A set theory within fuzzy logic

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Abstract

This paper proposes a possibility of developing an axiomatic set theory, as first-order theory within the framework of fuzzy logic in the style of Hájek's Basic fuzzy logic BL. In classical Zermelo-Fraenkel set theory, we use an analogy of the construction of a Boolean-valued universe—over a particular algebra of truth values—to show the non-triviality of our theory. We present a list of problems and research tasks.

1 Introduction

Fuzzy set theory originated with Zadeh's 1965 paper [32] and has been developed since by many authors, with various degrees of insight. It has evolved as an informal, non-axiomatic theory inside the classical informal set theory, a fuzzy subset of a crisp set being identified with its (crisp) characteristic function taking values in the real interval $[0, 1]$. Gradually one began to speak of "fuzzy logic" in a broad sense, and currently the term "fuzzy logic" is being used as synonymous with the term "fuzzy set theory". Recently, however, fuzzy logic has also been developed in a narrow sense as a fully fledged symbolic logical system (cf. [13] for a systematic presentation and historical remarks). Time seems ripe therefore to pose the question of the possibility to develop an axiomatic set theory within the framework of fuzzy logic (in the narrow sense). Any such system ought to be natural, sufficiently general, and should formalize the ways of thinking of those who use the naive fuzzy set theory. This paper is our first attempt to develop such a system—i.e., an axiomatic set theory based on fuzzy logic.

In our work we rely heavily on previous papers developing various systems of set theories as first-order theories in many-valued logics. Let us mention in passing that the first attempt in this area we know about is the work of Skolem and others ([22]-[26], [3]), trying to formulate a many-valued set theory with full comprehension (for each formula $\varphi(x)$ there is a set of all objects satisfying φ). This approach is not directly relevant to our aims. Next there is a series of papers trying to develop a theory generalizing the classical Zermelo-Fraenkel set theory, in a formally weaker logic—intuitionistic ([15], [8]) and later its strengthening presently referred to as Gödel logic ([27], [28], [29]). Finally, a similar approach based on linear logic is described in [21] (a paper which we find particularly inspiring): the author works over the so-called phase spaces as algebras of truth values and builds, using an analogy of the construction of a Boolean-valued universe (here over a phase-space), a class, with class operations evaluating formulas in the language $\{\in, =\}$, in which he verifies the chosen axioms of his set theory.

Having observed that the standard Łukasiewicz algebra enriched with the Δ operator is a particular phase space, we have studied this paper thinking of a more general approach employing (linearly ordered) BL-algebras with Δ; this should form a common basis for the approach of [27] and [28] and that of [21]. Our present proposal sketches the axiomatic system and construes a "BL-algebra-valued" universe, as the canonical "model" of the theory. This universe can give us at least a hint of what is natural to the theory and what is not; for example, it speaks strongly in favour of the axiom of support (cf. first [15]).

We start with the construction of a class V^L, define operations on it, and prove a handful of technicalities. We proceed by formulating the theory (as a set of axioms) and showing its validity in the constructed model. And we formulate a sort of a future research program.

This paper should be viewed as a report on a topic being worked on; the authors hope to carry their research in this area somewhat farther (it being the intended subject of the second author's PhD thesis) by a detailed inspection of the theory and by endowing it with some of the standard set-theoretic tools; besides we hope to establish it as a natural common platform for general discussions on the foundations of theory of fuzzy sets.

Regarding the nature of this paper we omit all the logical background, which is to be found elsewhere, as well as many technical details and some proofs. A more detailed and perhaps enhanced version of this material is to be found in a full paper to follow. Among the references we include all papers we have so far encountered on this subject, some of which (those not referred to in the text) we have *not yet* studied.

Acknowledgement. Partial support by the grant No. IAA1030004 of the Grant Agency of the Academy of Sciences of the Czech Republic is acknowledged.

2 A Model

For a broad description of the basic fuzzy logic BL and its extensions (including the addition of the Δ connective) see [13].

Definition 1 BL$\forall\Delta$ *is the logic given by (propositional and predicate) axioms and inference rules of BL$\forall$, together with the axioms and inference rule for the Δ connective (see [13]).*

Remark 1 For the definition of a BLΔ-algebra see [13].

Consider the classical ZFC. Fix a constant L for an arbitrary linearly ordered complete BL-algebra; write $L = (L, *, \Rightarrow, \wedge, \vee, 0, 1)$ (as usual, L denotes both the algebra and its support). In ZFC let us make the following construction: in analogy to the construction of a Boolean-valued universe over a complete Boolean algebra, we build the class V^L by ordinal induction.

$$V_0^L = \emptyset$$

$$V_{\alpha+1}^L = \{f : \mathrm{Fnc}(f) \,\&\, D(f) \subseteq V_\alpha^L \,\&\, R(f) \subseteq L\}$$

and for limit ordinals λ

$$V_\lambda^L = \bigcup_{\alpha<\lambda} V_\alpha^L$$

Here $\mathrm{Fnc}(x)$ is a unary predicate stating that x is a function, and $D(x)$ and $R(x)$ are unary functions assigning to x its domain and range, respectively.

Finally we put

$$V^L = \bigcup_{\alpha\in\mathrm{On}} V_\alpha^L$$

Observe that for $\alpha \leq \beta$, $V_\alpha^L \subseteq V_\beta^L$.

For any $x \in V^L$ we define $\rho(x) = \min\{\alpha : x \in V_\alpha^L\}$.

Using induction on ρ we define three binary functions from V^L into L, assigning to any tuple $x, y \in V^L$ the values $||x \in y||$, $||x \subseteq y||$ and $||x = y||$ (representing the "truth values" of the three predicates $\in$, $\subseteq$ and $=$):

$$||x \in y|| = \bigvee_{u\in D(y)} (||u = x|| * y(u)),$$

$$||x \subseteq y|| = \bigwedge_{u\in D(x)} (x(u) \Rightarrow ||u \in y||),$$

$$||x = y|| = \Delta||x \subseteq y|| * \Delta||y \subseteq x||.$$

We now use induction on the complexity of formulas to define for any formula $\varphi(x_1, \ldots, x_n)$—the free variables in φ being among $x_1, \ldots, x_n$—a corresponding n-ary function from $(V^L)^n$ into L. The induction steps admit the following cases (we just write φ for short):

φ is $\overline{0}$: then $||\varphi|| = 0$;
φ is $\psi \,\&\, \chi$; then $||\varphi|| = ||\psi|| * ||\chi||$;
φ is $\psi \to \chi$: then $||\varphi|| = ||\psi|| \Rightarrow ||\chi||$;
φ is $\psi \wedge \chi$: then $||\varphi|| = ||\psi|| \wedge ||\chi||$;
φ is $\psi \vee \chi$: then $||\varphi|| = ||\psi|| \vee ||\chi||$;
φ is $\Delta\psi$; then $||\varphi|| = \Delta||\psi||$:
φ is $\forall x\psi$: then $||\varphi|| = \bigwedge_{u\in V^L} \psi(x/u)$;
φ is $\exists x\psi$: then $||\varphi|| = \bigvee_{u\in V^L} \psi(x/u)$.

Here $\psi(x/u)$ is the result of substituting u for the variable x in ψ. We use the symbols $\wedge$, $\vee$, Δ for both logical connectives and operations in L.

Definition 2 *Let φ be a closed formula. We say that φ is valid in V^L iff $||\varphi = 1||$ is provable in ZFC.*

(As usual, we take closures of formulas containing free variables when considering their validity.)

The rest of this section contains some useful statements that might be of some interest.

Lemma 1 *(1) ZFC proves that for $u, x \in V^L$ the following expressions always have value 1: $u(x) \Rightarrow ||x \in u||$, $||u \subseteq u||$, $||u = u||$*
(2) ZFC proves for $u, v, w \in V^L$:
(i)$||u \in v|| * ||v \subseteq w|| \leq ||u \in w||$
(ii)$||u \subseteq v|| * ||v \subseteq w|| \leq ||u \subseteq w||$
(iii)$||u \subseteq v|| * ||v = w|| \leq ||u \subseteq w||$
(iv)$||u = v|| * ||v \subseteq w|| \leq ||u \subseteq w||$
(v)$||u \in v|| * ||v = w|| \leq ||u \in w||$
(vi)$||u = v|| * ||v = w|| \leq ||u = w||$
(vii)$||u = v|| * ||v \in w|| \leq ||u \in v||$

Proof by transfinite induction on ρ: (1) clear,
(2) (i) $||u \in v|| * ||v \subseteq w|| =$
$(\bigvee_{z\in D(v)}(||z = u|| * v(z))) * (\bigwedge_{y\in D(v)}(v(y) \Rightarrow ||y \in w||)) =$
$\bigvee_{z\in D(v)}(||z = u|| * v(z) * \bigwedge_{y\in D(v)}(v(y) \Rightarrow ||y \in w||)) \leq$
$\bigvee_{z\in D(v)}(||z = u|| * v(z) * (v(z) \Rightarrow ||z \in w||)) \leq$
$\bigvee_{z\in D(v)}(||z = u|| * ||z \in w||) =$
$\bigvee_{z\in D(v)}(||z = u|| * (\bigvee_{x\in D(v)}(||z = x|| * w(x))) =$
$\bigvee_{z\in D(v)} \bigvee_{x\in D(w)}(||z = u|| * ||z = x|| * w(x)) \leq$
$\bigvee_{x\in D(w)}(||x = u|| * w(x)) = ||u \in w||$

(ii) $\bigwedge_{x\in D(u)}(u(x) \Rightarrow ||x \in v||) * ||v \subseteq w|| \leq$
$\bigwedge_{x\in D(u)}(u(x) \Rightarrow (||x \in v|| * ||v \subseteq w||)) \leq$
$\bigwedge_{x\in D(u)}(u(x) \Rightarrow ||x \in w||) = ||u \subseteq w||$

(iii) $||u \subseteq v|| * ||v \subseteq w|| \leq ||u \subseteq w||$
$||u \subseteq v|| * \Delta||v \subseteq w|| \leq ||u \subseteq w||$
$||u \subseteq v|| * \Delta||v \subseteq w|| * \Delta||w \subseteq v|| \leq ||u \subseteq w||$

(iv) $||u \subseteq v|| * ||v \subseteq w|| \leq ||u \subseteq w||$
$\Delta||u \subseteq v|| * ||v \subseteq w|| \leq ||u \subseteq w||$
$\Delta||u \subseteq v|| * \Delta||v \subseteq u|| * ||v \subseteq w|| \leq ||u \subseteq w||$

(v) $||u \in v|| * ||v \subseteq w|| \leq ||u \in w||$
$||u \in v|| * \Delta||v \subseteq w|| \leq ||u \in w||$
$||u \in v|| * \Delta||v \subseteq w|| * \Delta||w \subseteq v|| \leq ||u \in w||$

(vi) $||u \subseteq v|| * ||v \subseteq w|| * ||v \subseteq u|| * ||w \subseteq v|| \leq ||u \subseteq w|| * ||w \subseteq u||$
$\Delta||u \subseteq v|| * \Delta||v \subseteq w|| * \Delta||v \subseteq u|| * \Delta||w \subseteq v|| \leq \Delta||u \subseteq w|| * \Delta||w \subseteq u||$

(vii) $||u = v|| * ||v \in w|| =$
$||u = v|| * \bigvee_{x\in D(w)}(||x = v|| * w(x)) =$
$\bigvee_{x\in D(w)}(||x = v|| * w(x) * ||u = v||) \leq$
$\bigvee_{x\in D(w)}(||x = u|| * w(x)) = ||u \in w||$

Lemma 2 (Substitution) *For any formula φ, ZFC proves: $\forall u, v \in V^L$, $||u = v|| * ||\varphi(u)|| \leq ||\varphi(v)||$.*

Proof by induction on the complexity of φ, using the previous lemma for φ atomic.

Lemma 3 (Bounded quantifiers) *$\forall x \in V^L$:*
$||\exists y(y \in x \,\&\, \varphi(y)|| = \bigvee_{y\in D(x)}(x(y) * ||\varphi(y)||)$;
$||\forall y(y \in x \to \varphi(y))|| = \bigwedge_{y\in D(x)}(x(y) \Rightarrow ||\varphi(y)||)$

Corollary 1 $||x \subseteq y|| = ||\forall u \in x(u \in y)|| = ||\forall u(u \in x \to u \in y)||$.

Remark 2 (1) We use the following auxiliary statements about BLΔ-algebras. Suppose $A, B, A_i, B_i, \ldots$ are (sequences of) elements of the algebra. Then:
(i) $\bigvee_i A_i * B = \bigvee(A_i * B)$
(ii) $\bigwedge_i A_i * B \leq \bigwedge(A_i * B)$
(iii) $\bigwedge_i(A_i \Rightarrow B_i) * C \leq \bigwedge_i(A_i \Rightarrow (B_i * C))$

(2) About linearly ordered complete BLΔ-algebras. Particular examples are the algebras on the real unit interval given by a continuous t-norm (these have the standard order of reals), among them the three famous ones given by Łukasiewicz, Gödel, and product t-norms. Known properties of linearly ordered BL-algebras ([2]) give the following: each completely linearly ordered (c. l. o.) BL-algebra is an ordered sum of copies of c. l. o. MV-algebras, G-algebras, and product algebras, and the latter are as follows (the degenerated one-element algebra disregarded):
- c. l. o. MV-algebras: the standard Łukasiewicz algebra on $[0, 1]$ and its finite n-element subalgebras, $n \geq 1$;
- c. l. o. G-algebras: any complete linear order determines a G-algebra;
- c. l. o. product algebras: the standard product algebra on $[0, 1]$, the two-element Boolean algebra, and the restriction of the standard product algebra to $\{0\} \cup \{(1/2)^n; n \geq 0\}$.

3 The theory FST

We introduce the theory FST in the language $\{\in, =\}$; we prefer to include $=$ among the basic predicates although it is definable via the axiom of extensionality, to be able to use shorthand formulations of some other axioms. Note in the following that ours is a "crisp" equality, i.e. admits only 0-1 interpretations in the (potential) models. This is forced by the fact that e.g. in Łukasiewicz and product logic, the axiom of separation together with equality axioms imply crispness of $=$ anyway. Moreover, under the "standard" formulation of extensionality, crispness of $=$ implies crispness of $\in$ (cf. [9]).

Definition 3 *FST is a first order theory in the language $\{\in, =\}$, with the following axioms:*
(i) (extensionality) $\forall x\forall y(x = y \equiv (\Delta(x \subseteq y) \,\&\, \Delta(y \subseteq x)))$
(ii) (empty set) $\exists x\Delta\forall y(y \notin x)$
(iii) (pair) $\forall x\forall y\exists z\Delta\forall u(u \in z \equiv (u = x \vee u = y))$
(iv) (union) $\forall x\exists z\Delta\forall u(u \in z \equiv \exists y(u \in y \,\&\, y \in x))$
(v) (infinity)$\exists z\Delta(\emptyset \in z \,\&\, \forall x \in z\exists y \in z(x \subseteq y \,\&\, x \in y))$
(vi) (separation) $\forall x\exists z\Delta\forall u(u \in z \equiv (u \in x \,\&\, \varphi(u, x))$, for any formula not containing z as a free variable
(vii) (collection) $\forall u \in x\exists z\,\varphi(u, z) \to \exists y\Delta\forall u \in x\exists z \in y\varphi(u, z))$ for any formula not containing y as a free variable
(viii) (crisp power) $\forall x\exists z\Delta\forall u(u \in z \equiv \Delta(u \subseteq x))$
(ix) (support) $\forall x\exists z(\mathrm{Crisp}(z) \,\&\, \Delta x \subseteq z))$,

where $\mathrm{Crisp}(x) \equiv \forall y\Delta(x \in y \vee \neg x \in y)$.

V^L is a model of ST We proceed by verifying that all axioms of FST (including logical axioms) are valid in V^L and that inference rules preserve validity. Having proved this we take the liberty to call V^L a *model* of ST. (More pedanticly, we have constructed an interpretation of FST (over $\mathcal{C}$) in ZFC.)

We omit all validity proofs for logical axioms and inference rules.

Lemma 4 *The set-theoretical axioms (i)-(ix) hold in V^L.*

We present the validity proof for some axioms:
(empty set)
Take an arbitrary $x \in V^L$ such that $\forall z \in D(x)\, x(z) = 0$. Then for any $y \in V^L$ we have $||y \in x|| = \bigvee_{z \in D(x)}(||y = z|| * x(z)) = 0$ and $||\neg y \in x|| = 1$. (Note that if x_1 and x_2 satisfy the above condition then $||x_1 = x_2|| = 1$, thus the empty set is unique modulo equality.)

(pair)
Let $x, y \in V^L$ be given. Set $\alpha = \max(\rho(x), \rho(y))$; then there is a $z \in V^L_{\alpha+1}$ such that $D(z) = \{x, y\}$ and $z(x) = z(y) = 1$. The set z has the desired properties: for arbitrary $u \in V^L$ we have $||u \in z|| = \bigvee_{v \in D(z)}(||v = u|| * z(v)) = ||x = u|| \vee ||y = u||$.

(union)
Define $D(z) = \bigcup_{y \in D(x)} D(y)$, and $z(u) = \bigvee_{\{y \in D(x): u \in D(y)\}}(x(y) * y(u))$. Then for $u \in D(y)$ and $y \in D(x)$ we have $x(y) * y(u) \leq z(u) \leq ||u \in z||$, therefore $\forall y \in D(x) \forall u \in D(y)(x(y) \leq y(u) \Rightarrow ||u \in z||)$. Then also $\forall y \in D(x)(x(y) \leq \bigwedge_{u \in D(y)}(y(u) \Rightarrow ||u \in z||))$. By a similar treatment, $1 \leq \bigwedge_{y \in D(x)}(x(y) \Rightarrow \bigwedge_{u \in D(y)}(y(u) \Rightarrow ||u \in z||))$, thus $||\forall y \in x \forall u \in y(u \in z)|| = 1$, and the obtained formula is $\mathcal{C}$-equivalent to the desired axiom.

(separation)
Suppose x is given. Define z so that $D(z) = D(x)$ and for $u \in D(z)$ set $z(u) = x(u) * ||\varphi(u)||$. It is enough to verify the validity of two implications: first we show, $||\forall u(u \in z \to (u \in x \,\&\, \varphi(u)))|| = 1$. The expression on the left is by Lemma 3 equal to $\bigwedge_{u \in D(z)} z(u) \Rightarrow (||u \in x|| * ||\varphi(u)||)$; this evaluates to 1 since by definition $z(u) = x(u) * ||\varphi(u)||$ and $x(u) \leq ||u \in x||$ for all $u \in D(z)$. The converse implication is $||\forall u((u \in x \,\&\, \varphi(u)) \to u \in z)|| = 1$; the expression on the left is by Lemma 3 equal to $\bigwedge_{u \in D(x)}(x(u) \Rightarrow (||\varphi(u)|| \Rightarrow ||u \in z||))$, which is $\bigwedge_{u \in D(x)}(x(u) * ||\varphi(u)|| \Rightarrow ||u \in z||)$, which is 1 since $\forall u \in D(x)\; x(u) * ||\varphi(u)|| = z(u)$ and $z(u) \leq ||u \in z||$.

(support)
For an arbitrary x take z such that $D(z) = D(x)$ and $z(u) = 1$ for all $u \in D(z)$. Then $||\mathrm{Crisp}(z)|| = 1$ and $\bigwedge_{u \in D(x)}(x(u) \Rightarrow ||u \in z||) = 1$ since $x(u) \leq z(u)$ and $z(u) \leq ||u \in z||$.

4 Conclusion

Let us stress one more time that we describe just a project of an axiomatic fuzzy set theory in its beginning, and that Shirahata's paper [21] has been used heavily. Trivially, FST is relatively consistent with respect to SFC since the former is weaker than the latter: strengthening the logic to Boolean our theory is a subtheory of ZFC in the classical sense. Our model shows that FST (in the fuzzy logic) is non-trivial: it does *not* prove *tertium non datur* and admits existence of non-crisp sets.

We now give a list of our task and questions:

(1) Is the interpretation of FST over $\mathcal{C}$ in ZFC faithful, i.e., is it true that FST $\vdash \varphi$ iff ZFC $\vdash ||\varphi|| = 1$?

(2) Which additional axioms might/should be added? Which must *not* be added since they imply crispness of everything? (Several examples of the latter are known, see e.g. [8].)

(3) How far can we depart from ZFC, e.g. allowing more comprehension, partially accepting Skolem's program?

(4) Inside FST, can we prove the existence of enough hereditarily crisp sets? Do they model the classical ZFC?

(5) In an analogy to [8, 18], investigate the crisp power $P(1)$ of $1 = \{0\}$ and show that it bears the structure of a linearly ordered BL-algebra.

(6) How do we define basic mathematical structures (natural, real, ordinal, cardinal numbers etc.) in FST?

(7) Which (if any) interesting facts follow from the choice of a particular underlying logic, especially Łukasiewicz, Gödel, and product logic?

(8) Is it possible to formalize facts about fuzzy control (and similar applications of the theory of fuzzy sets) in FST?

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