Distinguishing standard SBL-algebras with involutive negations by propositional formulas

Zuzana Haniková and Petr Savický
Institute of Computer Science, AS CR
182 07 Prague, Czech Republic,
Email: {zuzana,savicky}@cs.cas.cz

Abstract

Propositional fuzzy logics given by a combination of a continuous SBL t-norm with finitely many idempotents and of an involutive negation are investigated. A characterization of continuous t-norms which, in combination with different involutive negations, yield either isomorphic algebras or algebras with distinct and incomparable sets of propositional tautologies is presented.

1 Introduction

The aim of this section is to present the contents of this paper in the broader context of already existing results on the topic. For basic definitions, statements, and terminology used in this paper, the reader is recommended to consult Section 2.

1.1 Context and previous results

The paper [5] introduces the Strict Basic (Fuzzy) Logic SBL, the propositional logic given by standard BL-algebras[1] in which the negation definable in BL is the *strict*, also referred to as the Gödel, negation. It then investigates the expansions of SBL with a new connective $\sim$, whose axioms warrant that the interpretation on $[0,1]$ is a decreasing involution. The propositional calculus SBL$\sim$, consisting of axioms of the logic SBL and axioms for $\sim$, is shown to be complete w. r. t. the class of standard SBL-algebras with arbitrary involutive negations[2]. For any schematic extension **L** of SBL given by some standard algebra, the logic **L**$\sim$ is complete w. r. t. the class of standard algebras determined by **L** combined with arbitrary involutive negations; this is shown in [12].

The function $1-x$ is considered a prominent interpretation of $\sim$. [5] shows that G$\sim$ is complete w. r. t. the standard Gödel algebra $[0,1]_{\mathrm{G}}$ combined with $1-x$ (since all algebras for G$\sim$ on $[0,1]$ are isomorphic), while $\Pi\sim$ is shown not to be complete w. r. t. the standard product algebra $[0,1]_\Pi$

[1] i.e., BL-algebras on the real unit interval, $[0,1]$

[2] We use the term "involutive negation" and the notation $\sim$ for both the propositional connective and its interpretation on $[0,1]$

with $1-x$. The question of how to axiomatize $[0,1]_\Pi$ with $1-x$ has not been addressed in the paper [5].

Independently, the logic ŁΠ has been developed, starting with the paper [6]; this logic has the full expression power of both Łukasiewicz logic and product logic, it moreover includes Gödel logic and the Δ-projection; it has been shown in [3] that Łukasiewicz implication is definable in $\Pi\sim$, and adding an axiom stating its transitivity to $\Pi\sim$ yields an alternative complete axiomatics for the logic ŁΠ, which is, in fact, also a complete finite axiomatization for $[0,1]_\Pi$ with $1-x$. The standard product algebra with $1-x$, and the standard Gödel algebra with $1-x$, are apparently the only two instances of standard SBL$\sim$-algebras for which a complete axiomatization has been found.

In a recent paper [4] the authors propose a countable set of identities involving $\sim$ and investigate the lattice of varieties generated by (standard) SBL$\sim$-algebras for which various sets of these identities hold. They show that in the case of standard $\Pi\sim$-algebras, the lattice of subvarieties is of infinite height and width.

For the particular case of standard product algebra with involutive negations a stronger result on the width of the lattice is obtained in [8]. There, algebras on $[0,1]$ with the lattice operations, product t-norm, and arbitrary involutive negation are investigated. The result is that non-isomorphic algebras of this type give rise to incomparable sets of identities in the algebraic language, thus they generate incomparable varieties. The result is relevant also for residuated logics, and implies that the lattice of subvarieties is of uncountably infinite width.

1.2 Contents of this paper

Combining a continuous SBL t-norm $*$ with an involutive negation $\sim$ results in the algebra $\langle[0,1],*,\Rightarrow,0,\sim\rangle$, which will be shortly denoted $\langle *,\sim\rangle$. Each such algebra gives a semantics for the logic SBL$\sim$ and determines a set of propositional 1-tautologies in the language &, $\rightarrow$, 0, $\sim$. The set of propositional 1-tautologies of an SBL$\sim$-algebra A is denoted TAUT(A).

This work investigates standard SBL-algebras with arbitrary involutive negations and addresses the question whether for any two such non-isomorphic algebras, their sets of propositional 1-tautologies are different. This question arises naturally from the abovementioned results, having in mind the question of the existence of a finite axiomatization of, or a reasonable complexity bound for, the propositional logics determined by SBL$\sim$-algebras on $[0,1]$.

Definition 1. *Let $*$ be a continuous t-norm and $\sim_1$, $\sim_2$ two involutive negations. The two involutive negations $\sim_1$ and $\sim_2$ are* isomorphic *(w. r. t. the fixed $*$) iff $\langle *,\sim_1\rangle$ is isomorphic to $\langle *,\sim_2\rangle$.*

Any such isomorphism of $\langle *,\sim_1\rangle$ and $\langle *,\sim_2\rangle$ must obviously be an automorphism of $*$, and hence also of its residuum $\Rightarrow$, on $[0,1]$.

The above definition foreshadows the approach adopted in this paper, namely, that of fixing a continuous t-norm and adding arbitrary involutive negations.

Problem. Characterize all continuous t-norms $*$ for which the equivalence

$$\mathrm{TAUT}(\langle *,\sim_1\rangle)=\mathrm{TAUT}(\langle *,\sim_2\rangle) \iff \langle *,\sim_1\rangle \text{ is isomorphic to } \langle *,\sim_2\rangle \tag{1}$$

holds for arbitrary $\sim_1$ and $\sim_2$.

A related problem is to determine for which continuous t-norms $*$ satisfying (1), we additionally have

$$\begin{array}{c}\text{If } \ \mathrm{TAUT}(\langle *, \sim_1\rangle) \neq \mathrm{TAUT}(\langle *, \sim_2\rangle) \ \text{ then} \\ \mathrm{TAUT}(\langle *, \sim_1\rangle) \text{ and } \mathrm{TAUT}(\langle *, \sim_2\rangle) \text{ are incomparable.}\end{array} \tag{2}$$

For all cases considered in the current paper, where (1) is proved, we also have (2).

For simplicity, we distinguish between isomorphic copies of a t-norm because different isomorphic copies of a single t-norm might, in combination with the same involutive negation, yield different sets of tautologies. Therefore a 't-norm' means a particular function, not just a set of mutually isomorphic t-norms, as is often the case when contemplating logics in the language of BL. (In particular, the term 'standard product' is reserved for the particular t-norm which is multiplication of reals on $[0,1]$.) Naturally, however, for each type of ordinal sum it is sufficient to consider one representative in the problems above: validity of (1) and (2) for one t-norm $*$ plus all involutive negations is equivalent to the validity of (1) and (2) for the isomorphic copies of $*$ plus all involutive negations, since for any algebra $\langle *, \sim\rangle$ and any isomorphic copy $*'$ of $*$, there is an algebra $\langle *', \sim'\rangle$ isomorphic to $\langle *, \sim\rangle$.

In full generality the above problem remains open. As a consequence of [8], (1) and (2) are satisfied in the case of the standard product. The main result of the present paper is a characterization of continuous t-norms which satisfy (1) and (2) among SBL t-norms with a finite number of idempotents.[3] In other words, these t-norms consist of finite number of Π- and Ł-components, with Π as the initial component. It turns out that the number and the position of Π-components in the t-norm play a crucial role for the validity of (1) and (2). The results of the present paper extend [8] by investigating a broader class of continuous t-norms and also by giving examples of continuous t-norms which do not satisfy (1). The following theorem is proved:

Theorem 2. *Let $*$ be a finite ordinal sum of Ł- and Π-components, where the first component is Π. Then*

> *(i) If the ordinal sum determining $*$ is Π, $\Pi \oplus j.Ł$, or $\Pi \oplus i.Ł \oplus \Pi \oplus j.Ł$, for $i \geq 0$, $j > 0$, then (1) and (2) holds for $*$.*
>
> *(ii) Otherwise (if $*$ is of type $\Pi \oplus i.Ł \oplus \Pi$ or it contains at least three product components), (1) does not hold for $*$.*

Note that in the above theorem, (ii) occurs iff there is an involutive negation which maps two product components of $*$ onto each other.

Section 2 defines the semantics on $[0,1]$ of the propositional language used in this paper (i. e., &, $\to$, 0, Δ, $\sim$), gives references to the presentations of the axiomatic systems defining the logics BL, SBL, BLΔ, SBL$\sim$ etc., and presents some results concerning automorphisms of standard BL-algebras.

Section 3 presents an alternative proof of (1) and (2) for the case of standard product obtained by a slightly different method and in different logical setting than in [8].

Section 4 addresses continuous SBL t-norms with at least two product components, such that either one is at the beginning of the sum and another one is at its end, or they are both inner

[3] Note that by [7] the sets of propositional tautologies in the language of BL are distinct for each two distinct finite ordinal sums of Łukasiewicz and product components.

components. In any such case, there are non-isomorphic involutive negations which determine the same sets of propositional tautologies.

Section 5 shows that for all SBL t-norms which are finite sums of Ł's and Π's, except the above cases, non-isomorphic negations define incomparable sets of tautologies — i. e., (1) and (2) are satisfied.

2 Background

2.1 Basic semantical properties of the calculus

Axiomatic presentation of the propositional calculi BL, SBL, and SBL$\sim$ could be found in [5] (see also [3] for simplified axioms of $\sim$). For an extensive material on the logic BL, see [11].

We remark here that the basic connectives of the propositional calculus BL are the (strong) conjunction &, the implication $\to$, and the constant 0. From these one defines the negation ($\neg\varphi$ is $\varphi \to 0$), the (min-)conjunction ($\varphi \wedge \psi$ is $\varphi \,\&\, (\varphi \to \psi)$), the (max-)disjunction ($\varphi \vee \psi$ is $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$), the equivalence ($\varphi \equiv \psi$ is $(\varphi \to \psi) \,\&\, (\psi \to \varphi)$), and the constant 1 (1 is $0 \to 0$).

The general semantics of the propositional calculus BL is given by BL-algebras, with respect to which class of algebras BL has been shown to be complete in [10], see also [11]. BL-algebras on the real unit interval $[0,1]$ are called *standard* BL-algebras; propositional BL has been shown to be standard complete in [1]. It has also been shown in this paper that every BL-algebra on $[0,1]$ is uniquely determined by some continuous t-norm.

Definition 3. *A t-norm is a binary operation $*$ on $[0,1]$, satisfying the conditions:*
(i) $*$ *is commutative and associative*
(ii) $*$ *is non-decreasing in both arguments*
(iii) $1 * x = x$ *and* $0 * x = 0$ *for all* $x \in [0,1]$.

For a continuous t-norm $*$, we define $x \Rightarrow y = \max\{z \mid x * z \leq y\}$; the operation $\Rightarrow$ is called the *residuum* of $*$. A BL-algebra given by a continuous t-norm $*$ is the algebra $[0,1]_* = \langle [0,1], *, \Rightarrow, 0\rangle$, where $\Rightarrow$ is the residuum of $*$.

Any standard algebra gives a (standard) semantics for BL, as $*$ provides an interpretation for the (strong) conjunction &, whereas the residuum $\Rightarrow$ interprets the implication $\to$.

Three examples of continuous t-norms stand out:
– the Łukasiewicz t-norm (where $x * y$ is $\max(x + y - 1, 0)$),
– the Gödel t-norm (where $x * y$ is $\min(x, y)$),
– the product t-norm (where $x * y$ is $x.y$).

The following is the representation theorem for continuous t-norms, first proved in [13]. The set of all idempotents of $*$ is a closed subset of $[0,1]$. Its complement is a union of countably many pairwise disjoint open intervals; denote this set of intervals $\mathcal{I}_\mathrm{o}$. Let $[a,b] \in \mathcal{I}$ iff $(a,b) \in \mathcal{I}_\mathrm{o}$ (so $\mathcal{I}$ is the set of corresponding closed intervals for the intervals in $\mathcal{I}_\mathrm{o}$).

Theorem 4. (Representation theorem for continuous t-norms) *Let $*$ be any continuous t-norm.*
(i) *For each $[a,b] \in \mathcal{I}$, $*$ on $[a,b]$ is isomorphic either to the product t-norm (on $[0,1]$) or to Łukasiewicz t-norm (on $[0,1]$).*
(ii) *If for $x, y \in [0,1]$ there is no $[a,b] \in \mathcal{I}$ such that $x, y \in [a,b]$, then $x * y = \min(x,y)$.*

If two continuous t-norms are isomorphic via f, then so are their residua, and hence so are the two standard algebras determined by the two t-norms.

For each continuous t-norm, the maximal closed intervals which are isomorphic copies of the Łukasiewicz, Gödel, or product t-norm are called the *components* of the t-norm; we use the terms Ł-, G-, or Π-components and the three letters to denote the type of a component.

A standard algebra A with components $A_1, \ldots, A_n$ is often referred to as a (finite) *ordinal sum* $A = \bigoplus_{i=1}^{n} A_i$.

In any standard BL-algebra L, the interpretation of the definable negation $\neg$ is two-valued, or *strict* (yielding the value 1 for the argument 0, and the value 0 for all nonzero arguments), iff L is an SBL-algebra. This occurs iff the algebra does not have an initial Ł-component (i. e., if there is no initial component, or the initial component is G or Π). Those t-norms which determine the strict negation are, throughout this paper, referred to as SBL t-norms.

The language of BL (and SBL) can be expanded with additional connectives, as indeed it has been in many cases. [5] added the Δ-projection and the involutive negation $\sim$.

Following the established usage, we use the symbol Δ for both logical connective and its interpretation on $[0,1]$ similarly as for $\sim$. (Whereas the BL-connectives & and $\rightarrow$ correspond to the interpretations $*$ and $\Rightarrow$.)

The semantics of Δ in any standard algebra (and generally in any BL-chain) is as follows: $\Delta(1) = 1$ and $\Delta(x) = 0$ for $x < 1$.

The semantics of an involutive negation on [0,1] is any decreasing involution, i. e., a function $\sim : [0,1] \longrightarrow [0,1]$ such that $x < y$ implies ${\sim}y < {\sim}x$ for all $x, y \in [0,1]$, and for all $x \in [0,1]$ we have $\sim\sim x = x$. We will need the following simple properties of involutive negations.

Lemma 5. *Let $\sim$ be a decreasing involution on $[0,1]$. Then $\sim$ is a bijection on $[0,1]$, the graph of $\sim$ is symmetric w. r. t. the diagonal, and $\sim$ is continuous.*

See e. g. [14] for a proof using the name strong negation for involutive negation.

For an involutive negation $\sim$ on $[0,1]$, its *fixed point* a is the unique value x for which ${\sim}x = x$.

We will also need the following lemma from [5].

Lemma 6. *Let $0 < a_0 < \cdots < a_k < 1$ be reals. Then there is a decreasing involution $\sim$ on $[0,1]$ such that ${\sim}(a_i) = a_{k-i}$ for $i = 0, \ldots, k$.*

Interestingly, when $\sim$ is added to SBL, the Δ connective is definable: $\Delta\varphi$ is $\neg{\sim}\varphi$.

2.2 Automorphisms of standard algebras

The following theorem follows from more general statements, e.g., the lemma due to Hion to be found in [9], 4.1.6.

Theorem 7. $f : [0,1] \longrightarrow [0,1]$ *is an automorphism of the standard product algebra iff $f(x) = x^r$ for some real $r > 0$.*

The next theorem can be found in [2], Corollary 7. 2. 6.

Theorem 8. *The standard Łukasiewicz algebra has no nontrivial automorphisms.*

We address automorphisms of continuous t-norms which are finite ordinal sums of Ł- and Π-components. Let $*$ be a continuous t-norm. Obviously, any automorphism of $*$ must be an identity on all of its idempotents. For a component $[b, c]$ of $*$, denote $f : [0, 1] \longrightarrow [b, c]$ the mapping which defines $*$ on $[b, c]$ as an isomorphic copy of some $*'$ on $[0, 1]$ (by assumption, $*'$ is either Ł or Π).

Lemma 9. *The restrictions of automorphisms of $*$ to $[b, c]$ have the form $fgf^{-1}(x)$, where g is an automorphism of $*'$ on $[0, 1]$ and f is as above.*

Proof. We show that each fgf^{-1} is an automorphism on $[b, c]$: assuming $x, y \in [b, c]$, we have $fgf^{-1}(x * y) = fg(f^{-1}(x) *' f^{-1}(y)) = f(gf^{-1}(x) *' gf^{-1}(y)) = fgf^{-1}(x) * fgf^{-1}(y)$. To see that all automorphisms of $*$ on $[b, c]$ are obtained as images of automorphisms of $*'$ on $[0, 1]$, assume h is an automorphism on $[b, c]$, and define $g(x) = f^{-1}hf(x)$; then g is an automorphism of $*'$, and for $x \in [b, c]$, $h(x) = fgf^{-1}(x)$. □

Therefore, in case $[b, c]$ is an Ł-component, the restriction of any automorphism of $*$ to $[b, c]$ is the identity (by Theorem 8) and in case it is a Π-component, the restrictions are exactly r-powers with respect to $*$ (by Theorem 7). In more detail, for any real $r \geq 0$, the r-power of $x \in [b, c]$ with respect to $*$, is $x^{r(*)} = f[(f^{-1}(x))^r]$. Obviously, if $x \in (b, c)$ is fixed, then x^r is a function of r, which is a bijection between R^+ and (b, c). Note that for a natural n we have $x^{n(*)} = x * \ldots * x$ (n times). We write simply x^r instead of $x^{r(*)}$ where the component $[b, c]$ and the operation $*$ is clear from the context.

Corollary 10. *Let $*$ be a finite sum of Ł-components and Π-components; denote k the number of Π-components in the sum. Any automorphism f is uniquely determined by a vector $r_1, \ldots, r_k$ of positive real numbers, s. t. $f(x) = x$ if x is idempotent or an element of an Ł-component, and $f(x) = x^{r_i}$ if x belongs to the i-th Π-component.*

3 The case of standard product

This section contains a proof of the fact that the standard product t-norm satisfies the conditions (1) and (2). This statement is a consequence of the result [8], where distinguishing inequalities for each two non-isomorphic algebras of the type $\langle [0, 1], 0, 1, \wedge, \vee, *, \sim \rangle$ are presented. Within BL with $\sim$, these inequalities can be expressed by propositional formulas.

We present an alternative proof, using a family of distinguishing propositional formulas in the language of BL with $\sim$. The proof introduces the β functions, whose generalization is needed in Section 4.

Throughout this section, $*$ denotes the standard product t-norm.

For an arbitrary involutive negation $\sim$, denote a its fixed point and for every positive real x let the function $\beta_\sim(x)$ be the solution of the equation $\sim(a^x) = a^{\beta_\sim(x)}$. Clearly, the function $\beta_\sim(x)$ maps positive reals to positive reals, is continuous, strictly decreasing, satisfies $\beta_\sim(1) = 1$ and $\beta_\sim(\beta_\sim(x)) = x$.

Lemma 11. *Let $\sim_1, \sim_2$ be two involutive negations. Then $\sim_1$ is isomorphic to $\sim_2$ iff $\beta_{\sim_1} = \beta_{\sim_2}$.*

Proof. Assume a_i is the fixed point of $\sim_i$ for $i = 1, 2$ and denote $r = \ln a_2 / \ln a_1$, i.e., $a_2 = a_1^r$.

If $\sim_1$ is isomorphic to $\sim_2$, then by Theorem 7, there is a positive s such that the isomorphism is via z^s, i.e., $(\sim_1 z)^s = \sim_2(z^s)$ holds for all $z \in [0, 1]$. By substituting $z = a_1$, we obtain

$a_1^s = (\sim_1 a_1)^s = \sim_2 a_1^s$, which implies $a_2 = a_1^s$ and $s = r$. It follows that $a_2^{\beta_{\sim_1}(x)} = (a_1^{\beta_{\sim_1}(x)})^r = (\sim_1 a_1^x)^r = \sim_2 (a_1^x)^r = \sim_2 a_2^x = a_2^{\beta_{\sim_2}(x)}$. Consequently, $\beta_{\sim_1} = \beta_{\sim_2}$.

From the assumption $\beta_{\sim_1} = \beta_{\sim_2}$, we obtain for every $x > 0$ $\ln(\sim_1 a_1^x)/\ln a_1 = \ln(\sim_2 a_2^x)/\ln a_2$. This yields $(\sim_1 a_1^x)^r = \sim_2 a_2^x = \sim_2 (a_1^x)^r$. Thus, $(\sim_1 z)^r = \sim_2(z^r)$ holds for all $z \in (0,1)$ and, by continuity, for all $z \in [0,1]$. Since z^r is an automorphism of the product, $\sim_1$ and $\sim_2$ are isomorphic (via z^r). $\square$

Consider the following two types of formulas with positive integer parameters i, j, r, and s:
$\Phi(i/j, r/s)$ is $\Delta(q \equiv \sim q) \,\&\, \Delta(z^j \equiv q^i) \to \Delta(q^r \to (\sim z)^s)$, and
$\Phi'(i/j, r/s)$ is $\Delta(q \equiv \sim q) \,\&\, \Delta(z^j \equiv q^i) \to \Delta((\sim z)^s \to q^r)$.

Theorem 12. *$\Phi(i/j, r/s)$ is a tautology of $\langle *, \sim \rangle$ iff $r/s \geq \beta_\sim(i/j)$.*
$\Phi'(i/j, r/s)$ is a tautology of $\langle *, \sim \rangle$ iff $r/s \leq \beta_\sim(i/j)$.

Proof. We address the former case, the other one being analogous.

Owing to the presence of Δ's, the implication in Φ behaves classically (i. e., is valid iff the succedent holds for all evaluations satisfying the antecedent); so it is sufficient to investigate the validity of $e(q)^r \leq (\sim e(z)^s)$ under the conditions $e(q) = a$ and $e(z) = e(q)^{i/j}$, for arbitrary e.

Thus (performing the two substitutions) the tautologousness of Φ reduces to the validity of the inequality $a^{r/s} \leq \sim(a^{i/j})$; or equivalently, of the inequality $r/s \geq \beta_\sim(i/j)$, using the definition of the β-function. $\square$

Corollary 13. $\beta_\sim(i/j) = \inf\{r/s; \Phi(i/j, r/s)$ *is a tautology of* $\langle *, \sim \rangle\}$
$= \sup\{r/s; \Phi'(i/j, r/s)$ *is a tautology of* $\langle *, \sim \rangle\}$.

These results imply that in case of the product t-norm the equivalence (1) from Section 1 is satisfied.

Theorem 14. $\mathrm{TAUT}(\langle *, \sim_1 \rangle) = \mathrm{TAUT}(\langle *, \sim_2 \rangle)$ *iff* $\langle *, \sim_1 \rangle$ *is isomorphic to* $\langle *, \sim_2 \rangle$.

The two theorems below follow from the results in [8]; it is sufficient to rewrite the identities in the language of [8], which distinguish the non-isomorphic involutive negations, into logical formulas, using the equivalence connective. We present a different proof based on previous results of this section.

Proof. The right-to-left implication holds. For the reverse implication, assume $\mathrm{TAUT}(\langle *, \sim_1 \rangle) = \mathrm{TAUT}(\langle *, \sim_2 \rangle)$. Corollary 13 implies that the functions $\beta_{\sim_1}$ and $\beta_{\sim_2}$ coincide on positive rational numbers. Since they are continuous, they are equal. Then, by Lemma 11, $\langle *, \sim_1 \rangle$ is isomorphic to $\langle *, \sim_2 \rangle$. $\square$

Moreover, the sets of tautologies of non-isomorphic negations are incomparable by inclusion:

Theorem 15. *If $\langle *, \sim_1 \rangle$ is not isomorphic to $\langle *, \sim_2 \rangle$, then* $\mathrm{TAUT}(\langle *, \sim_1 \rangle)$ *and* $\mathrm{TAUT}(\langle *, \sim_2 \rangle)$ *are incomparable.*

Proof. Assume $\langle *, \sim_1 \rangle$ is non-isomorphic to $\langle *, \sim_2 \rangle$. By Lemma 11, $\beta_{\sim_1} \neq \beta_{\sim_2}$. Since the two functions are continuous, there is a positive rational number i/j such that $\beta_{\sim_1}(i/j) \neq \beta_{\sim_2}(i/j)$. W. l. o. g. assume that $\beta_{\sim_1}(i/j) < \beta_{\sim_2}(i/j)$. Then there is a positive rational number r/s such that $\beta_{\sim_1}(i/j) < r/s < \beta_{\sim_2}(i/j)$. By Theorem 12, $\Phi(i/j, r/s) \in \mathrm{TAUT}(\langle *, \sim_1 \rangle) \setminus \mathrm{TAUT}(\langle *, \sim_2 \rangle)$, while $\Phi'(i/j, r/s) \in \mathrm{TAUT}(\langle *, \sim_2 \rangle) \setminus \mathrm{TAUT}(\langle *, \sim_1 \rangle)$. $\square$

Let us add the following result concerning the "standard" negation $1 - x$. Albeit the set of formulas $\Phi(i/j, r/s)$ and $\Phi'(i/j, r/s)$ which are tautologies of standard product with $1 - x$ characterize this algebra up to an isomorphism, it is not an axiomatization (i. e., its deductive closure is not the set of *all* tautologies of standard product with $1 - x$). The reason for this is that the characterization up to an isomorphism holds only among algebras on $[0, 1]$ with involutive negation.

Theorem 16. *The set of formulas $\Phi(i/j, r/s)$ and $\Phi'(i/j, r/s)$ which are tautologies of the standard product with $1 - x$ do not form an axiomatization of the set of tautologies of this algebra.*

Proof. Let $1 - (1/2)^{i/j} = (1/2)^{\beta(i/j)}$, so β determines the "standard" involutive negation $1 - x$.

Let F denote the set of all formulas $\Phi(i/j, r/s)$ for $r/s > \beta(i/j)$ and all formulas $\Phi'(i/j, r/s)$ for $r/s < \beta(i/j)$. In other words, F is the set of formulas of type Φ and Φ' which are valid in $\langle *, 1 - x\rangle$.

Let ψ be the formula $\sim(x^2 \,\&\, y) \,\&\, \sim y \to (\sim(x \,\&\, y))^2$. Then ψ is a 1-tautology of $\langle *, 1 - x\rangle$, since it represents the inequality $(1 - xy)^2 \geq (1 - x^2 y)(1 - y)$, which is equivalent to $y(x - 1)^2 \geq 0$.

We show that if $F_0 \subseteq F$ is an arbitrary finite subset of F, then there is an involutive negation $\sim$ such that $F_0 \subseteq \mathrm{TAUT}(\langle *, \sim\rangle)$, but $\psi \notin \mathrm{TAUT}(\langle *, \sim\rangle)$. As a consequence, ψ is not provable from F.

Fix F_0 to be a finite subset of F. Let us consider only $\sim$, whose fixed point is $a = 1/2$. Formula $\Phi(i/j, r/s) \in F_0$ is a tautology iff $a^{r/s} \leq \sim(a^{i/j})$. Formula $\Phi'(i/j, r/s) \in F_0$ is a tautology iff $a^{r/s} \geq \sim(a^{i/j})$. Let us restrict ourselves to negations such that $\sim(a^{i/j}) = 1 - a^{i/j}$ for all pairs i, j occurring in F_0. As $1 - x$ satisfies all formulas in F_0, so does any such $\sim$. This determines the values of $\sim$ on a finite set of arguments.

Now choose b and c s. t. $0 < b < c < 1/2$ and neither $[b, c]$ nor $[1 - c, 1 - b]$ includes any arguments on which $\sim$ has already been fixed. On $[0, 1/2]$ except (b, c) put $\sim x = 1 - x$. Denote $d = \sqrt{bc}$; then $d \in (b, c)$. Choose $\sim d$ arbitrarily but so as to satisfy

$$1 - c < \sim d < \sqrt{(1 - b)(1 - c)} < 1 - b\ . \tag{3}$$

This is possible since $1 - c < 1 - b$, thus $1 - c < \sqrt{(1 - b)(1 - c)} < 1 - b$. Define $\sim$ as a linear function on $[b, d]$ and $[d, c]$. It follows from (3) that the resulting function is decreasing. On $[1/2, 1]$ define $\sim$ to be involutive.

It remains to show that ψ does not hold in $\langle *, \sim\rangle$, i. e., the inequality

$$(\sim(xy))^2 \geq (\sim(x^2 y))(\sim y) \tag{4}$$

does not hold for some x and y. Choose $x = \sqrt{b/c}$ and $y = c$. Then $xy = d$ and $x^2 y = b$. The inequality (4) then yields $(\sim d)^2 \geq (1 - b)(1 - c)$, which contradicts (3). $\square$

4 Ordinal sums with indistinguishable negations

Throughout this section, we fix a continuous t-norm $*$ which is a finite sum of Ł- and Π-components, and with at least two Π-components $[c, d]$ and $[d', c']$ (where $c < d$ and $d' < c'$), s. t. $d \leq d'$ and either $c > 0$ and $c' < 1$, or $c = 0$ and $c' = 1$. Note that specifically in this section, it is not necessary to assume that the first component is a product component. We show that for any such continuous t-norm $*$ there are two non-isomorphic involutive negations which have the same set of 1-tautologies, i. e., the equivalence (1) from Section 1 is not satisfied for $*$.

Let us consider an arbitrary involutive negation $\sim$ which map the components $[c,d]$ and $[d',c']$. As a generalization of the β-function introduced in Section 3, we define, for any $x \in (c,d)$ and any positive $r \in \mathrm{R}$, the function $\gamma(x,r)$ s. t. $\sim(x^r) = (\sim x)^{\gamma(x,r)}$[4]. The value x is referred to as the *base*. For each choice of $x \in (c,d)$, the function $\gamma(x,r)$ is continuous, strictly decreasing and maps R^+ onto R^+. Moreover, the following holds.

Lemma 17. *For every $x \in (c,d)$ and $s \in \mathrm{R}^+$, we have $\gamma(x^s,r) = \gamma(x,rs)/\gamma(x,s)$.*

Proof. Using the definition of γ successively with bases x, x^s, and x, we obtain $(\sim x)^{\gamma(x,s)\cdot\gamma(x^s,r)} = (\sim(x^s))^{\gamma(x^s,r)} = \sim(x^{rs}) = (\sim x)^{\gamma(x,rs)}$. Hence, $\gamma(x,s)\cdot\gamma(x^s,r) = \gamma(x,rs)$. □

Definition 18. *For every $\sim$ mapping $[c,d]$ to $[d',c']$, let $F(\sim)$ denote the set of the one-argument functions $\gamma(x)(r)$ obtained from $\gamma(x,r)$ by fixing the argument $x \in (c,d)$ in all possible ways.*

Theorem 19. *Let $\sim_1$ and $\sim_2$ be two negations mapping $[c,d]$ and $[d',c']$ onto each other. Assume that $\sim_1$ and $\sim_2$ are isomorphic and that they coincide outside $[c,d]$ and $[d',c']$. Then $F(\sim_1) = F(\sim_2)$.*

Proof. Let f be an isomorphism of $\sim_1$ and $\sim_2$; by definition f is an automorphism of $*$ and $f(\sim_1 x) = \sim_2 f(x)$ for all $x \in [0,1]$. By Corollary 10, f is a power w. r. t. $*$ on each product component and it is an identity on all idempotents. As the two negations coincide outside $[c,d]$ and $[d',c']$, we may assume that f is an identity outside these two components. Denote r the exponent of the power on $[c,d]$ and s the one on $[d',c']$. In particular, $(\sim_1 x)^s = \sim_2(x^r)$ for $x \in [c,d]$.

For any positive $q \in \mathrm{R}$ and any $x \in (c,d)$, the definition of $\gamma_2(x^r,q)$ implies

$$\sim_2(x^{rq}) = (\sim_2 x^r)^{\gamma_2(x^r,q)}.$$

Using the identity $\sim_2(y^r) = (\sim_1 y)^s$, we obtain

$$(\sim_1 x^q)^s = (\sim_1 x)^{s.\gamma_2(x^r,q)}$$

and, hence, $\sim_1 x^q = (\sim_1 x)^{\gamma_2(x^r,q)}$. Since we also have $\sim_1 x^q = (\sim_1 x)^{\gamma_1(x,q)}$, the functions $\gamma_1(x)(q)$ and $\gamma_2(x^r)(q)$ coincide. This holds for arbitrary $x \in (c,d)$, and x^r ranges over the whole (c,d). □

Definition 20. *For every $\sim$ mapping $[c,d]$ to $[d',c']$, let $G(\sim)$ denote the set of restrictions of the functions from $F(\sim)$ to finite domains. In other words, $G(\sim)$ is the set of all k-tuples of pairs of the form*

$$(\langle r_1, \gamma(x,r_1)\rangle, \ldots, \langle r_k, \gamma(x,r_k)\rangle)$$

for all $\gamma \in F(\sim)$, all $k \in \mathrm{N}$, all $x \in (c,d)$ and all positive $r_1, \ldots, r_k$.

Theorem 21. *Assume $\sim_1$ and $\sim_2$ map $[c,d]$ and $[d',c']$ onto each other, are equal outside these two components and $G(\sim_1) = G(\sim_2)$. Then* $\mathrm{TAUT}(\langle *, \sim_1\rangle) = \mathrm{TAUT}(\langle *, \sim_2\rangle)$.

[4] See the section 2.2 for r-powers in any product component.

Proof. Denote $A_1 = \langle *, \sim_1 \rangle$ and $A_2 = \langle *, \sim_2 \rangle$. Let φ be a formula which is not valid in A_1, i. e., there is an evaluation e_1 in A_1 s. t. $e_1(\varphi) < 1$. Using e_1, we define a non-satisfying evaluation e_2 in A_2 for φ.

Let $\varphi_1, \ldots, \varphi_m$ be all subformulas of φ. For all $i = 1, \ldots, m$, let $a_i = e_1(\varphi_i)$. Fix an arbitrary $a \in (c, d)$. For each $i = 1, \ldots, m$ such that $a_i \in (c, d) \cup (d', c')$, let r_i be the number satisfying either $a_i = a^{r_i}$ or $a_i = (\sim_1 a)^{r_i}$.

Let M be the number of pairs of indices $k, l \in \{1, \ldots, m\}$, such that $a_k \in (c, d)$, $a_l \in (d', c')$ and $\sim_1 a_k = a_l$. For each of these pairs, we have $\sim_1(a^{r_k}) = (\sim_1 a)^{r_l}$ and also $r_l = \gamma_1(a, r_k)$. Considering all of these pairs k, l together, we obtain a finite set of conditions $r_{l_1} = \gamma_1(a, r_{k_1}), \ldots, r_{l_M} = \gamma_1(a, r_{k_M})$.

As M is finite, we have $\langle \langle r_{k_1}, \gamma_1(a, r_{k_1}) \rangle, \ldots, \langle r_{k_M}, \gamma_1(a, r_{k_M}) \rangle \rangle \in G(\sim_1)$. Using the assumption of the theorem, we obtain a $b \in (c, d)$ such that $\langle \langle r_{k_1}, \gamma_2(b, r_{k_1}) \rangle, \ldots \langle r_{k_M}, \gamma_2(b, r_{k_M}) \rangle \rangle \in G(\sim_2)$. In other words, $r_{l_i} = \gamma_2(b, r_{k_i})$, $i = 1, \ldots, M$.

Let s, t be such that $b = a^s$, $\sim_2 b = (\sim_1 a)^t$. Moreover, let

$$f(x) = \begin{cases} x^s & \text{if } x \in (c, d) \\ x^t & \text{if } x \in (d', c') \\ x & \text{otherwise} \end{cases}$$

The function f is an automorphism of $*$ and for every $r \in \mathrm{R}^+$ satisfies $f(a^r) = b^r$ and $f((\sim_1 a)^r) = (\sim_2 b)^r$.

Consider a complete evaluation e_2 for φ in A_2, defined for each $i = 1, \ldots, m$ by $e_2(\varphi_i) = f(e_1(\varphi_i))$; we show that e_2 is a sound complete evaluation for φ in A_2, s. t. $e_2(\varphi) < 1$. The fact that $e_2(\varphi) < 1$ follows from $e_1(\varphi) < 1$ and the definition of f. Since f is an automorphism of $*$, e_2 is sound on subformulas which have the form of the conjunction or the implication. It is also sound for involutive negations whose (both) arguments are evaluated with values outside $(c, d) \cup (d', c')$, since there f is identical. If φ_i is $\sim\varphi_j$ and $e(\varphi_i), e(\varphi_j) \in (c, d) \cup (d', c')$, either the pair (i, j) or the pair (j, i) is one of the M pairs (k, l) considered above, which satisfy $r_l = \gamma_1(a, r_k)$ and hence $r_l = \gamma_2(b, r_k)$. We have $e_2(\varphi_l) = f(e_1(\varphi_l)) = f(\sim_1 e_1(\varphi_k)) = f(\sim_1(a^{r_k})) = f((\sim_1 a)^{r_l}) = ((\sim_1 a)^{r_l})^t = (\sim_2 b)^{r_l} = (\sim_2 b)^{\gamma_2(b, r_k)} = \sim_2(b^{r_k}) = \sim_2 f(a^{r_k}) = \sim_2 f(e_1(\varphi_k)) = \sim_2 e_2(\varphi_k)$. This implies $e_2(\varphi_i) = \sim_2 e_2(\varphi_j)$ independently of the mapping between the two tuples (k, l) and (i, j), since $\sim_2$ is involutive. Hence, e_2 is a valid non-satisfying evaluation of φ in A_2. □

In the following, we construct $\sim_1$ and $\sim_2$ which are equal outside $[c, d]$ and $[d', c']$, map these two components onto each other, and such that $F(\sim_1) \neq F(\sim_2)$ and $G(\sim_1) = G(\sim_2)$.

Consider all finite sequences of two symbols A, B, ordered lexicographically assuming $A < B$ and appended into a single sequence $P = A\,B\,AA\,AB\,BA\,BB\,AAA\ldots$ Symbols in P will be indexed from 0. So, we consider P as a function $P : N \to \{A, B\}$ in such a way that P is $P(0), P(1), \ldots$. Using P, define two sequences P_1 and P_2, infinite in both directions, namely, $P_1 = \ldots AAAAB\,P$ and $P_2 = \ldots AAABB\,P$. In terms of functions, P_1 and P_2 extend P to functions defined on all integers in such a way that for $i = 1, 2$ and $j \geq 0$, we have $P_i(j) = P(j)$, $P_i(-1) = B$, $P_1(-2) = A$, $P_2(-2) = B$ and for $j \leq -3$, we have $P_i(j) = A$. Note that the leftmost B in P_1 is followed by A, while the leftmost B in P_2 is followed by B. Hence, there is no integer a such that for all j the identity $P_1(j) = P_2(j + a)$ is satisfied.

Choose two distinct continuous functions $g_A, g_B : [0, 1] \longrightarrow [0, 1]$ and a constant C s. t.
(1) $g_A(0) = g_B(0) = g_A(1) = g_B(1) = 0$;

(2) for $z \in (0,1)$, $g_A(z) > 0$ and $g_B(z) > 0$;
(3) the real-valued functions $g_A(z) - C\,z$ and $g_B(z) - C\,z$ on the interval $[0,1]$ are strictly decreasing.

Denoting $\lfloor z \rfloor$ the integer part of z (the floor function), define for $i = 1, 2$ and a real z

$$\delta_i(z) = \begin{cases} g_A(z - \lfloor z \rfloor) \text{ if } P_i(\lfloor z \rfloor) = A \\ g_B(z - \lfloor z \rfloor) \text{ if } P_i(\lfloor z \rfloor) = B \end{cases}$$

Lemma 22. *(i) There are no $a, b \in \mathrm{R}$ such that, for every $z \in \mathrm{R}$, $\delta_1(z) = \delta_2(z + a) + b$.*
(ii) For every $z_1, \ldots, z_m \in \mathrm{R}$, there is an integer a such that for every $j = 1, \ldots, m$ we have $\delta_1(z_j) = \delta_2(z_j + a)$.

Proof. (i) Assume $\delta_1(z) = \delta_2(z + a) + b$ for every z and fixed a, b. The minimum of both δ_1 and δ_2 is zero. Since the minima of both sides of $\delta_1(z) = \delta_2(z + a) + b$ coincide, we have $b = 0$. The left-hand side achieves its minimum iff z is integer, the right-hand does iff $z + a$ is integer, hence a must be integer. This implies $P_1(u) = P_2(u + a)$ for all integers u, a contradiction.

(ii) Let S be the shortest interval of integers containing $\lfloor z_j \rfloor$ for all $j = 1, \ldots, m$. The subsequence $P_1(S)$ of P_1 has infinitely many occurrences in P_2. Choose one of them and let a be an integer such that for all $u \in S$, $P_1(u) = P_2(u + a)$. For every $j = 1, \ldots, m$, we have $z_j - \lfloor z_j \rfloor = z_j + a - \lfloor z_j + a \rfloor$ and $P_1(\lfloor z_j \rfloor) = P_2(\lfloor z_j + a \rfloor)$, implying $\delta_1(z_j) = \delta_2(z_j + a)$. $\square$

Using the functions δ_1 and δ_2, we now define the two negations $\sim_1$ and $\sim_2$ with the desired properties. Let $\overline{\delta_i}(r) = \exp(\delta_i(\log r) - C\,\log r)$. Both $\overline{\delta_i}$ are decreasing and continuous on $(0, \infty)$.

Definition 23. *Choose arbitrary $x_0, y_0 \in (c, d)$ and $x_0', y_0' \in (d', c')$ and define $\sim_1$, $\sim_2$ on $(c, d) \cup (d', c')$ as follows. Let*

$$\sim_1 x_0 = x_0', \quad \sim_2 y_0 = y_0'$$

and for any $r \in \mathrm{R}^+$ let

$$\sim_1(x_0^r) = (\sim_1 x_0)^{\overline{\delta_1}(r)}, \quad \sim_2(y_0^r) = (\sim y_0)^{\overline{\delta_2}(r)}.$$

By continuity, $\sim_i c = c'$ and $\sim_i d = d'$; on the complement of $(c, d) \cup (d', c')$, $\sim_i$ is linear, connecting the points $[0, 1]$, $[c, c']$, the points $[d, d']$, $[d', d]$, and the points $[c', c]$, $[1, 0]$.

Note that the definition of $\sim_i$ guarantees that for every r, $\gamma_1(x_0, r) = \overline{\delta_1}(r)$ and $\gamma_2(y_0, r) = \overline{\delta_2}(r)$.

Theorem 24. *Let $\sim_1$ and $\sim_2$ be the negations from Definition 23. Then $F(\sim_1) \neq F(\sim_2)$ and $G(\sim_1) = G(\sim_2)$.*

Proof. Assume $F(\sim_1) = F(\sim_2)$. By definition of $\sim_1$, this implies that there is $y \in (c, d)$ such that $\overline{\delta_1}(r) = \gamma_1(x_0, r) = \gamma_2(y, r)$ for all $r \in \mathrm{R}^+$. Let $y = y_0^s$, where y_0 is the basis used to define $\sim_2$. By Lemma 17, we have $\gamma_2(y, r) = \gamma_2(y_0, rs)/\gamma_2(y_0, s) = \overline{\delta_2}(rs)/\overline{\delta_2}(s)$.

Combining the above we get $\overline{\delta_1}(r) = \overline{\delta_2}(rs)/\overline{\delta_2}(s)$. Using the definition of $\overline{\delta_i}$, taking logarithms of both sides and simplifying, we obtain

$$\delta_1(\log\, r) = \delta_2(\log\, r + \log\, s) - \delta_2(\log\, s)$$

for a fixed s and all $r \in \mathrm{R}^+$. By Lemma 22(i), this is not possible. Consequently, $F(\sim_1) \neq F(\sim_2)$.

In order to prove $G(\sim_1) = G(\sim_2)$, let $(\langle r_1, \gamma_1(x, r_1)\rangle, \ldots \langle r_m, \gamma_1(x, r_m)\rangle)$ be an arbitrary element of $G(\sim_1)$. We will show that it belongs to $G(\sim_2)$, by finding a suitable base $y = y_0^t$ to be used in $\gamma_2(y, r)$.

Assume $x = x_0^s$. Then by Lemma 17 we have $\gamma_1(x, r_i) = \gamma_1(x_0, r_i s)/\gamma_1(x_0, s) = \overline{\delta_1}(r_i s)/\overline{\delta_1}(s)$ for $i = 1, \dots, m$. We need t such that $y = y_0^t$ satisfies $\gamma_1(x, r_i) = \gamma_2(y, r_i)$ for $i = 1, \dots, m$. By Lemma 17, this is equivalent to $\overline{\delta_1}(r_i s)/\overline{\delta_1}(s) = \overline{\delta_2}(r_i t)/\overline{\delta_2}(t)$. By taking logarithms and simplifying, we obtain

$$\delta_1(\log\, r_i + \log\, s) - \delta_1(\log\, s) = \delta_2(\log\, r_i + \log\, t) - \delta_2(\log\, t)\ . \tag{5}$$

Let a be the integer guaranteed by Lemma 22(ii) for $z_i = \log r_i + \log s$, $i = 1, \dots, m$ and $z_{m+1} = \log s$. For $t = \exp(a + \log s)$, the equality (5) is achieved for all $i = 1, \dots, m$. Hence, $(\langle r_1, \gamma_1(x, r_1)\rangle, \dots \langle r_m, \gamma_1(x, r_m)\rangle) = (\langle r_1, \gamma_2(y, r_1)\rangle, \dots \langle r_m, \gamma_2(y, r_m)\rangle) \in G(\sim_2)$.

The above argument yields $G(\sim_1) \subseteq G(\sim_2)$. The opposite direction is analogous. $\square$

Corollary 25. *Let $\sim_1$ and $\sim_2$ be the negations from Definition 23. Then, $\sim_1$ and $\sim_2$ are not isomorphic and* $\mathrm{TAUT}(\langle *, \sim_1\rangle) = \mathrm{TAUT}(\langle *, \sim_2\rangle)$.

Proof. Combine Theorems 19, 21 and 24. $\square$

5 Ordinal sums with distinguishable negations

The standard SBL-algebras considered in this section are ordinal sums with at least two components, finitely many idempotents, and such that there are no two product components which might be mapped onto each other by any involutive negation. One may verify that these are ordinal sums of the types $\Pi \oplus j.$Ł and $\Pi \oplus i.$Ł $\oplus\, \Pi \oplus j.$Ł ($j > 0$ in both cases). For any finite ordinal sum, n is the total number of its components.

We will show that for each such algebra, any two non-isomorphic involutive negations yield incomparable sets of propositional 1-tautologies; i. e., algebras of the two types given above satisfy the equivalence (1) and the condition (2) from Section 1.

This goal will be achieved by fixing a particular dense set of definable elements in each component (the notion of definability will be specified and analyzed further on), and comparing the values of each negation on definable elements against other definable elements, using propositional formulas.

After some preliminary technical work, the main result of this section is stated in Subsection 5.4, Theorem 51. For each pair of non-isomorphic negations $\sim_1$ and $\sim_2$ a distinguishing formula is found in this theorem. The proof is carried out by considering several possible cases expressed as properties of the pair $\sim_1$ and $\sim_2$. For simplicity of formulations in Subsections 5.1, 5.2 and 5.3, we work with several different classes C of negations to which $\sim_1$ and $\sim_2$ are assumed to belong. Some of the notions introduced in the preliminary subsections are motivated only by their use in the proof of Theorem 51. Hence, it is recommended to combine reading of the preliminary subsections with reading the proof of Theorem 51.

5.1 Definability of truth values

Definition 26. *(i) For an algebra A and a formula $\varphi(x_1, \dots, x_n)$ let*

$$E^1_\varphi(A) = \{\langle v_1, \dots, v_n\rangle \in A^n : \varphi(x_1/v_1, \dots, x_n/v_n) = 1\}$$

(ii) For an algebra A and a given value $r \in A$, a formula $\varphi(x_1, \dots, x_n)$ defines *the value r in A in the variable x_i iff $E^1_\varphi(A)$ is non-empty and for any $\langle v_1, \dots, v_n\rangle \in E^1_\varphi(A)$ we have $v_i = r$. A value for which such a defining formula exists is* definable *in A (by φ in x_i).*

In the following, fix an arbitrary continuous t-norm $*$ of type either $\Pi \oplus j.\text{Ł}$ or $\Pi \oplus i.\text{Ł} \oplus \Pi \oplus j.\text{Ł}$ ($j > 0$), denote $a_0, \ldots, a_n$ its idempotents.

Definition 27. *Consider a class C of involutive negations. We say that a value s is* definable for the class C *(in short, C-definable), iff there is a propositional formula φ (possibly including $\sim$) which defines s in each algebra $\langle *, \sim \rangle$, $\sim \in C$.*

It is our goal to find, for each component $[a_{i-1}, a_i]$ of $*$, a dense set S_i of C-definable elements, for a conveniently chosen and sufficiently broad class C of involutive negations containing $\sim_1$ and $\sim_2$.

First, we define idempotent elements for $*$ and the class of all involutive negations. Note that the defining formula does not use $\sim$ directly, but uses the definable connective Δ, whose semantics is the same for all involutive negations.

Let $\sim$ be an arbitrary involutive negation and let $A = \langle *, \sim \rangle$ be the SBL$\sim$-algebra given on $[0, 1]$ by $*$ and $\sim$.

Let $\mathrm{Id}(x_0, \ldots, x_n)$ denote the formula

$$\bigwedge_{i=0}^{n} (x_i \,\&\, x_i \equiv x_i) \,\&\, \bigwedge_{i=0}^{n-1} \neg\Delta(x_{i+1} \to x_i)$$

Lemma 28. $E^1_{\mathrm{Id}}(A) = \{\langle a_0, \ldots, a_n \rangle\}$.

Proof. For a 1-satisfying evaluation $b_0, \ldots, b_n$, the first conjunct warrants that all b_i are idempotents. If the second conjunct gets the value 1, then for all i, $\Delta(b_{i+1} \Rightarrow b_i) = 0$, i.e., it is not the case that $b_{i+1} \leq b_i$; hence $b_{i+1} > b_i$ for $i = 0, \ldots, n-1$. Thus, it must be that $b_i = a_i$ for $i = 0, \ldots, n$. On the other hand, the values $a_0, \ldots, a_n$ 1-satisfy φ by a simple computation. □

Thus, the formula $Id(x_0, \ldots, x_n)$ defines the i-th idempotent in x_i, for the class of all involutive negations.

Definition 29. *Let I be a set of indices of components in $*$ and denote $U(I)$ the set of all idempotents of $*$ and all elements of $[a_{i_k-1}, a_{i_k}]$, $i_k \in I$. We say that two negations $\sim_1$, $\sim_2$ are* I-equivalent *(w. r. t. $*$) iff, whenever $x \in U(I)$, the following holds:*
(i) either both $\sim_1 x$ and $\sim_2 x$ are in $U(I)$, or both $\sim_1 x$ and $\sim_2 x$ are in the complement of $U(I)$;
(ii) if $\sim_1 x$ and $\sim_2 x$ are in $U(I)$, then $\sim_1 x = \sim_2 x$;
(iii) if $\sim_1 x$ and $\sim_2 x$ are in the complement of $U(I)$, then there is an $l \in \{1, \ldots, n\}$ s. t. $a_{l-1} < \sim_1 x, \sim_2 x < a_l$.

Observation 30. *For any I, the relation of being I-equivalent (w. r. t. $*$) forms an equivalence on the class of involutive negations.*

Proof. For a fixed I, we show transitivity. Assume $\sim_1$ is I-equivalent to $\sim_2$ and so is $\sim_2$ to $\sim_3$. Then if $x \in U(I)$, we have the following.
(i) $\sim_1 x \in U(I) \iff \sim_2 x \in U(I) \iff \sim_3 x \in U(I)$.
(ii) If $\sim_1 x \in U(I) \,\&\, \sim_3 x \in U(I)$, we have also $\sim_2 x \in U(I)$. Hence $\sim_1 x = \sim_2 x = \sim_3 x$.
(iii) If $\sim_1 x \notin U(I) \,\&\, \sim_3 x \notin U(I)$, we have also $\sim_2 x \notin U(I)$. Then, all three values $\sim_1 x, \sim_2 x, \sim_3 x$ belong to the interior of the same component, which is not in I. □

Theorem 31. *Let C be a class of involutive negations and let I be a set of indices of components s. t. for each $i \in I$, a dense set of elements of $[a_{i-1}, a_i]$ is definable for C. For any two negations $\sim_1$ and $\sim_2$ in C that are not I-equivalent (w. r. t. $*$), $\mathrm{TAUT}(\langle *, \sim_1\rangle)$ and $\mathrm{TAUT}(\langle *, \sim_2\rangle)$ are incomparable.*

In order to prove of Theorem 31, we need the following two lemmas.

Lemma 32. *Let C be a class of involutive negations and let I be a set of indices of components s. t. for each $i \in I$, a dense set of elements of $[a_{i-1}, a_i]$ is definable for C. Let $x \in U(I)$. For two negations $\sim_1$ and $\sim_2$ in C, assume $\sim_1 x < \sim_2 x$ and let y be such that $\sim_1 x \leq y \leq \sim_2 x$. Then one can find an x' definable for C, such that $\sim_1 x' < y < \sim_2 x'$.*

Proof. If x is C-definable, we are done; otherwise x is in the interior of a component of $*$ (since all idempotents are C-definable), and by definition of $U(I)$ this component contains a dense set of C-definable elements. Since it cannot be the case that $\sim_1 x = y = \sim_2 x$, assume first that $\sim_1 x < y \leq \sim_2 x$. Then, since both negations are decreasing, there is a C-definable x' such that $\sim_1 y < x' < x$ and, hence, $\sim_1 x' < y \leq \sim_2 x < \sim_2 x'$.

In case $\sim_1 x \leq y < \sim_2 x$, we find x' such that $x < x' < \sim_2 y$ and, hence, $\sim_1 x' < \sim_1 x \leq y < \sim_2 x'$. □

Lemma 33. *Let C be a class of involutive negations and let x and y be C-definable. For two negations $\sim_1$ and $\sim_2$ in C, assume $\sim_1 x < \sim_2 x$ and $\sim_1 x \leq y \leq \sim_2 x$. Then, $\mathrm{TAUT}(\langle *, \sim_1\rangle)$ and $\mathrm{TAUT}(\langle *, \sim_2\rangle)$ are incomparable.*

Proof. Let x and y be C-definable by formulas $\varphi(\overline{x})$ and $\psi(\overline{y})$ respectively[5]. First, assume $\sim_1 x \leq y < \sim_2 x$ and consider formulas

$$\Delta[\varphi(\overline{x}) \,\&\, \psi(\overline{y})] \longrightarrow \Delta(\sim\overline{x} \to \overline{y}) \tag{6}$$

$$\Delta[\varphi(\overline{x}) \,\&\, \psi(\overline{y})] \longrightarrow \Delta(\overline{y} \to \sim\overline{x}) \,\&\, \neg\Delta(\overline{y} \to \sim\overline{x}) \tag{7}$$

Formula (6) is valid only for $\sim_1$ and formula (7) is valid only for $\sim_2$.

Now, assume $\sim_1 x < y \leq \sim_2 x$ and consider formulas

$$\Delta[\varphi(\overline{x}) \,\&\, \psi(\overline{y})] \longrightarrow \Delta(\sim\overline{x} \to \overline{y}) \,\&\, \neg\Delta(\sim\overline{x} \to \overline{y}) \tag{8}$$

$$\Delta[\varphi(\overline{x}) \,\&\, \psi(\overline{y})] \longrightarrow \Delta(\overline{y} \to \sim\overline{x}) \tag{9}$$

Formula (8) is valid only for $\sim_1$ and formula (9) is valid only for $\sim_2$. □

P r o o f of Theorem 31. Fix any two involutive negations $\sim_1$ and $\sim_2$ in C that are not I-equivalent w. r. t. $*$. Then by the definition of I-equivalence, there is an $x \in U(I)$ which violates at least one of the conditions (i)–(iii). Fix such an $x \in U(I)$. We find the distinguishing formulas for each of the three possible cases. W. l. o. g. we may assume $\sim_1 x < \sim_2 x$. In each case, we find a C-definable y satisfying $\sim_1 x \leq y \leq \sim_2 x$. Using Lemma 32, we obtain a C-definable x' satisfying $\sim_1 x' \leq y \leq \sim_2 x'$. Then, by Lemma 33, $\mathrm{TAUT}(\langle *, \sim_1\rangle)$ and $\mathrm{TAUT}(\langle *, \sim_2\rangle)$ are incomparable.

[5] The variable representing a value in a defining formula is denoted by the same letter as the value itself, with a bar.

First, assume $\sim_1 x \in U(I)$ and $\sim_2 x \notin U(I)$. Then, there is $j \in \{0, \dots, n-1\}$ s. t. $\sim_1 x \leq a_j < \sim_2 x < a_{j+1}$ and we use $y = a_j$. If $\sim_1 x \notin U(I)$ and $\sim_2 x \in U(I)$, then there is $j \in \{0, \dots, n-1\}$ s. t. $a_j < \sim_1 x < a_{j+1} \leq \sim_2 x$ and we use $y = a_{j+1}$.

Second, assume $\sim_1 x$, $\sim_2 x \in U(I)$, and $\sim_1 x \neq \sim_2 x$. If one of $\sim_1 x, \sim_2 x$ is an idempotent, we use it as y. Otherwise, the interval $(\sim_1 x, \sim_2 x)$ has a nonempty intersection with the interior of a component contained in $U(I)$ and this component contains a dense set of C-definable elements. At least one of them belongs to $(\sim_1 x, \sim_2 x)$ and we use it as y.

Finally, assume $\sim_1 x$, $\sim_2 x$ are in the complement of $U(I)$, but not in the same component. Then there is a k s. t. $\sim_1 x < a_k < \sim_2 x$ and we use $y = a_k$. $\square$

5.2 Definable values in Ł-components

This section shows that a dense set is definable in $[0,1]_{\text{Ł}}$ (in the language of BL, i.e., without $\sim$) and in any Ł-component of any standard SBL-algebra which is a finite sum of Ł's and Π's, for the class of (all) involutive negations.

Definability of rational numbers in Łukasiewicz logic follows from [11], Lemma 3. 3. 11. The following lemma presents the corresponding formulas in a conjunction.

Lemma 34. *Let A denote the standard Łukasiewicz algebra $[0,1]_{\text{Ł}}$ and let $k, m \in N$, $k < m$. Let $\lambda_{(k,m)}(x,y)$ denote the propositional formula $\neg x^m \equiv x^m \,\&\, y \equiv \neg x^{2k}$. Then $E^1_{\lambda_{(k,m)}}(A) = \{\langle 1 - 1/(2m), k/m \rangle\}$.*

Proof. Note that $\lambda_{(k,m)}(x,y)$ is satisfied iff $x^m = 1 - m(1-x) = 1/2$ and $1 - y = 1 - 2k(1-x) = 1 - k/m$. See also [11], Lemma 3. 3. 11. $\square$

In the following, consider a standard algebra B given by $*$ and let $[c,d]$ be an Ł-component of B. Introduce the following formula-translating function adding two propositional variables $\overline{c}$ and $\overline{d}$, where φ, ψ are arbitrary formulas and θ a propositional atom:

$$\begin{aligned}
0^{[c,d]} &= \overline{c}\ ; \\
\theta^{[c,d]} &= \theta \wedge \overline{d} \vee \overline{c}\ ; \\
(\varphi \,\&\, \psi)^{[c,d]} &= \varphi^{[c,d]} \,\&\, \psi^{[c,d]}\ ; \\
(\varphi \to \psi)^{[c,d]} &= (\varphi^{[c,d]} \to \psi^{[c,d]}) \wedge \overline{d}\ .
\end{aligned}$$

Further, denote $f : [0,1]_{\text{Ł}} \longrightarrow [c,d]$ the isomorphism between the Łukasiewicz t-norm on $[0,1]$ and $*$ on $[c,d]$[6]. Assume $\alpha(\overline{c}, \overline{d}, z_1, \dots, z_q)$ is a formula which defines the values c, d in its first two variables in B for the class of all involutive negations.

For every formula $\varphi(x_1, \dots, x_p)$, define $\varphi[c,d](\overline{c}, \overline{d}, z_1, \dots, z_q, x_1, \dots, x_p)$ to be the formula

$$\alpha(\overline{c}, \overline{d}, z_1, \dots, z_q) \wedge (\varphi^{[c,d]} \equiv \overline{d}) \wedge \bigwedge_{i=1}^{p} (\overline{c} \to x_i \wedge x_i \to \overline{d}).$$

Lemma 35. *Let A denote the standard Łukasiewicz algebra, $\varphi(x_1, \dots, x_p)$ be a propositional formula, B, f, c, d, α and $\varphi[c,d]$ be as above. If φ defines the values $v_1, \dots, v_p$ in A, then $\varphi[c,d]$ defines the values $f(v_1), \dots, f(v_p)$ in B (in the variables $x_1, \dots, x_p$).*

[6]The isomorphism f is determined uniquely, since Łukasiewicz t-norm has no nontrivial automorphisms.

Proof. First, observe that for any subformula $\chi(x_{i_1}, \ldots, x_{i_k})$ of φ and any vector of values $w_1, \ldots, w_k$ from $[0, 1]$, we have $\chi^{[c,d]}(x_{i_1}/f(w_1), \ldots, x_{i_k}/f(w_k)) = f(\chi(x_{i_1}/w_1, \ldots, x_{i_k}/w_k))$.

If $\langle v_1, \ldots, v_p\rangle \in E^1_\varphi(A)$, then $\varphi^{[c,d]}(x_1/f(v_1), \ldots, x_p/f(v_p)) = d$ in B. Moreover, $f(v_i) \in [c, d]$ for $i = 1, \ldots, p$. Let $\langle c, d, b_1, \ldots, b_q\rangle$ be a satisfying assignment of α. Then the vector $\langle c, d, b_1, \ldots, b_q, f(v_1), \ldots, f(v_p)\rangle$ is in $E^1_{\varphi[c,d]}(B)$.

If $\langle c, d, b_1, \ldots, b_q, u_1, \ldots, u_p\rangle \in E^1_{\varphi[c,d]}(B)$, we have $u_i \in [c, d]$, so there are w_i such that $u_i = f(w_i)$. Since $\langle c, d, b_1, \ldots, b_q, f(w_1), \ldots, f(w_p)\rangle \in E^1_{\varphi[c,d]}(B)$, also $\varphi^{[c,d]}(x_1/f(w_1), \ldots, x_p/f(w_p)) = d$ in B, which implies $\varphi(x_1/w_1, \ldots, x_p/w_p) = 1$ in A and $w_i = v_i$ for $i = 1, \ldots, p$. $\square$

The above formulas make it possible to define the value $f(k/m)$ in any Ł-component $[a_{i-1}, a_i]$, $i = 1, \ldots, n$, where k, m are integers and f is the corresponding isomorphism between $[0, 1]_Ł$ and $[a_{i-1}, a_i]$. For simplicity, $f(k/m)$ is referred to as "k/m in $[a_{i-1}, a_i]$".

Corollary 36. *Assume the i-th component $[a_{i-1}, a_i]$ of $*$ is an Ł-component. Then the value k/m in $[a_{i-1}, a_i]$ is definable in the variable y by the formula*

$$Id(x_0, \ldots, x_n)\&\lambda_{(k,m)}[x_{i-1}, x_i](x, y)$$

for all involutive negations.

As a consequence of Theorem 31, we are able to give a distinguishing formula for each two negations that are not I-equivalent for I being the set of indices of all Ł-components of $*$. We use the term Ł+Id for $U(I)$ and the term Ł+Id-equivalence for I-equivalence under this particular choice of the parameter I.

Observation 37. *The relation of being Ł+Id-equivalent (w. r. t. $*$) preserves classes of isomorphism.*

Proof. Assume that $\sim$ and $\sim'$ are isomorphic. Then, $\sim' x = f(\sim f^{-1}(x))$ for some automorphism f of $*$. We have to prove that $\sim$ and $\sim'$ are Ł+Id equivalent. Whenever $x \in Ł + \mathrm{Id}$, we have $f^{-1}(x) = x$. If moreover $\sim x$ is in Ł+Id, we have $\sim' x = f(\sim x) = \sim x$. Otherwise, since f preserves components, $\sim' x = f(\sim x)$ and $\sim x$ belong to the interior of the same product component. Thus $\sim$ and $\sim'$ are Ł+Id-equivalent. $\square$

5.3 Definable values in Π-components

In order to find a dense definable set in each Π-component, one can use the fixed point of an involutive negation, provided it is located in one of the product components, or one can employ an involutive negation to map some of the definable values in Ł-components into Π-components; one then forms positive rational powers w. r. t. $*$ of the obtained value. Both ways make the dense set depend on the negation involved.

We show that a sufficiently broad class of negations can be chosen, for which a dense set is definable for the first product component, and possibly also for the second one (if there is one).

For the rest, fix an arbitrary $*$ of relevant type. First we address the initial product component of the sum, Π_1. We use the term Ł+Id+Π_1 for $U(I)$ and the term Ł+Id+Π_1-equivalence for I-equivalence for I being the indices of all Ł-components and the first Π-component.

It is not true that any two negations that are not Ł+Id+Π_1-equivalent can be distinguished by a formula; note that two isomorphic copies of the same negation can be Ł+Id+Π_1-nonequivalent, because of the existence of non-trivial automorphisms of product.

Theorem 38. *Let C_0 be an L+Id-equivalence class of involutive negations (w. r. t. $*$). Then, there is a $C_1 \subseteq C_0$ s. t. elements of C_1 represent all classes of isomorphism in C_0, and there is a dense set of elements of Π_1 which is definable for C_1.*

Proof. We split the proof into several steps.

Fix a class C_0 of negations which are Ł+Id-equivalent w. r. t. $*$. Consider the last component $[a_{n-1}, 1]$, which is an Ł-component.

Observation 39. *For each two involutive negations $\sim_1$ and $\sim_2$ in C_0 we have $\sim_1 a_1 > a_{n-1}$ iff $\sim_2 a_1 > a_{n-1}$. Moreover, if both inequalities hold, then $\sim_1 a_1 = \sim_2 a_1$.*

Proof. Follows directly from the assumption of C_0 being a class of Ł+Id-equivalent negations, since each $x \geq a_{n-1}$ is in Ł+Id and both a_1 and a_{n-1} are in Ł+Id. □

Denote l_0 the maximum of a_{n-1} and the value of each $\sim \in C_0$ on a_1. By the previous observation this is well defined since there is at most one value $\sim a_1 > a_{n-1}$ for all $\sim \in C_0$.

Choose l_1 to be a definable[7] value in $(l_0, 1)$. Note that for any $\sim \in C_0$, we have $\sim l_1 \in (0, a_1)$; that is, the value is in Π_1.

Fix $\sim_0 \in C_0$, and let $y_0 = \sim_0 l_1$.

Lemma 40. *For any $\sim \in C_0$ there is an isomorphic copy $\sim' \in C_0$ s. t. $\sim' l_1 = y_0$.*

Proof. Let r_1 be such that $(\sim l_1)^{r_1} = y_0$. Let f be an automorphism of $*$ defined by $f(x) = x$ on all components except of Π_1 and $f(x) = x^{r_1}$ for $x \in \Pi_1$ (cf. Corollary 10). Let $\sim' x = f(\sim f^{-1}(x))$ for any x. If $x \in [a_{n-1}, 1]$, which is an Ł-component, we have $f^{-1}(x) = x$. Hence, $\sim' l_1 = f(\sim f^{-1}(l_1)) = (\sim l_1)^{r_1} = y_0$. It follows that $\sim'$ has the required properties. □

Definition 41. *Let $C_1 = \{\sim \in C_0 : \sim l_1 = y_0\}$.*

By the previous lemma C_1 includes at least one representative from each class of isomorphism on C_0.

Since for all $\sim \in C_1$ the values $\sim l_1$ coincide, we can use these values to obtain a dense set in Π_1 which is C_1-definable, by forming all positive rational powers w. r. t. $*$.

Corollary 42. *If $\varphi(y, z_1, \ldots, z_k)$ is the formula defining l_1 in the variable y in the last Ł-component (for the class of all involutive negations), then*

$$\varphi(y, z_1, \ldots, z_k) \mathbin{\&} (z^q \equiv (\sim y)^p)$$

defines z to be the p/q-th power of $\sim l_1$ in the first Π-component for all negations in C_1.

By combining Lemma 40, Definition 41 and Corollary 42, we obtain proof of Theorem 38. □

In the rest of Section 5.3, we restrict our attention to standard algebras of type $\Pi \oplus i.\text{Ł} \oplus \Pi \oplus j.\text{Ł}$, $j > 0$, and the definability of dense sets in the second product component, Π_2. Assume $*$ of this type is fixed.

[7] Since we choose a value in an Ł-component, we may assume it is definable for the class of all involutive negations.

Theorem 43. *Let D_0 be an L+Id+Π_1-equivalence class of involutive negations (w. r. t. $*$). Then, there is a subclass $D_1 \subseteq D_0$ which represents all classes of isomorphism in D_0 (in fact, it contains exactly one element from each class of isomorphism in D_0) and there is a dense definable set in Π_2 for the involutive negations in D_1.*

Proof. The proof is split into several statements.

Observation 44. *At least one of the following statements hold:*
(i) The fixed point of all negations in D_0 is in the interior of Π_2.
(ii) There is a value in L+Id+Π_1, definable for D_0, s. t. all negations in D_0 map this value into the interior of Π_2.

Proof. If the fixed point of all $\sim\in D_0$ is in the interior of Π_2, then (i) is satisfied. Otherwise, fix $\sim\in D_0$ such that its fixed point is in Ł+Id+Π_1 and consider the values $\sim a_{i+2}, \sim a_{i+1}$, where a_{i+1} and a_{i+2} are the delimiting idempotents of Π_2. Note that $\sim a_{i+2}, \sim a_{i+1}$ belong to Ł+Id+Π_1. Since all the negations in D_0 are Ł+Id+Π_1-equivalent, all of them have the same values on a_{i+1} and a_{i+2}, which are either both below, or both above the interior of Π_2. Hence, there is a value inside $(\sim a_{i+2}, \sim a_{i+1})$, that is definable. This value satisfies the properties required by (ii). ☐

The construction of the set D_1 depends on whether (i) or (ii) is satisfied in Observation 44. First assume (i) is satisfied. Fix $\sim_1\in D_0$, and let b_1 be its fixed point. We have $b_1\in\Pi_2$.

Lemma 45. *For each $\sim\in D_0$ there is an isomorphic copy $\sim'\in D_0$ s. t. the fixed point of $\sim'$ is b_1.*

Proof. Let $\sim\in D_0$ and let b be its fixed point. By definition of D_0, any isomorphism of two negations in D_0 is the identity on Π_1. We define an isomorphic copy $\sim'$ of $\sim$ by specifying the exponent r_2 which determines the automorphism f of $*$ on Π_2. Let r_2 be s. t. $b^{r_2} = b_1$ and let $\sim' x = f(\sim f^{-1}(x))$. Then, $\sim' b_1 = f(\sim f^{-1}(b_1)) = f(\sim b_1^{1/r_2}) = f(\sim b) = f(b) = b_1$. ☐

Definition 46. *Let $D_1 = \{\sim\in D_0 : \sim b_1 = b_1\}$.*

Corollary 47. *The formula $(\sim z \equiv z)\&(x^q \equiv z^p)$ defines the value $b_1^{p/q}$ in Π_2 for each $\sim\in D_1$.*

Combining Lemma 45, Definition 46, and Corollary 47, we obtain the dense set needed for the proof of Theorem 43 in case (i).

Now, assume (ii) is satisfied in Observation 44 and let x_2 be the definable value guaranteed by (ii). Fix $\sim_2\in D_0$, and let $y_2 = \sim_2 x_2$.

Lemma 48. *For any $\sim\in D_0$ there is an isomorphic copy $\sim'\in D_0$ s. t. $\sim' x_2 = y_2$.*

Proof. Similarly as above, we define the isomorphic copy $\sim'$ of $\sim$ by specifying the value r_2 which determines the automorphism f of $*$ on Π_2. Let us choose r_2 such that $(\sim x_2)^{r_2} = y_2$ and let $\sim' x = f(\sim f^{-1}(x))$. Since $x_2\in$ Ł+Id+Π_1, we have $\sim' x_2 = f(\sim f^{-1}(x_2)) = f(\sim x_2) = (\sim x_2)^{r_2} = y_2$ as required. ☐

Definition 49. *Let $D_1 = \{\sim\in D_0 : \sim x_2 = y_2\}$.*

Corollary 50. *If $\varphi(y, z_1, \ldots, z_k)$ is the formula defining y to be x_2 in one of the Ł-components or Π_1 component, then*

$$\varphi(y, z_1, \ldots, z_k) \mathbin{\&} (x^q \equiv (\sim y)^p)$$

defines x to be the p/q-th power of $\sim x_2$ in Π_2 for all negations in D_1.

Combining Lemma 48, Definition 49, and Corollary 50, we obtain the dense set needed for the proof of Theorem 43 in case (ii). □

5.4 Main result of Section 5

Theorem 51. *Let $*$ be a continuous t-norm of type either $\Pi \oplus j.Ł$ or $\Pi \oplus i.Ł \oplus \Pi \oplus j.Ł$, where $j > 0$ in both cases. For any two negations $\sim_1$ and $\sim_2$ that are not isomorphic w. r. t. $*$, $\mathrm{TAUT}(\langle *, \sim_1\rangle)$ and $\mathrm{TAUT}(\langle *, \sim_2\rangle)$ are incomparable.*

In other words, any continuous t-norm of types $\Pi \oplus j.Ł$ or $\Pi \oplus i.Ł \oplus \Pi \oplus j.Ł$, for $j > 0$, satisfies the conditions (1) and (2) from Section 1.

Proof. Fix $*$ of one of the abovementioned types, and let $\sim_1$, $\sim_2$ be two arbitrary involutive negations non-isomorphic w. r. t. $*$.

Assume first that $\sim_1$, $\sim_2$ are not Ł+Id-equivalent (i. e., I-equivalent for I being the set of indices of all Ł-components of $*$). By results of Subsection 5.2, a dense set is definable in each Ł-component for the class of all involutive negations (cf. Corollary 36). Hence, Theorem 31 applies for C being the class of all involutive negations and for I being the set of indices of all Ł-components. Consequently, distinguishing formulas for $\mathrm{TAUT}(\langle *, \sim_1\rangle)$ and $\mathrm{TAUT}(\langle *, \sim_2\rangle)$ are obtained.

If, on the other hand, $\sim_1$ and $\sim_2$ are within the same Ł+Id-equivalence class C_0 of involutive negations, then by Theorem 38 there is a class $C_1 \subseteq C_0$ representing all classes of isomorphism in C_0, and such that a dense set is definable in the first Π-component Π_1 for all negations in C_1. We find isomorphic copies $\sim_1'$ of $\sim_1$ and $\sim_2'$ of $\sim_2$ s. t. $\sim_1', \sim_2' \in C_1$.

If $\sim_1'$ and $\sim_2'$ are not Ł+Id+Π_1-equivalent (i. e., I-equivalent for I being the set of indices of all Ł-components and the first Π-component of $*$), then Theorem 31 applies for C being C_1 and for I being the set of indices of all Ł-components and the first Π-component. Consequently, the required distinguishing formulas are obtained.

Assume $\sim_1'$ and $\sim_2'$ are within the same Ł+Id+Π_1-equivalence class D_0. It follows from the assumption of $\sim_1'$ and $\sim_2'$ being non-isomorphic that there are two product components, Π_1 and Π_2, in $*$. Then by Theorem 43 there is a class $D_1 \subseteq D_0$ representing all classes of isomorphism in D_0, and such that a dense set is definable in the second Π-component for D_1. We find isomorphic copies $\sim_1''$ of $\sim_1'$ and $\sim_2''$ of $\sim_2'$ s. t. $\sim_1'', \sim_2'' \in D_1$. It follows that $\sim_1''$, $\sim_2''$ are not equivalent for I being the set of indices of all components in $*$. Theorem 31 then applies for C being the class D_1 of involutive negations, and for I being the set of indices of all components in $*$. Consequently, the required distinguishing formulas are obtained. □

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