and that of FST. The first difference consists in connectives used in the formulations of axioms: LZFZ uses the lattice conjuction, while FST uses the strong conjunction (which does not admit contraction). Second, FST lacks the axiom of choice or its equivalent, as it relates not to classical ZFC but to classical ZF. In order to fully interpret lattice-valued set theory in FST, one has to incorporate some version of choice into FST, and also to make another amendment in the formulation of the axiom $\in$-induction in FST, which seems to be too weak to prove Titani's version.

2 Lattice-valued set theory

Titani [16] defines her lattice-valued logic L on complete lattices where she introduces a basic implication, which is two-valued and represents the lattice ordering, and a corresponding negation. Lattice-valued logic has the logical symbols $\wedge$ (interpreted as lattice meet), $\vee$ (interpreted as lattice join), $\rightarrow$, $\neg$, $\forall$, $\exists$. Other definable connectives include 1 standing for $\varphi \rightarrow \varphi$, 0 standing for $\neg 1$, and $\Box$, where $\Box\varphi$ stands for $1 \rightarrow \varphi$.

A Gentzen-style proof system for L is a modification of the classical Gentzen calculus LK. First, define $\Box$-closed (Bc) formulas as follows:

(1) formulas of the form $\varphi \rightarrow \psi$, $\neg\varphi$ are $\Box$-closed
(2) if φ, ψ are $\Box$-closed, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$
(3) if $\varphi(x)$ is $\Box$-closed with a free variable x, then $\forall x\varphi(x)$ and $\exists x\varphi(x)$ are $\Box$-closed
(4) any $\Box$-closed formula is obtained by application of (1)–(3)
A sequence of formulas Γ is $\Box$-closed iff it is composed solely from $\Box$-closed formulas.

Axioms of L are sequents[I)] of the form $\varphi \longrightarrow \varphi$. The rules for L are obtained from the rules of LK, which are subject to the following restrictions:

(a) The structural rules of Weakening, Contraction, and Interchange are as in LK.

(b) In the structural rule of Cut below, Δ or Π or φ must be $\Box$-closed.

$$\frac{\Gamma \longrightarrow \Delta, \varphi \qquad \varphi, \Pi \longrightarrow \Lambda}{\Gamma, \Pi \longrightarrow \Delta, \Lambda}$$

(c) The rules $\wedge$:left as in LK, while in $\wedge$:right below, Δ must be $\Box$-closed or (φ, ψ) must be $\Box$-closed:

$$\frac{\Gamma \longrightarrow \Delta, \varphi \qquad \Gamma \longrightarrow \Delta, \psi}{\Gamma \longrightarrow \Delta, \varphi \wedge \psi}$$

(d) The rules $\vee$:right as in LK, while in $\vee$:left below, Γ must be $\Box$-closed or (φ, ψ) must be $\Box$-closed:

$$\frac{\varphi, \Gamma \longrightarrow \Delta \qquad \psi, \Gamma \longrightarrow \Delta}{\varphi \vee \psi, \Gamma \longrightarrow \Delta}$$

(e) In the rule $\rightarrow$:left below left, Δ and Π must be $\Box$-closed; while in the rule $\rightarrow$:right below right, Γ and Δ must be $\Box$-closed.

$$\frac{\Gamma \longrightarrow \Delta, \varphi \qquad \psi, \Pi \longrightarrow \Lambda}{\varphi \rightarrow \psi, \Gamma, \Pi \longrightarrow \Delta, \Lambda} \qquad \frac{\varphi, \Gamma \longrightarrow \Delta, \psi}{\Gamma \longrightarrow \Delta, \varphi \rightarrow \psi}$$

(f) In the rule $\neg$:left below left, Δ or φ must be $\Box$-closed; while in the rule $\neg$:right below right, Γ and Δ must be $\Box$-closed or φ must be $\Box$-closed.

$$\frac{\Gamma \longrightarrow \Delta, \varphi}{\neg\varphi, \Gamma \longrightarrow \Delta} \qquad \frac{\varphi, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg\varphi}$$

(g) The rule $\forall$:left as in LK; while in the rule $\forall$:right below, Δ must be $\Box$-closed or $\varphi(a)$ must be $\Box$-closed, where a is a free variable not occurring in the lower sequent:

$$\frac{\Gamma \longrightarrow \Delta, \varphi(a)}{\Gamma \longrightarrow \Delta, \forall x \varphi(x)}$$

(h) The rule $\exists$:right as in LK; while in the rule $\exists$:left below, Γ must be $\Box$-closed or $\varphi(a)$ must be $\Box$-closed, where a is a free variable not occuring in the lower sequent:

[I)] A sequent is a formal expression $\Gamma \longrightarrow \Delta$, where Γ, Δ are finite sequences of formulas.

$$\frac{\varphi(a), \Gamma \longrightarrow \Delta}{\exists x\varphi(x), \Gamma \longrightarrow \Delta}$$

The semantics of a first-order language $\mathcal{L}$ over L is given by a complete lattice $\mathbf{L}$, a non-empty universe M, and for each n-ary predicate symbol P in the language $\mathcal{L}$, a function $r_P : M^n \longrightarrow \mathbf{L}$. Then $\mathbf{M} = \langle M, (r_P)_{P \text{ predicate symbol}}\rangle$ is a structure for $\mathcal{L}$.[II)] An evaluation v is a mapping assigning a value from M to each free variable. The truth value of a formula φ in $\mathbf{M}$ under an evaluation v is as follows:

$$\begin{aligned}
\|P(a_1, \ldots, a_n)\|^{\mathbf{L}}_{\mathbf{M},v} &= r_P(v(a_1), \ldots, v(a_n)) \\
\|\varphi \wedge \psi\|^{\mathbf{L}}_{\mathbf{M},v} &= \inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}, \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}\} \\
\|\varphi \vee \psi\|^{\mathbf{L}}_{\mathbf{M},v} &= \sup\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}, \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}\} \\
\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} &= 1 \text{ if } \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \leq \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}, 0 \text{ otherwise} \\
\|\neg\varphi\|^{\mathbf{L}}_{\mathbf{M},v} &= 1 \text{ if } \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = 0, 0 \text{ otherwise} \\
\|\forall x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} &= \inf_{v \equiv_x v'} \{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}\} \\
\|\exists x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} &= \sup_{v \equiv_x v'} \{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}\}
\end{aligned}$$

A sequent $\Gamma \longrightarrow \Delta$ is *valid* in lattice-valued logic L iff $\inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} : \varphi \in \Gamma\} \leq \sup\{\|\psi\|^{\mathbf{L}}_{\mathbf{M},v} : \psi \in \Delta\}$ for every choice of $\mathbf{L}$, $\mathbf{M}$, and v. In [13], strong completeness of L w. r. t. (models over) complete lattices has been shown:

Theorem 2.1. (Strong Completeness) *A sequent is provable in lattice-valued logic iff it is valid in it.*

Lattice-valued set theory LZFZ has the following axioms:

GA1. Equality $\forall u, v(u = v \wedge \varphi(u) \to \varphi(v))$
GA2. Extensionality $\forall u, v(\forall x(x \in u \equiv x \in v) \to u = v)$
GA3. Pairing $\forall u, v\exists z\forall x(x \in z \equiv (x = u \vee x = v))$
GA4. Union $\forall u\exists z\forall x(x \in z \equiv \exists y(y \in u \wedge x \in y))$
GA5. Power set $\forall u\exists z\forall x(x \in z \equiv x \subseteq u)$
GA6. Infinity $\exists u(\exists x(x \in u) \wedge \forall x(x \in u \to \exists y \in u(x \in y)))$
GA7. Separation $\forall u\exists v\forall x(x \in v \equiv x \in u \wedge \varphi(x))$
GA8. Collection $\forall u\exists v(\forall x \in u\exists y\varphi(x, y) \to \forall x \in u\exists y(\Box y \in v \wedge \varphi(x, y)))$
GA9. $\in$-induction $\forall x(\forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x\varphi(x)$
GA10. Zorn $\mathrm{Gl}(u) \wedge \forall v(\mathrm{Chain}(v, u) \to \bigcup v \in u) \to \exists z\mathrm{Max}(z, u)$, where

$$\begin{aligned}
\mathrm{Gl}(u) &\text{ is } \forall x(x \in u \to \Box(x \in u)) \\
\mathrm{Chain}(v, u) &\text{ is } v \subseteq u \wedge \forall x, y(x, y \in v \to x \subseteq y \vee y \subseteq x) \\
\mathrm{Max}(z, u) &\text{ is } z \in u \wedge \forall x(x \in u \wedge z \subseteq x \to z = x)
\end{aligned}$$

GA11. Axiom of $\Diamond$ $\forall u\exists z\forall x(x \in z \equiv \Diamond(x \in u))$

[II)] This definition is taken from [13], where function symbols are not considered.

3 Fuzzy set theory over BL$\forall\Delta$

In this section we describe the logic BL$\forall\Delta$ and the theory FST, which is a fuzzy set theory over BL$\forall\Delta$. See [8] for details on the logic and [11] for development of the set theory; some degree of familiarity with both is an advantage in technical details to follow.

The logic BL$\forall\Delta$ is the first-order Basic Fuzzy Logic expanded with the Δ-connective. The basic logical symbols in the language of BL$\forall\Delta$ are

(i) The logical connectives of BLΔ: the (strong) conjunction &, its residuated implication $\rightarrow$, the unary Δ connective and the constant $\overline{0}$.

(ii) Quantifiers $\forall$ and $\exists$

(iii) The symbol $=$ denoting equality.[III)]

Definable connectives include the negation ($\neg\varphi$ stands for $\varphi \rightarrow 0$), the min-conjunction ($\varphi \wedge \psi$ stands for $\varphi \,\&\, (\varphi \rightarrow \psi)$), the max-disjunction ($\varphi \vee \psi$ stands for $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$), the equivalence ($\varphi \equiv \psi$ stands for $(\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)$), and the constant 1, standing for $0 \rightarrow 0$.

We list some shorthand:

$$\begin{aligned} \varphi \rightarrow^{\Delta} \psi &\text{ is } \Delta(\varphi \rightarrow \psi) \\ \varphi \equiv^{\Delta} \psi &\text{ is } (\varphi \rightarrow^{\Delta} \psi) \,\&\, (\psi \rightarrow^{\Delta} \varphi) \\ (\exists x \in y)\varphi &\text{ is } \exists x(x \in y \,\&\, \varphi) \\ (\forall x \in y)\varphi &\text{ is } \forall x(x \in y \rightarrow \varphi) \end{aligned}$$

A language of a particular theory has object constants, predicate symbols and function symbols. Object variables $\mathcal{V}$ are denoted x, y, z, w etc.

The Hilbert-style axioms and rules of BL$\forall\Delta$ are listed below, in the following order: (A1)–(A7) axioms of propositional BL; (Δ1)–(Δ5) axioms for the Δ-connective; ($\forall$1)–($\forall$3) axioms for quantifiers; (E1)–(E5) axioms for equality.

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi \,\&\, \psi) \rightarrow \varphi$
(A3) $(\varphi \,\&\, \psi) \rightarrow (\psi \,\&\, \varphi)$
(A4) $(\varphi \,\&\, (\varphi \rightarrow \psi)) \rightarrow (\psi \,\&\, (\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \,\&\, \psi) \rightarrow \chi)$
(A5b) $((\varphi \,\&\, \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $0 \rightarrow \varphi$
(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
(Δ2) $\Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$
(Δ3) $\Delta\varphi \rightarrow \varphi$
(Δ4) $\Delta\varphi \rightarrow \Delta\Delta\varphi$
(Δ5) $\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$
($\forall$1) $\forall x\varphi(x) \rightarrow \varphi(t)$ (t substitutable for x in φ)

[III)] Inclusion of $=$ among logical symbols is a technical device

(∃1) $\varphi(t) \to \exists x\varphi(x)$ (t substitutable for x in φ)
(∀2) $\forall x(\chi \to \varphi) \to (\chi \to \forall x\varphi)$ (x not free in χ)
(∃2) $\forall x(\varphi \to \chi) \to (\exists x\varphi \to \chi)$ (x not free in χ)
(∀3) $\forall x(\varphi \vee \chi) \to (\forall x\varphi \vee \chi)$ (x not free in χ)
(E1) (reflexivity) $\forall x(x = x)$
(E2) (symmetry) $\forall xy(x = y \to y = x)$
(E3) (transitivity) $\forall xyz(x = y \,\&\, y = z \to x = z)$
(E4) (congruence) $\forall xyz(x = y \,\&\, z \in x \to z \in y)$
(E5) (congruence) $\forall xyz(x = y \,\&\, y \in z \to x \in z)$
Deduction rules of BL∀Δ are modus ponens, Δ-generalization: from φ derive $\Delta\varphi$, and ∀-generalization: from φ derive $\forall x\varphi$.

The semantics of a first-order language $\mathcal{L}$ over BL∀Δ is given by an **L**-structure **M**, for **L** a BL∀Δ-chain. $\mathbf{M} = (M, (r_P)_{P \text{ predicate symbol}}, (m_c)_{c \text{ constant}}, (f_F)_{F \text{ function symbol}})$ has a non-empty universe M, for each n-ary predicate symbol P of $\mathcal{L}$ an **L**-fuzzy n-ary relation $r_P : M^n \to L$, for each constant c of $\mathcal{L}$ an element $m_c \in M$, and for each n-ary function symbol F of $\mathcal{L}$ a function $f_F : M^n \longrightarrow M$ (see [8], [9]).

An **M**-evaluation of object variables is a mapping $v : \mathcal{V} \longrightarrow M$. For two evaluations v, v', we write $v \equiv_x v'$ if $v(y) = v'(y)$ for each variable distinct from x. The value $\|t\|_{\mathbf{M},v}$ of a term t under evaluation v is defined inductively: $\|x\|_{\mathbf{M},v} = v(x)$, $\|c\|_{\mathbf{M},v} = m_c$, $\|F(t_1,\ldots,t_n)\|_{\mathbf{M},v} = f_F(\|t_1\|_{\mathbf{M},v},\ldots,\|t_n\|_{\mathbf{M},v})$ for an n-ary function symbol F and terms $t_1,\ldots,t_n$.

The truth value $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ of a formula φ given by an **L**-structure **M** and an evaluation v in **M** is as follows:
$\|P(t_1,\ldots,t_n)\|^{\mathbf{L}}_{\mathbf{M},v} = r_P(\|t_1\|_{\mathbf{M},v},\ldots,\|t_n\|_{\mathbf{M},v})$
$\|\varphi \,\&\, \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} * \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$
$\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$
$\|0\|^{\mathbf{L}}_{\mathbf{M},v} = 0$
$\|\Delta\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \Delta\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$
$\|\forall x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{v\equiv_x v'} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}$
$\|\exists x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{v\equiv_x v'} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}$

Since **L** need not be complete lattice, this value may be undefined; the **L**-structure **M** is *safe* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for each φ and v. The truth value of a formula φ of a predicate language $\mathcal{L}$ in a safe **L**-structure **M** for $\mathcal{L}$ is

$$\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \bigwedge_{v \text{ an } \mathbf{M}-\text{evaluation}} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$$

We call an interpretation **M** of $\mathcal{L}$ *admissible* if all the axioms for = are 1-true in **M**.

Let T be a theory over BL∀Δ, **L** a BL∀Δ-chain and **M** a safe admissible **L**-structure for the language of T. **M** is an **L**-model of T iff all axioms of T are 1-true in **M**, i. e., $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$ for each $\varphi \in T$. A formula φ of a predicate language $\mathcal{L}$ is an **L**-tautology iff $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ for each safe admissible **L**-structure **M**.[IV)]

[IV)] It is essential to remark that the set of tautologies of all BLΔ-chains whose lattice ordering is complete is non-arithmetical. Therefore, the theories of models over such algebras are non-axiomatizable (and the

Theorem 3.1. (Strong Completeness) *Let T be a theory over* BL$\forall\Delta$ *and let φ be a formula of the language of T. Then T proves φ iff for each* BL$\forall\Delta$*-chain* $\mathbf{L}$ *and each safe admissible* $\mathbf{L}$*-model* $\mathbf{M}$ *of T, φ is* 1*-true in* $\mathbf{M}$.

Fuzzy Set Theory (FST) is a theory over BL$\forall\Delta$ with the language $\{\in\}$ (note that $=$ is a logical symbol). It has the following axioms:[V)]
(extensionality) $\forall xy(x = y \equiv (\Delta(x \subseteq y) \,\&\, \Delta(y \subseteq x)))$
(empty set) $\exists x\Delta\forall y\neg(y \in x)$
(pair) $\forall xy\exists z\Delta\forall u(u \in z \equiv (u = x \vee u = y))$
(union) $\forall x\exists z\Delta\forall u(u \in z \equiv \exists y(u \in y \,\&\, y \in x))$
(weak power) $\forall x\exists z\Delta\forall u(u \in z \equiv \Delta(u \subseteq x))$
(infinity) $\exists z\Delta(\emptyset \in z \,\&\, \forall x \in z(x \cup \{x\} \in z))$
(separation) $\forall x\exists z\Delta\forall u(u \in z \equiv (u \in x \,\&\, \varphi(u, x)))$ (z not free in φ)
(collection) $\forall x\exists z\Delta[\forall u \in x\exists v\, \varphi(u, v) \to \forall u \in x\exists v \in z\varphi(u, v)]$ (z not free in φ)
($\in$-induction) $\Delta\forall x(\forall y \in x\varphi(y) \to \varphi(x)) \to \Delta\forall x\varphi(x)$
(support) $\forall x\exists z(\mathrm{Crisp}(z)\&\Delta(x \subseteq z)))$

We define an interpretation of FST in ZF. Let $\mathbf{L}$ be a linearly ordered complete BLΔ-algebra. Define a BLΔ-valued universe $V^{\mathbf{L}}$ over $\mathbf{L}$ (denote $L^+ = L - \{0\}$).
$V_0^{\mathbf{L}} = \{\emptyset\}$;
$V_{\alpha+1}^{\mathbf{L}} = \{f : \mathrm{Fn}(f) \,\&\, \mathrm{Dom}(f) \subseteq V_\alpha^{\mathbf{L}} \,\&\, \mathrm{Rng}(f) \subseteq L^+\}$ for any ordinal α;
$V_\lambda^{\mathbf{L}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbf{L}}$ for limit ordinals λ; finally, put
$V^{\mathbf{L}} = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^{\mathbf{L}}$.
Define evaluation of formulas in $V^{\mathbf{L}}$. For $u, v \in V^{\mathbf{L}}$ let
$\|u \in v\| = v(u)$ if $u \in D(v)$, otherwise 0,
$\|u = v\| = 1$ if $u = v$, otherwise 0.
In the usual way, use induction on complexity of formulas to define, for any FST formula $\varphi(x_1, \dots, x_n)$, an n-ary function assigning to an n-tuple $u_1, \dots, u_n \in V^{\mathbf{L}}$ the value $\|\varphi(u_1, \dots, u_n)\|$.

Definition 3.2. *Let φ be a closed formula in the language of FST. We say that φ is valid in $V^{\mathbf{L}}$ iff $\|\varphi\| = 1$ is provable in ZF.*

Theorem 3.3. *Let φ be a closed formula provable in FST. Then φ is valid in $V^{\mathbf{L}}$.*

Lemma 3.4. *The following formulas are provable in* BL$\forall\Delta$*:*

1. $\Delta(\varphi \vee \neg\varphi) \equiv \Delta(\varphi \to \Delta\varphi)$
2. $\Delta(\alpha \,\&\, \beta) \equiv (\Delta\alpha \,\&\, \Delta\beta)$
3. $\Delta\varphi \wedge \psi \to \Delta\varphi \,\&\, \psi$
4. $\forall x\Delta\varphi \equiv \Delta\forall x\varphi$
5. $(\Delta(\varphi \vee \neg\varphi) \,\&\, (\varphi \to \Delta\psi)) \to \Delta(\varphi \to \psi)$

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logic BL$\forall\Delta$ is not complete w. r. t. complete BLΔ-chains).
[V)] Cf. [11] for details on the choice of axioms.

Definition 3.5. (i) *A formula* φ *is* crisp *in* FST *iff* FST $\vdash \varphi \vee \neg\varphi$
(ii) *In* FST*, we define* $\mathrm{Crisp}(x) \equiv \forall u \Delta(u \in x \vee \neg u \in x)$.

In item (i) of the previous definition, it is an equivalent condition to prefix a Δ to the provability in FST (since in BL$\forall\Delta$, φ and $\Delta\varphi$ are equiprovable for any φ). It follows from Lemma 3.4 that a provably equivalent definition of a crisp set in item (ii) is $\mathrm{Crisp}(x) \equiv \Delta\forall u(u \in x \to \Delta(u \in x))$.

Next we define an interpretation of ZF in FST, by way of restricting quantifiers to a class H of hereditarily crisp sets.

Definition 3.6. (i) $\mathrm{HCT}(x) \equiv \mathrm{Crisp}(x) \,\&\, \forall u \in x(\mathrm{Crisp}(u) \& u \subseteq x)$
(ii) $\mathrm{H}(x) \equiv \mathrm{Crisp}(x) \,\&\, \exists x'(\mathrm{HCT}(x) \,\&\, x \subseteq x')$.

We write $x \in \mathrm{H}$ for $\mathrm{H}(x)$. FST proves that both of the above classes are crisp and H is also transitive. One defines a translation φ^{H} to each formula φ of ZF by restricting quantification to the class H. Then it can be shown that H is an inner model of ZF in FST:

Theorem 3.7. *Let* φ *be a formula in the language of ZF. Then* ZF $\vdash \varphi$ *entails* FST $\vdash \varphi^{\mathrm{H}}$.

Working in FST, we define a few notions from classical ZF that will be needed.

Definition 3.8. (i) $\langle u, v\rangle = \{\{u\}, \{u, v\}\}$
(ii) $\mathrm{Rel}(r) \equiv \forall x \in r\, \exists u, v(x = \langle u, v\rangle)$
(iii) $\mathrm{Dom}(r) = \{x : \exists y(\langle x, y\rangle \in r)\}$
(iv) $\mathrm{Rng}(r) = \{x : \exists y(\langle y, x\rangle \in r)\}$
(v) $\mathrm{CrispFn}(f) \equiv \mathrm{Rel}(f) \,\&\, \mathrm{Crisp}(f) \,\&\, \forall x \in \mathrm{Dom}(f)(\langle x, y\rangle \in f \,\&\, \langle x, z\rangle \in f \to y = z)$.

In order to obtain a suitable definition of ordinal numbers in FST, we rely on Theorem 3.7. Recall the classical definition:

$$\begin{aligned}\mathrm{Ord}_0(x) \equiv &(\mathrm{i})\forall y \in x(y \subseteq x)\\ &(\mathrm{ii})\forall y, z \in x(y \in z \vee y = z \vee z \in y)\\ &(\mathrm{iii})\forall q \subseteq x(\neg(q = \emptyset) \to \exists y \in q(y \cap q = \emptyset))\end{aligned}$$

If $x \in \mathrm{H}$, then $\mathrm{Ord}_0(x) \equiv \mathrm{Ord}_0^{\mathrm{H}}(x)$, and $\mathrm{Ord}_0(x)$ is crisp. We define ordinal numbers to be those sets in H for which $\mathrm{Ord}_0^{\mathrm{H}}$ is satisfied. That is,

Definition 3.9. $\mathrm{Ord}(x) \equiv x \in \mathrm{H} \,\&\, \mathrm{Ord}_0(x)$.

Theorem 3.7 yields that for any formula φ, if
ZF $\vdash \forall x_1, \ldots, x_n(\mathrm{Ord}_0(x_1)\& \ldots \&\mathrm{Ord}_0(x_n) \to \varphi)$, then
FST $\vdash \forall x_1, \ldots, x_n(\mathrm{Ord}(x_1)\& \ldots \&\mathrm{Ord}(x_n) \to \varphi^{\mathrm{H}})$.

4 Interpreting Titani's set theory

We interpret Titani's logic L of complete lattices in BL$\forall\Delta$ by introducing a translation function $(\star)$, taking formulas of the logic L as arguments and producing formulas of BL$\forall\Delta$

as values. Using this function, we translate provable sequents in L to provable formulas in BL$\forall\Delta$.[VI)]

Further, we also show that the $\star$-translations of Titani's set-theoretic axioms are provable in FST under the translation $\star$, with the possible exceptions of $\in$-induction and Zorn's lemma. Note that the translation does not concern extralogical symbols.

The translation $\star$ is defined as follows:

$$\begin{aligned} \varphi^\star &\text{ is } \varphi \text{ for } \varphi \text{ atomic} \\ (\varphi \wedge \psi)^\star &\text{ is } \varphi^\star \wedge \psi^\star \\ (\varphi \vee \psi)^\star &\text{ is } \varphi^\star \vee \psi^\star \\ (\varphi \to \psi)^\star &\text{ is } \Delta(\varphi^\star \to \psi^\star) \\ (\neg\varphi)^\star &\text{ is } \Delta\neg\varphi^\star \\ (\forall x\varphi)^\star &\text{ is } \forall x\varphi^\star \\ (\exists x\varphi)^\star &\text{ is } \exists x\varphi^\star \end{aligned}$$

We write $\varphi \to^\Delta \psi$ for $\Delta(\varphi \to \psi)$, and similarly $\neg^\Delta\varphi$ for $\Delta\neg\varphi$. We write $\varphi \equiv^\Delta \psi$ for $(\varphi \to^\Delta \psi) \wedge (\psi \to^\Delta \varphi)$, which is BL$\Delta$-equivalent to $(\varphi \to^\Delta \psi) \,\&\, (\psi \to^\Delta \varphi)$.

Note that $x \to^\Delta y$ in a BL$\forall\Delta$-chain is the characteristic function of its lattice ordering, and $\neg^\Delta x$ is $x \to^\Delta 0$; thus the evaluation of the $\star$-formulas in any BL$\forall\Delta$-chain $\mathbf{L}$ is fully determined by $\mathbf{L}$'s lattice operations.

Also, note that $(\Box\varphi)^\star$ is $\Delta\varphi$.

Theorem 4.1. *Let $\psi_1, \ldots, \psi_n, \varphi$ be sentences over* L. *Assume* $\vdash_{\mathrm{L}} \psi_1, \ldots, \psi_n \longrightarrow \varphi$. *Then* $\vdash_{\mathrm{BL}\forall\Delta} \psi_1^\star \wedge \cdots \wedge \psi_n^\star \to \varphi^\star$.

Proof. Completeness theorems for both L and BL$\forall\Delta$ allow for posing equivalent semantic conditions for each of the above provabilities. Using these, the theorem can be equivalently reformulated as follows:
For the given sentences $\psi_1, \ldots, \psi_n, \varphi$ over L, if for any structure $\mathbf{M}$ over any complete lattice $\mathbf{L}$ we have $\|\psi_1\|^{\mathbf{L}}_{\mathbf{M}} \wedge \cdots \wedge \|\psi_n\|^{\mathbf{L}}_{\mathbf{M}} \le \|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$ in $\mathbf{L}$, then for any safe structure $\mathbf{M}'$ over any BLΔ-chain $\mathbf{L}'$ we have $\|\psi_1^\star\|^{\mathbf{L}'}_{\mathbf{M}'} \wedge \cdots \wedge \|\psi_n^\star\|^{\mathbf{L}'}_{\mathbf{M}'} \le \|\varphi^\star\|^{\mathbf{L}'}_{\mathbf{M}'}$ in $\mathbf{L}'$.

We prove a variant. Assume that for some L-sentences $\psi_1, \ldots, \psi_n, \varphi$ and some safe structure $\mathbf{M}_0$ over a BLΔ-chain $\mathbf{L}_0 = \langle L_0, \wedge, \vee, *, \Rightarrow, \Delta, 0, 1\rangle$, we have $\|\psi_1^\star\|^{\mathbf{L}_0}_{\mathbf{M}_0} \wedge \cdots \wedge \|\psi_n^\star\|^{\mathbf{L}_0}_{\mathbf{M}_0} > \|\varphi^\star\|^{\mathbf{L}_0}_{\mathbf{M}_0}$. We take $\mathbf{L}_1 = \langle L_0, \wedge, \vee, 0, 1\rangle$ to be the lattice-reduct of $\mathbf{L}_0$. Let $\mathbf{M}_1$ be a new structure over $\mathbf{L}_1$, which inherits its universe and predicates from $\mathbf{M}_0$. It is immediate that $\|\psi_1^\star\|^{\mathbf{L}_1}_{\mathbf{M}_1} \wedge \cdots \wedge \|\psi_n^\star\|^{\mathbf{L}_1}_{\mathbf{M}_1} > \|\varphi^\star\|^{\mathbf{L}_1}_{\mathbf{M}_1}$ in $\mathbf{L}_1$, and therefore $\|\psi_1\|^{\mathbf{L}_1}_{\mathbf{M}_1} \wedge \cdots \wedge \|\psi_n\|^{\mathbf{L}_1}_{\mathbf{M}_1} > \|\varphi\|^{\mathbf{L}_1}_{\mathbf{M}_1}$ in $\mathbf{L}_1$. Now we take $\mathbf{L}_2$ to be a (lattice-)completion of $\mathbf{L}_1$, which means $\mathbf{L}_1$ is embeddable into $\mathbf{L}_2$ and the embedding preserves arbitrary joins and meets. Thus all formulas evaluate in $\mathbf{M}_1$ over $\mathbf{L}_2$ in exactly the same way as they do in $\mathbf{M}_1$ over $\mathbf{L}_1$. Hence $\|\psi_1\|^{\mathbf{L}_2}_{\mathbf{M}_1} \wedge \cdots \wedge \|\psi_n\|^{\mathbf{L}_2}_{\mathbf{M}_1} > \|\varphi\|^{\mathbf{L}_2}_{\mathbf{M}_1}$ in $\mathbf{L}_2$. Then $\mathbf{M}_1$ is a structure over a complete lattice $\mathbf{L}_2$ and $\mathbf{M}_2 \not\models \psi_1 \wedge \cdots \wedge \psi_n \to \varphi$, which is the desired conclusion. $\Box$

[VI)] We fix the set-theoretic language $\{\in, =\}$ as the language of formulas of both L and BL$\forall\Delta$. In the following theorem, the set-theoretic language can be replaced, w. l. o. g., by any language without constants or function symbols.

As already remarked in [13] on behalf of an anonymous referee, this interpretation is not faithful: for example, the formula expressing distributivity of a lattice, $((\varphi \vee \psi) \wedge \chi) \to ((\varphi \wedge \chi) \vee (\psi \wedge \chi))$, while provable in BL$\forall\Delta$ (and in BL alone), is not provable in lattice-valued logic L. As also pointed out in that paper, the same translation function yields an interpretation of L in Gödel logic with Δ immediately, given the fact that Gödel logic is complete w. r. t. models over complete lattices.

Let φ be a set-theoretic statement. What does it mean that φ is provable in LZFZ over L? In accordance with [15], we take the statement "φ is provable in LZFZ" to mean GA1, ..., GA11 $\longrightarrow \varphi$ is a provable sequent in L. Then, in order to show the provability of $\varphi^\star$ in FST, it is sufficient to show that the $\star$-translations of the axioms GA1,...,GA11 are provable in FST also.

Now let us take Titani's axioms of set theory $LZFZ$ (lattice-valued Zermelo-Fraenkel with Zorn's lemma) one by one and prove their translations in FST.

Lemma 4.2. *(*Equality*, GA1) For any φ,* FST $\vdash (x = y \wedge \varphi(x) \to^\Delta \varphi(y)$.[VII)]

Proof. In FST, for each φ we have $\vdash (\Delta(x = y) \,\&\, \varphi(x)) \to \varphi(y)$ (a consequence of equality axioms in FST, including crispness of $=$). Hence the following chain of implications is provable: $(x = y \wedge \varphi(x)) \to (\Delta(x = y) \wedge \varphi(x)) \to (\Delta(x = y) \,\&\, \varphi(x)) \to \varphi(y)$. (The second implication is due to Lemma 3.4). ☐

Lemma 4.3. *(*Extensionality*, GA2)* FST $\vdash \forall u, v(\forall x(x \in u \equiv^\Delta x \in v) \to^\Delta u = v)$

Proof. It is sufficient to prove $\forall x(x \in u \equiv^\Delta x \in v) \to u = v$ and then to apply Δ- and $\forall$-generalization. The statement follows from extensionality in FST. ☐

Lemma 4.4. *(*Pairing*, GA3)* FST $\vdash \forall x, y\exists z\forall u(u \in z \equiv^\Delta (u = x \vee u = y))$

Proof. Equivalent to pair axiom in FST by trivial handling of Δ. ☐

Lemma 4.5. *(*Union*, GA4)* FST $\vdash \forall u\exists z\forall x(x \in z \equiv^\Delta \exists y(y \in u \wedge x \in y)$

Proof. Equivalently, $\forall u\exists z\Delta\forall x(x \in z \equiv (\exists y)(y \in u \wedge x \in y)$. This differs from our union only by having $\wedge$ instead of $\&$ in the condition on the right. Take u; to each $y \in u$ associate its support y'. Then by collection, there exists a z_0 such that $y \in u \to y' \in z_0$. Now take $\bigcup z_0$, observe $\exists y(y \in u \wedge x \in y) \to x \in \bigcup z_0$. Therefore, setting $z = \{x : x \in \bigcup z_0 \,\&\, \exists y(y \in u \wedge x \in y)$, we obtain the set which is claimed by Titani's axiom. ☐

Lemma 4.6. *(*Power*, GA5)* FST $\vdash \forall u\exists z\forall x(x \in z \equiv^\Delta x \subseteq^\Delta y)\}$

This translation is weak power in FST verbatim.

Lemma 4.7. *(*Infinity*, GA6.)* FST $\vdash \exists u(\exists x(x \in u) \wedge \forall x(x \in u \to^\Delta \exists y(y \in u \wedge x \in y)))$.

Proof. This is trivially provable in FST: the set ω of natural numbers in the class H satisfies the axiom. ☐

Lemma 4.8. *(*Separation*, GA7.)* $\forall u\exists v\forall x(x \in z \equiv^\Delta (x \in u \wedge \varphi(x))$.

[VII)] Where possible, we omit the $\star$-index with φ since the statements hold in general.

Proof. Again, this differs from ours by $\wedge$ instead of $\&$. But note that $(x \in u \wedge \varphi(x))$ is $(x \in u \,\&\, (x \in u \to \varphi(x))$, where the second conjunct can be taken as the separation formula; then Titani's separation follows from ours. $\square$

Lemma 4.9. *(*COLLECTION*, GA8.)* $(\forall u)(\exists v)[(\forall x)(x \in u \to^{\Delta} (\exists y)\varphi(x,y)) \to^{\Delta}$
$\to^{\Delta} (\forall \alpha)(x \in u \to^{\Delta} (\exists y)(\Delta(y \in v) \wedge \varphi(x,y))]$.

Proof. Collection schema in FST spells as follows:
$\forall u \exists v[\forall x(x \in u \to \exists y \varphi(x,y)) \to^{\Delta} \forall x(x \in u \to \exists y \in v \varphi(x,y))]$.
First observe that (due to idempotence of Δ), collection in FST entails
$\forall u \exists v[\forall x(x \in u \to^{\Delta} \exists y \varphi(x,y)) \to^{\Delta} \forall x(x \in u \to^{\Delta} \exists y \in v \varphi(x,y))]$.
Then replace v by a crisp support $\hat{v}$:
$\forall u \exists \hat{v}[\forall x(x \in u \to^{\Delta} \exists y \varphi(x,y)) \to^{\Delta} \forall x(x \in u \to^{\Delta} \exists y(\Delta(y \in \hat{v}) \,\&\, \varphi(x,y))]$. Due to Lemma 3.4, the last strong conjunction can be equivalently replaced by a min-conjunction, i. e., $(\Delta(y \in \hat{v}) \wedge \varphi(x,y))$. This yields Titani's collection. $\square$

For GA9, see below.

The axiom GA10, a variant of Zorn's lemma in LZFZ, has no counterpart in FST. This is because Titani relates her theory to ZFC, while authors of this paper work in ZF. In order to fully interpret LZFZ, it would be necessary to adopt at least the $\star$-translation of GA10 into FST; while it would need a more thorough discussion to identify a suitable version of Zorn for FST, we remark here that the $\star$-translation of GA10 is valid in $V^{\mathbf{L}}$ when Zorn's lemma is assumed externally.

Lemma 4.10. *(*AXIOM OF $\Diamond$*, GA11)* FST $\vdash (\forall u)(\exists z)(\forall t)(t \in z \equiv^{\Delta} \neg^{\Delta}\Delta\neg^{\Delta}(t \in u))$

Proof. First observe that $\neg^{\Delta}\Delta\neg^{\Delta}\varphi$ is equivalent to $\neg\Delta\neg\varphi$. For u given, let v be a support (i. e., $(\mathrm{Crisp}(v) \,\&\, \Delta u \subseteq v)$. Observe $\neg\Delta\neg(t \in u) \to t \in v$. (Indeed, $t \in u \to t \in v$, therefore $\neg(t \in v) \to \neg(t \in u)$ and $\Delta\neg(t \in v) \to \Delta\neg(t \in u)$ and $\neg\Delta\neg(t \in u) \to \neg\Delta\neg(t \in v)$; the last succedent is equivalent to $t \in v$ as v is a crisp set). Separate from v: $z = \{t \in v : \neg\Delta\neg(x \in u)\}$. Then $t \in z \equiv \neg\Delta\neg(t \in u)$ by the last observation. This satisfies Titani's axiom. $\square$

5 On the axiom of $\in$-induction

Titani's $\in$-induction GA9 is an axiom schema in LZFZ for all its formulas. The following schema will be called Titani's $\in$-induction in FST, or simply Titani's induction (TI):

$$\forall x[\forall y(y \in x \to^{\Delta} \varphi(y)) \to^{\Delta} \varphi(x)] \to^{\Delta} \forall x \varphi(x)$$

This is the $\star$ translation of GA9 except for the fact that we apply the schema to any formula φ of FST, rather than just to $\star$-translations of LZFZ formulas. Obviously, the consequent may be equivalently replaced by $\Delta\forall x\varphi(x)$. The FST-native $\in$-induction then differs only by having $\forall y(y \in x \to \varphi(y))$ instead of $\forall y(y \in x \to^{\Delta} \varphi(y))$ in its antecedent. It is easy to prove that Titani's induction implies FST-induction, using properties of Δ.

It is an open problem whether FST can prove Titani's induction. Another open problem is whether FST can prove all formulas that are valid in $V^{\mathbf{L}}$. In this section we do not offer solutions to either of the problems, but we show that if the latter is solved in the affirmative, then so is the former; in particular, Titani's $\in$-induction is valid in $V^{\mathbf{L}}$. Then we study a rather natural statement which, in FST, is provably equivalent to Titani's $\in$-induction. Namely, we formulate an axiom saying that each set is obtained from the empty set by iterating the application of the weak power axiom of FST along ordinal numbers in FST.

Lemma 5.1. *For any FST-formula φ, Titani's induction is valid in $V^{\mathbf{L}}$.*

Proof. Suppose $\|\forall x[\forall y(y \in x \to^{\Delta} \varphi(y)) \to^{\Delta} \varphi(x)] \to^{\Delta} \forall x\varphi(x)\| < 1$ in $V^{\mathbf{L}}$; then it must be the case that $\|\forall x\Delta[\forall y\Delta(y \in x \to \varphi(y)) \to \varphi(x)]\| = 1$ and $\|\forall x\varphi(x)\| < 1$; hence, there is a least ordinal α s. t. $\exists a \in V^{\mathbf{L}}_{\alpha}(\|\varphi(a)\| < 1)$ and α is not a limit ordinal. If $\alpha = 0$, then $a = \emptyset$, but we have $\bigwedge_{b\in V^{\mathbf{L}}}(\|b \in \emptyset\| \Rightarrow \|\varphi(b)\|) = 1$. If α has a predecessor, then $\bigwedge_{b\in V^{\mathbf{L}}} \Delta(\|b \in a\| \Rightarrow \|\varphi(b)\|) = 1$, since for $b \in \mathrm{Dom}(a)$ we have $\|\varphi(b)\| = 1$ because of the rank of b, and for $b \notin \mathrm{Dom}(a)$ we have $\|b \in a\| = 0$ by definition. In both cases, the value of the antecedent is $1 \Rightarrow \|\varphi(a)\| = \|\varphi(a)\| < 1$, which contradicts the assumption. □

Definition 5.2. *The* iterated weak power *property is as follows:*
$\mathrm{ItWP}(f) \equiv \mathrm{CrispFn}(f) \,\&\, \mathrm{Dom}(f) \in \mathrm{Ord} \,\&\, f(\emptyset) = \emptyset \,\&\, \forall\alpha \in \mathrm{Ord}(\alpha \neq \emptyset \,\&\, \alpha \in \mathrm{Dom}(f) \to f(\alpha) = \bigcup_{\beta\in\alpha} WP(f(\beta)))$

Lemma 5.3. *(1)* ItWP *is a crisp notion:* $\mathrm{ItWP}(f) \equiv \Delta\mathrm{ItWP}(f)$.
(2) $[\mathrm{ItWP}(f) \,\&\, \mathrm{ItWP}(g) \,\&\, \mathrm{Dom}(f) \leq \mathrm{Dom}(g)] \to \Delta(f \subseteq g)$

Proof. (1) obvious. (2) Put $x = \{\alpha \in \mathrm{Dom}(f) | f(\alpha) = g(\alpha)\}$; we show $x = \mathrm{Dom}(f)$. As both f and $\mathrm{Dom}(x)$ are crisp, we may prove by contradiction. If $\mathrm{Dom}(f) \setminus x \neq \emptyset$ then it is a crisp set of ordinals with a least element, a successor ordinal $\alpha + 1$ (for some α). But then $f(\alpha) = g(\alpha)$ implies $f(\alpha + 1) = g(\alpha + 1)$, contradiction. □

Lemma 5.4. $(\forall\alpha)(\exists f)(\mathrm{ItWP}(f) \,\&\, \mathrm{Dom}(f) = \alpha)$.

Proof. As above. □

Definition 5.5. *(i) For each $\alpha \in$ Ord let $\hat{V}_\alpha$ be the unique (crisp) set z such that* $\exists f(\mathrm{ItWP}(f) \,\&\, \alpha \in \mathrm{Dom}(f) \,\&\, f(\alpha) = z)$.
(ii) Axiom of iterated weak power: $(\forall x)(\exists\alpha)(x \in \hat{V}_\alpha)$ *(IWP)*

Theorem 5.6. FST + (TI) *proves* (IWP).

Proof. Let $\varphi(x)$ stand for $(\exists\alpha)(x \in \hat{V}_\alpha)$. Clearly, $\varphi(\emptyset)$ holds. The following is a provable chain of implications in FST:
$(\forall^{\Delta} y \in x)(\exists\alpha)(y \in \hat{V}_\alpha) \to^{\Delta} (\exists z)(\forall^{\Delta} y \in x)(\exists\alpha \in z)(y \in \hat{V}_\alpha) \to^{\Delta}$
$\to^{\Delta} (\exists z \subseteq \mathrm{Ord}, \mathrm{Crisp})(\forall^{\Delta} y \in x)(\exists\alpha \in z)(y \in \hat{V}_\alpha) \to^{\Delta}$
$\to^{\Delta} (\exists\beta)(\forall^{\Delta} y \in x)(y \in \hat{V}_\beta) \to^{\Delta}$
$\to^{\Delta} (\exists\beta)(\Delta(x \subseteq \hat{V}_\beta)) \to^{\Delta}$
$\to^{\Delta} (\exists\beta)(x \in \hat{V}_{\beta+1})$.
This gives (IWP) by Titani's induction. □

Lemma 5.7. *FST+(IWP) proves $\forall x\exists\alpha(\alpha$ least s. t. $x \in \hat{V}_\alpha)$.*

Proof. Given x, let β be such that $x \in \hat{V}_\beta$ and let $z = \{\alpha | \alpha \leq \beta \,\&\, x \in \hat{V}_\alpha\}$; x is a crisp non-empty set of ordinals and hence it has a least element. □

Definition 5.8. *For each x, $\tau(x)$ is the least α s. t. $x \in \hat{V}_\alpha$.*

Lemma 5.9. *FST+(IWP) proves $y \in x \to \tau(y) < \tau(x)$.*

Proof. Let $\tau(x) = \alpha + 1$, thus $x \in WP(\hat{V}_\alpha)$, $\Delta(x \subseteq \hat{V}_\alpha)$, thus $y \in x \to y \in \hat{V}_\alpha \to \tau(y) \leq \alpha \to \tau(y) < \tau(x)$. □

Theorem 5.10. *FST+(IWP) proves Titani's $\in$-induction.*

Proof. We want to prove $\forall x[\forall y(y \in x \to^\Delta \varphi(y)) \to^\Delta \varphi(x)] \to^\Delta \forall x \varphi(x)$. Since both the antecedent and the succedent are crisp, we may prove by contradiction. Indeed,

$$
\begin{array}{ll}
(\exists x)\neg\Delta\varphi(x) \to^\Delta & \text{(smallest type)} \\
\to^\Delta (\exists x)[\neg\Delta\varphi(x) \,\&\, (\forall y)(\tau(y) < \tau(x) \to \Delta\varphi(y)] \to^\Delta & \text{(crispness of } \tau, <) \\
\to^\Delta (\exists x)[\neg\Delta\varphi(x) \,\&\, (\forall y)\Delta(\tau(y) < \tau(x) \to \Delta\varphi(y)] \to^\Delta & \text{(Lemma 5.9)} \\
\to^\Delta (\exists x)[\neg\Delta\varphi(x) \,\&\, (\forall y)\Delta(y \in x \to \varphi(y))] \to^\Delta & \\
\to^\Delta \neg\forall x(\forall y\Delta(y \in x \to \varphi(y)) \to^\Delta \varphi(x)). &
\end{array}
$$

□

6 Concluding remarks

It remains open whether (IWP) is provable in FST as it stands (without strengthening $\in$-induction). In any case, Titani's induction (TI) can be adopted as an additional axiom in FST (instead of the current axiom of $\in$-induction), on the ground of its validity in $V^{\mathbf{L}}$. Clearly, both FST and FST + (IWP) can be seen as conservative extensions of ZF understood as the theory of hereditarily crisp sets (by adding the law of the excluded middle, the logic collapses to classical and FST becomes ZF). Likewise, the axiom (GA10)*—Zorn's lemma—can be adopted as an additional axiom in FST. Adopting these two additional axioms is a final step toward a full interpretation of LZFZ in FST.

Titani has proved a completeness of LZFZ with respect to the lattice-valued universe in her paper; an analogous result for FST is still to be proved. Needless to say, comparison with Titani's works is extremely interesting and inspiring for developing set theory over various fuzzy logics.

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