On the complexity of propositional logics with an involutive negation

Zuzana Haniková
Institute of Computer Science
Czech Academy of Sciences
182 07 Prague, Czech Republic
zuzana@cs.cas.cz

Abstract

Let C be the propositional calculus given by a standard SBL-algebra; $C_\sim$ is obtained from C by adding an involutive negation, with axioms and deduction rules as in [4]. Then $C_\sim$ is coNP-complete.

Keywords: Basic Fuzzy Logic (BL), Strict Basic Fuzzy Logic (SBL), involutive negation, $\sim$-standard completeness, computational complexity

1 Introduction

This work studies the computational complexity of certain type of propositional calculi with an involutive negation. All of the calculi are extensions of the Basic Fuzzy Logic (BL), introduced by Hájek and investigated minutely in [6], [7], [3], and elsewhere. It was shown in [3] that the propositional logic BL is complete w. r. t. t-tautologies, the formulas which are tautologies of all *standard* algebras (BL-algebras on the real interval [0,1], given by continuous t-norms, also called t-algebras). By the main result of [2], the propositional BL is coNP-complete.

Three important schematic extensions of BL, namely the Łukasiewicz, Gödel, and product logics, have been investigated thoroughly. Among other results, propositional tautologies of each of these calculi are coNP-complete ([7] gives references).

In general, [8] shows that the logic of an arbitrary standard algebra is coNP-complete. Recently a finite axiomatization of any such logic has been given in [5].

The focus of this paper is the Strict Basic Fuzzy Logic (SBL) and its schematic extensions, endowed with an involutive negation as an additional connective. SBL, the Gödel logic, and the product logic with an involutive negation have been investigated in [4].

We consider calculi C of individual standard SBL-algebras, with an involutive negation $\sim$; the calculus (given by axioms and rules from [4]) is denoted $C_\sim$. Section 2 goes briefly over relevant results of [4], adding a generalization of their standard completeness result for the product logic with involutive negation to any calculus $C_\sim$. As a main result of this paper we prove in Section 3 that any $C_\sim$ is coNP-complete.

2 Logics with an involutive negation

We assume the reader is familiar with BL-algebras and the propositional BL; [7] is a suitable source of information.

SBL is an extension of BL introduced in [4]; the *chains* (linearly ordered algebras) in the variety generated by SBL are exactly those chains whose *defined negation* function is *strict*, i. e., $\neg x = 1$ for $x = 0$, otherwise $\neg x = 0$. One way to achieve this is to add the schema $\varphi \wedge \neg\varphi \to 0$ (the axiom $\Pi 2$) to BL. Note that SBL does not expand the language of BL. Standard completeness of SBL follows from [3].

The language of SBL is expanded with a unary connective $\sim$. The standard semantics (on [0,1]) is a decreasing involution, i. e., a function $\sim: [0,1] \longrightarrow [0,1]$, s. t. for all $x, y \in [0,1]$, $\sim\sim x = x$ and if $x \leq y$, then $\sim y \leq \sim x$. A particular example of a decreasing involution is the function $1 - x$ on $[0,1]$. (In the standard Łukasiewicz algebra $[0,1]_{\mathrm{L}}$, this is the truth function of the definable negation $\neg x = x \Rightarrow 0$.) As the interplay of the two negations is best observed when the defined negation is strict, we restrict our attention to extensions of SBL.

Interestingly, in case $\sim$ is introduced to SBL-chains, the Δ-projection is definable: Δx is $\neg \sim x$ (cf. [4], p. 109).

We give axioms and a deduction rule for the new connective $\sim$, using also Δ as shortcut (cf. [4], Definition 3).

Definition 1 *Axioms of* $\mathrm{SBL}_\sim$ *are the axioms of SBL plus*

($\sim$1) $\sim\sim \varphi \equiv \varphi$

($\sim$2) $\neg\varphi \to \sim \varphi$

($\sim$3) $\Delta(\varphi \to \psi) \to \Delta(\sim \psi \to \sim \varphi)$

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$

(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

Deduction rules of $\mathrm{SBL}_\sim$ *are modus ponens and necessitation for* Δ*: from* φ *derive* $\Delta\varphi$.

The calculus SBL with $\sim$ will be denoted $\mathrm{SBL}_\sim$.

Definition 2 *An* $\mathrm{SBL}_\sim$*-algebra is an algebra* $\mathbf{L} = \langle L, \cap, \cup, *, \Rightarrow, \sim, 0, 1\rangle$*, where* $\langle L, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ *is an* SBL*-algebra,* $\sim$ *is unary, and the axioms (*$\sim$1*)–(*$\sim$3*), (*Δ1*), (*Δ2*), (*Δ5*) are* $\mathbf{L}$*-tautologies.*

[4] proves subdirect representation for $\mathrm{SBL}_\sim$-algebras and, consequently, the following completeness theorem:

Theorem 1 (Completeness theorem for $\mathrm{SBL}_\sim$**)** *For a formula in the language of* $\mathrm{SBL}_\sim$*, the following are equivalent:*

(i) $\mathrm{SBL}_\sim \vdash \varphi$

(ii) φ *is an* $\mathbf{L}$*-tautology for each* $\mathrm{SBL}_\sim$*-algebra* $\mathbf{L}$

(iii) φ *is an* $\mathbf{L}$*-tautology for each* $\mathrm{SBL}_\sim$*-chain* $\mathbf{L}$

Let C be a schematic extension of propositional SBL (possibly void). By $C_\sim$ we denote the calculus of $\mathrm{SBL}_\sim$ plus the additional schemata of C. Subdirect representation theorem for $C_\sim$ is immediate (factor algebras of a $C_\sim$-algebra are $C_\sim$-algebras).

Two examples of standard completeness results: $\mathrm{G}_\sim$ (Gödel logic with involutive negation) has in [4] been proved complete w. r. t. the standard Gödel algebra with $1-x$ for the involutive negation. (In fact, all standard Gödel algebras with involutive negation are isomorphic). [4] also shows that the calculus $\Pi_\sim$ fails to capture the tautologies of $[0,1]_\Pi$ with $1-x$. For $\Pi_\sim$ it obtains completeness w. r. t. all algebras obtained from standard product by adding *any* decreasing involution. These algebras are called *semi-standard* $\Pi_\sim$-algebras in [4]. The result is due to the following lemma (cf. [4], Lemma 10):

Lemma 1 *Let* $0 < a_0 < \ldots < a_k < 1$ *be reals. Then there is a decreasing involution* $\sim$ *on* $[0,1]$ *such that* $\sim a_i = a_{k-i}$ *for* $i = 0, \ldots, k$.

In fact distinguishing between $1-x$ and other decreasing involutions on $[0,1]$ means also distinguishing between $[0,1]_\Pi$ and its isomorphic copies. But this is only possible when considering a particular algebra (e.g., the standard Π-algebra). When speaking of, e.g., a standard algebra of type $Ł \oplus \Pi$, or even of a class of algebras like SBL, one usually only considers the *type* of the sum. An attempt to generalize the concept of semi-standardness to, say, SBL, will fail: any standard SBL-algebra with an arbitrary decreasing involution has an isomorphic standard SBL copy with $1-x$.

In this paper we use the terms (suggested by an anonymous referee) $\sim$*-standard algebras* and $\sim$*-standard completeness*. A $C_\sim$-algebra is $\sim$-standard iff its reduct to the BL language is a standard C-algebra (the decreasing involution may be arbitrary).

It turns out that one can easily prove $\sim$-standard completeness for any $C_\sim$ provided C is generated by a standard algebra. The following theorem combines the results of [4] with those of [1] and [5].

Theorem 2 *Let* C *be the logic of some standard algebra* $\mathbf{A}$*. Then* $C_\sim$ *is* $\sim$*-standard complete: provable formulas are exactly tautologies of all* $\sim$*-standard* $C_\sim$*-algebras.*

Proof: Assume φ is a formula in the language of $C_\sim$, but not provable in $C_\sim$. Then there is a $C_\sim$-chain $\mathbf{L}$ in which φ does not hold; fix an evaluation e in

$\mathbf{L}$ s. t. $e(\varphi) < 1$. Let $a_1, \ldots, a_m \in L$ be a complete evaluation of φ under e.

If $\sim a_i = a_j$ for some $1 \leq i, j \leq m$, then we say (a_i, a_j) form a pair. Note that all pairs defined by φ and e in $\mathbf{L}$ are *nested*, i.e., for any two pairs (a_i, a_j) and (a_k, a_l) if $a_i < a_k$, then $a_l < a_j$.

We need to embed the finite partial subalgebra on $\{0,1\} \cup \{a_1, \ldots, a_m\}$, with operations $*$, $\Rightarrow$ and 0 only, into $\mathbf{A}$. To this end, recall the decomposition of any BL-chain into a sum of linearly ordered Wajsberg hoops ([1]). Considering a finite subsum $\mathbf{L}_1 \oplus \ldots \oplus \mathbf{L}_{m'}$ of $\mathbf{L}$ (all hoops which host the finite partial subalgebra), we may find a corresponding subsum of hoops in $\mathbf{A}$ such that: if $\mathbf{L}_i$ is an MV-algebra, then $\mathbf{A}_i$ is the standard Ł-algebra, if $\mathbf{L}_i$ is cancellative then $\mathbf{A}_i$ is either the cancellative hoop which is the semi-open interval of the standard Π-algebra, or the standard Ł-algebra, and if $\mathbf{L}_i$ is 2 then $\mathbf{A}_i$ is 2 or the standard Ł-algebra. Thus $\mathbf{A}$ embeds the finite partial algebra.

Thanks to Lemma 1 there is a decreasing involution in $\mathbf{A}$ which heeds the pairing of values prescribed by φ. Thus $\mathbf{A}$ offers a complete evaluation of φ which gives value less that 1. QED

Note that SBL is complete w. r. t. individual standard algebras, so this theorem also says that $\mathrm{SBL}_\sim$ is $\sim$-standard complete.

This means that we may work with $\sim$-standard tautologies of some standard algebra and results will carry over to the calculi.

3 Main result

Observation 1

(i) If C is coNP-hard, then so is $C_\sim$.

(ii) Any schematic extension of SBL is coNP-hard.

The main task is to show containment in coNP for any $C_\sim$ in case C is the logic of some standard SBL-algebra. (Note that this includes SBL.)

Theorem 3 *Let C be the logic of some standard SBL-algebra. Then $C_\sim$ is in coNP.*

The proof of this statement relies on the coNP completeness result for C, presented in [8], and the task is to take into account the added connective $\sim$. Here we present the simplest case of the algorithm, considering an extension $C_\sim$ where C is given by a standard SBL-algebra $\mathbf{A}$ which is a *finite* ordinal sum of Ł-, G-, and Π-segments (in the sense of the Mostert-Shields representation theorem of [9]).

Fix an algebra $\mathbf{A}$ which is a finite sum of Ł-, G-, and Π-segments. Let n be its cardinality (the number of segments in the sum). For a propositional formula φ, denote $m := |\varphi|$, the number of subformulas in φ.

The following algorithm has built-in information about the algebra $\mathbf{A}$ in question. Each input is a propositional φ (in the language of $\mathrm{SBL}_\sim$). The output is 'yes' iff this formula is not a tautology of $\mathbf{A}$, that is, iff there is an evaluation e in $\mathbf{A}$ s. t. $e(\varphi) < 1$; otherwise the output is 'no'.

We claim that the algorithm works in polynomial time (in m) on a nondeterministic machine. This is easy to see for all steps but the last one; proof that this step is an NP subroutine has been given in [2] for the standard Łukasiewicz algebra and in [8] for the standard product algebra (it is obvious for the standard Gödel algebra). From this the coNP-containment for the set of A-tautologies follows.

`// algorithm for finite sum A`

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`nameSubformulas()` Fix an arbitrary enumeration of all subformulas of φ; for m being the number of subformulas in φ, introduce variables $x_0 \ldots, x_{m-1}$, and assign the variable x_i to the subformula φ_i of φ.

`cutpointVariables()` Introduce variables $z_0, \ldots, z_n$ for the cutpoints of A, enumerated in such a way that z_i represents that i-th cutpoint of A in the ordering of reals; z_0 is intended for 0 and z_n is intended for 1.

Set $C = \{z_0, \ldots, z_n\} \cup \{x_0, \ldots, x_{m-1}\}$.

`guessOrder()` Guess a linear ordering $\preceq$ of variables in C: write the variables down in a sequence, for each consequent pair a and b decide whether $a \approx b$ or $a \prec b$.

`checkOrder()` Check that $\preceq$ satisfies some basic natural conditions: first, if $i < j$, then $z_i < z_j$ (i.e., the natural strict ordering of the z-variables (given by the cutpoints they represent) must be preserved); second, any variable assigned to the constant 0 must be $\approx$-equal to z_0, the variable denoting the smallest cutpoint.

`checkInvolution()` for φ_i, φ_j subformulas of φ, where $i,j \in \{0,\ldots,m-1\}$, if φ_i is $\sim \varphi_j$, put down a pair $\{x_i,x_j\}$. Do this for every occurrence of $\sim$ in φ. Check that the pairs are *mutually exclusive* (up to $\approx$) and that they are *nested*.

`checkExternal()` Check external soundness of $\preceq$: for φ_i, φ_j subformulas of φ ($i,j \in \{0,\ldots,m-1\}$),

– if $\varphi_i \& \varphi_j$ is a subformula φ_k of φ for some $k \in \{0,\ldots,m-1\}$ and, for some $l \in \{0,\ldots,m-1\}$, we have $x_i \preceq z_l \preceq x_j$, then $x_k \approx x_i$;

– if $\varphi_i \to \varphi_j$ is a subformula φ_k of φ for some $k \in \{0,\ldots,m-1\}$ and $x_i \preceq x_j$, then $x_k \approx z_n$;

– if $\varphi_i \to \varphi_j$ is a subformula φ_k of φ for some $k \in \{0,\ldots,m-1\}$ and for some $l \in \{0,\ldots,m-1\}$, we have $x_j \prec z_l \preceq x_i$, then $x_k \approx x_j$.

`checkInternal()` Check internal soundness of $\preceq$ for each interval $[z_i,z_{i+1}]$, $i = 0,\ldots,n-1$ in $\preceq$. Consider variables in i. Construct a system $\mathcal{S}_i$ of equations and inequalities; $\mathcal{S}_i$ is intially empty. For each subformula φ_l which is $\varphi_j\&\varphi_k$ (or $\varphi_j \to \varphi_k$), if x_j and x_k are in i, check x_l is also in i and put equation $x_j * x_k = x_l$ (or $x_j \Rightarrow x_k = x_l$ respectively) into $\mathcal{S}_i$. Further, put all equations and inequalities defined by $\preceq$ for the variables in i into $\mathcal{S}_i$. Check whether the system $\mathcal{S}_i$ has a solution in the i-th segment of A, such that z_i and z_{i+1} evaluate to the cutpoints delimiting the i-th segment of A.

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It is worth noticing that the soundness of $\preceq$ w. r. t. the identities given by φ can be, in case of $\sim$, checked syntactically and independently of the check of soundness for $*$ and $\Rightarrow$. This is due to Lemma 1, which says that on a basis of a set of pairs of mutually negated elements assembled in the `checkInvolution()` step, one can always find a suitable involutive negation function on $\mathbf{A}$.

[8] proves that there are only countably many varieties (propositional calculi) generated by standard algebras. A modification of the above algorithm for infinite ordinal sums (and for SBL) with involutive negation will be obtained by adding steps as in [8].

The algorithm does not use the assumption that the algebra $\mathbf{A}$ is an SBL-algebra; it could be any standard algebra. Then the set of formulas in question would be the tautologies of all $\sim$-standard algebras obtained from $\mathbf{A}$. We prefer working with SBL-algebras with involutive negations because there are nice calculi for them.

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References

[1] P. Aglianò and F. Montagna. Varieties of BL-algebras: general properties. to appear in Journal of Pure and Applied Algebra, 2003.

[2] Matthias Baaz, Petr Hájek, Franco Montagna, and Helmut Veith. Complexity of t-tautologies. *Ann. Pure Appl. Logic*, 113(1-3):2–11, 2002.

[3] Roberto Cignoli, Francesc Esteva, Lluís Godo, and Antoni Torrens. Basic Fuzzy Logic is the logic of continuous t-norms and their residua. *Soft Comput.*, 4:106–112, 2000.

[4] Francesc Esteva, Lluís Godo, Petr Hájek, and Mirko Navara. Residuated fuzzy logic with an involutive negation. *Archive for Mathematical Logic*, 39:103–124, 2000.

[5] Francesc Esteva, Lluís Godo, and Franco Montagna. Equational characterization of the subvarieties of BL generated by t-norm algebras. draft.

[6] Petr Hájek. Basic fuzzy logic and BL-algebras. *Soft Comput.*, 2:124–128, 1998.

[7] Petr Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 1998.

[8] Zuzana Haniková. A note on the complexity of propositional tautologies of individual t-algebras. *Neural Network World*, (5):453–460, 2002.

[9] P. S. Mostert and A. L. Shields. On the structure of semigroups on a compact manifold with boundary. *Annals of Math.*, 65:117–143, 1957.