

Petr Hájek: a scientific biography

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1 Introduction

Petr Hájek is a renowned Czech logician, whose record in mathematical logic spans half a century. His results leave a permanent imprint in all of his research areas, which can be roughly delimited as set theory, arithmetic, fuzzy logic and reasoning under uncertainty, and information retrieval; some of his results have enjoyed successful applications. He has, throughout his career, worked at the Academy of Sciences of the Czech Republic¹, starting as a postgraduate student at the Institute of Mathematics in 1962. At present, he is a senior researcher at the Institute of Computer Science.

Petr's scientific career is well captured by the books he (co)authored:

- P. Vopěnka, P. Hájek: *The Theory of Semisets*. Academia Praha/North Holland Publishing Company, 1972.
- P. Hájek, T. Havránek: *Mechanizing Hypothesis Formation: Mathematical Foundations of a General Theory*. Springer, Berlin, 1978.
- P. Hájek, T. Havránek, M. Chytil: *Metoda GUHA: Automatická tvorba hypotéz*, Academia, Praha, 1983. (in Czech).
- P. Hájek, T. Havránek, R. Jiroušek: *Uncertain Information Processing in Expert Systems*. CRC Press, Boca Raton, 1992.
- P. Hájek, P. Pudlák. *Metamathematics of First-Order Arithmetic*. Springer Verlag, 1993.
- P. Hájek: *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.
- P. Cintula, P. Hájek, C. Noguera (eds.): *Handbook of Mathematical Fuzzy Logic*. College Publications, London, 2011.

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Apart from these books, Petr Hájek is the (co)author of more than 350 research papers, textbooks and popular articles; his works are frequently cited with the number of citations approaching 3000. He taught logic at the Faculty of Mathematics and Physics, Charles University in Prague, where he was appointed full professor in 1997, and at the Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University. He also taught at the Vienna University of Technology, where he was appointed honorary professor in 1994. For the timespan of four decades, he has been running a weekly seminar of applied mathematical logic, and he co-founded another seminar on mathematical logic that is still being run at the Institute of Mathematics.

He has served as a member of committees and editorial boards and has been a long-time member of the Union of Czech Mathematicians and Physicists. Since 1993, he has been a member of the Association for Symbolic Logic. During 1999 to 2003 he was the President of Kurt Gödel Society; he was reelected in 2009 and is currently serving his second term. Since 1996 he has been a member of the Learned Society of the Czech Republic. During 1993 to 2005 he was a member of the Scientific Council of the Academy of Sciences of the Czech Republic. His awards include the Bolzano medal from the Academy of Sciences in 2000, a medal of the Minister of Education of the Czech Republic in 2002, the *De scientiae et humanitate optime meritis* medal from the Academy of Sciences in 2006, the Medal of Merit from the President of the Czech Republic in 2006, and the Josef Hlávka medal in 2009.

Apart from the pursuit of mathematics, Petr Hájek is an organist. He graduated from the Academy of Performing Arts in Prague and was, for a considerable period of time, organist on Sundays at the protestant St. Clemens Church in Prague; since childhood years he has been a member of the Evangelical Church of Czech Brethren. He is married, has two children and a grandson. He is fluent in several languages, including German, English, and Polish.

Petr Hájek is generally viewed as a very friendly and modest person, known for his readiness to help and listen to others. Many colleagues consider him their teacher. He is respected for his principles, not least among these, his stands during the totalitarian era, when he would not enter the communist party or cooperate with the State Security (known under the acronym StB). In consequence, for a long period of time, he was prevented from advancing his career or travelling abroad.

The few above paragraphs condense Petr Hájek's life to a very modest space, collecting the highlights of his professional career. This may be sufficient for many readers. Still, in this biographical essay, I will try to offer somewhat more: to record an appropriate context for events; to mention people that Petr encountered; and to answer some why-questions. I must emphasize that, though I can contribute a knowledge of Petr based on personal acquaintance, being younger I have only met him in his "fuzzy period". Thus in the earlier periods I rely on documents and recollections of others. By nature this is a professional

biography, thus it will not delve into Petr’s private life.

2 Early years and set theory

Petr Hájek was born in Prague on February 6, 1940; after him, two girls were subsequently born into the family. His mother was a private language teacher and his father worked in Papirografia Praha; the family lived in the Prague quarter of Žižkov. In supplement to the usual education, Petr received a musical one: he took piano lessons in a public school of arts. The family were religious, being members of the Evangelical Church of Czech Brethren and frequenting a church near their home; it was a natural decision for young Petr to start to study the organ, with a view of, one day, being able to play it at services, thus contributing his skill to the community.

In June 1957 Petr completed his secondary education by graduating from a local high school, namely, Jedenáctiletá střední škola v Praze, Sladkovského náměstí (currently Gymnázium Karla Sladkovského). At that time, Petr was deliberating his future, deciding between mathematics and music.

The final decision was to make mathematics his main pursuit, and the young Petr commenced his studies at the newly established Faculty of Mathematics and Physics of Charles University in Prague. He finished in 1962, submitting a master thesis in algebra, written under the guidance of Vladimír Kořínek, a well known algebraist. Even though Petr was an excellent student, it was out of the question for him to get a position at the Faculty: authorities declared it undesirable that a religious person such as himself have any contact with students. At that time, upon graduating from the University, students were “assigned” employment roughly in the area of study. The exact process of assignation varied, but its results were often cumbersome: it was not uncommon for Prague residents to be assigned to the outskirts of the country. This time, however, Petr was lucky: in 1962, he obtained a position at the Institute of Mathematics of the Czechoslovak Academy of Sciences. This was also the commencement of his postgraduate training, which, at that time in our country, was called *aspirantura*, and those who successfully completed it were honoured by the title *candidate of sciences* (CSc).

Petr started his studies under the guidance of Ladislav S. Rieger, a professor at the Czech Technical University in Prague and a distinguished logician. He introduced Petr to contemporary results in mathematical logic, and recommended some essential reading. To appreciate what Rieger’s agenda was like, see for example [28]. He also conducted a seminar in mathematical logic; one of the attendees was Petr Vopěnka. Unfortunately, Rieger passed away in 1963. In his essay *Prague set theory seminar* (see [35]), Petr Vopěnka writes: “. . . Then [after Rieger’s death], I decided to start a new seminar in axiomatic set theory, intended mainly for students. The students who enlisted were (in alphabetical order) Bohuslav Balcar, Tomáš Jech, Karel Hrbáček, Karel Příkrý, Antonín Sochor, Petr

Štěpánek and some others. We were joined by Lev Bukovský from Bratislava, and, last but not least, Rieger’s doctoral student, Petr Hájek. The main target of the seminar was to study non-standard models of Gödel–Bernays set theory.” The seminar took place at the Faculty of Mathematics and Physics, where Vopěnka worked throughout.

Petr Vopěnka is often considered to have been Petr Hájek’s advisor. While there is no doubt that Vopěnka actually advised Petr Hájek’s in many respects, and was his teacher, it was Karel Čulík who was appointed the advisor after Rieger’s death. Čulík, at that time employed in the Institute of Mathematics, was an excellent mathematician with a broad scope of interests, and, like so many of his colleagues, not in grace of the authorities; he finally left Czechoslovakia in 1976 (see [14]). Petr Hájek submitted his thesis, ‘Models of set theory with individuals’, in 1964 (see [10]), and defended it a year later.

Subsequently to his thesis, Petr Hájek published a considerable number of papers on set theory; many of them were about the role of the axiom of foundation. Some were coauthored by colleagues from Vopěnka’s seminar. Some favourite publishing options included *Commentationes Mathematicae Universitatis Carolinae*, a mathematical journal published by Charles University since 1960; *Časopis pro pěstování matematiky* (‘Journal for fostering mathematics’), published by the Union of Czech Mathematicians and Physicists; or *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, where many papers of Vopěnka’s group were published in English or in German. Meanwhile, Petr was not neglecting music, and continued to study the organ, under the guidance of Jaroslava Potměšilová, a distinguished Czech organist.

Vopěnka’s set theory seminar was a great success: it brought together a group of young researchers² who shared a common topic of interest and who contributed substantially to the set-theoretical agenda of the period. Even today, there are very few students of logic in Prague who have never heard about Vopěnka’s seminar and are not aware of many of the participants’ contributions to mathematical logic, given in the course of their lives. Still, even though Vopěnka himself achieved lasting results in (what he refers to as) Cantor’s set theory, he was rather uncomfortable with its progress. In particular, the independence results of the late nineteen sixties seemed for Vopěnka to highlight an element of arbitrariness in choosing set-theoretic axioms which was beyond his endurance (see [36]). Vopěnka is, primarily, a mathematician. For him, investigations of formal theories and relations inbetween them (the term ‘metamathematics’ is often used) is an interesting, but secondary pursuit; a formal theory does not constitute the objects that form the subject matter of mathematics, but merely tries to capture them, more or less conveniently. He has always had strong preconceptions of the universe of mathematical discourse; in particular, his concern was the phenomenon of infinity. Vopěnka’s view was that Cantor’s set theory was cumbersome in capturing this phenomenon, having closed many doors that should have remained open.

²In 1963, Petr Vopěnka was twenty-eight, and most of the attendants were undergraduate students.

The theory of semisets, written by Petr Vopěnka and Petr Hájek (neither of the authors was fluent in English at that time and the book was translated from Czech by T. Jech and G. Rousseau), was published simultaneously by North Holland Publishers and by Academia in Prague in 1972 (see [37]). This book is a result of an intense study of the construction of models for set theory, to which Vopěnka contributed significantly during the sixties. A *semiset* is a subclass of a set; the theory of semisets is formally obtained by modifying the axioms of NBG in such a way that they admit (but do not prove) the existence of proper semisets. The theory of sets extends the theory of semisets by simply positing that all semisets are sets; this extension is conservative in the sense that it does not add any new statements about sets. The book develops both theories (i.e., of semisets and of sets) along each other, exploring their mathematics and presenting many results on them, highlighting the differences. It sets great store by *interpretations* (also called ‘syntactical models’ in the text), typically sought as a means of obtaining relative consistency statements; interpretability later—during his arithmetic years—became the flagship of Petr Hájek’s research.

Perhaps it is worth stressing at this point that, while Vopěnka and Hájek joined forces to make a significant step aside from the mainstream of research in mathematical logic, both were, at the same time, excellent and very active researchers in the classical line. Interestingly, the mindsets of these two researchers seem to be very different: with a little exaggeration, one might say that from Vopěnka’s view, Petr Hájek is a formalist, whereas from Hájek’s view, Petr Vopěnka is a foundationist. Looking at Petr Hájek’s works, one notices that very early on he gives a set of axioms and rules; without these, it would be unthinkable to continue. In Petr Vopěnka’s works, some axioms will, reluctantly and almost apologetically, be given halfway through the text. From this aspect, the book on semisets is an interesting synthesis of these two approaches operating together. Although excellently thought of and docilely written, the book never attracted a wide audience.

Some years later, Petr Vopěnka wrote another book and brought up another generation of students. This book, called ‘An introduction to mathematics in alternative set theory’ (see [34]), was published in Bratislava in 1979, having been translated into Slovak language by Pavol Zlatoš. While Vopěnka’s alternative set theory can be seen as a continuation of some ideas present in *The Theory of Semisets*, it departs much further from the classical line and, one may say, offers a remedy to some of its alleged misconceptions. A notorious example of a semiset in alternative set theory is the collection of natural numbers n such that n grains of wheat do not form a heap; this property delimits a class, but the class is not a set, thus it is a proper semiset. Perhaps this example may sketch how semisets, among other things, model the vagueness phenomenon. Prior to this publication, Vopěnka had been running a second installment of his set-theoretic seminar, which was dedicated to developing and working in the alternative set theory. Again the seminar was very

popular among its contemporaries.³ Among the former attendants of the seminar, one can find Karel Čuda, Josef Mlček, Jiří Sgall, Antonín Sochor, Kateřina Trlifajová, Alena Vencovská, Blanka Vojtášková and Jiří Witzany. While Vopěnka's alternative set theory is still a popular concept among Czech logicians, from a more global point of view it seems to have shared the fate of many other hitherto proposed alternatives to the classical conception of mathematics (logic, set theory etc.): it was trampled underfoot the crowd that pursued the classical direction.

A more focused view of Petr Vopěnka's personality and achievements can be found in [29], an introductory paper to a special issue of *Annals of Pure and Applied Logic* dedicated to himself.

The years spent with Vopěnka's group at the Faculty of Mathematics and Physics brought another major change into Petr's life: he met his second wife, Marie, among the people who frequented the seminar. They were married in 1969, after Petr had spent a semester visiting his colleague and lifelong friend, Gert Müller, in Heidelberg. Petr Vopěnka was a witness at the wedding. Petr cooperated with Marie and they coauthored several papers; a glimpse into their life together can be found in [20].

3 Arithmetic

In the beginning of the seventies, Petr Hájek was still deeply engaged in set theory; however, he also seemed open to starting a new line of research. Alluring new topics presented themselves at that time; in particular, computational complexity was established as a new research area. A bit later, exciting new incompleteness results appeared in the form of natural combinatorial statements independent of Peano arithmetic. A first-hand account of the echoes these great currents had in Prague, and a lot more, is presented in Pavel Pudlák's essay [25].

During this busy period, Petr also enlisted as a student⁴ at the Music Faculty of the Academy of Performing Arts in Prague, where his subject was the organ and his tutor was Jiří Reinberger, a Czech organ virtuoso, teacher and composer. Petr obtained his degree, and continued his engagement as an organist in the St. Kliment church.

Pavel Pudlák became Petr's student in mid seventies, in particular, he wrote his master thesis under Petr's supervision, on a subject in finite model theory. The scope of Pudlák's interests was rather broad, ranging over algebra, combinatorics, and computational complexity. After some time elapsed, and some deliberation, he and Petr arrived at a decision to make arithmetic the object of their joint study, in the late seventies. Petr had had a previous acquaintance with Andrzej Mostowski in Warsaw, with whom the topic had a long tradition and around whom a working group formed itself gradually (including Zofia

³The first installment of Vopěnka's seminar dispersed after 1968.

⁴Because of his employment, the form was a distance study.

Adamowicz and Roman Kossak, see [1]). Poland is a neighbouring country and it was relatively easy to travel there; this was a lucky circumstance, owing to which Polish and Czech logicians were able to meet frequently and share knowledge.

Another person with whom Petr shared his interest in arithmetic was his wife, Marie. She was a member of Petr Vopěnka's department, and her thesis, defended in 1969, concerned binumerations of arithmetic, extending earlier results of Feferman [7]. This inspired Petr to give a course for students on the topic at the Faculty of Mathematics and Physics, in the early seventies.

In the late seventies, Petr gained another student, Vítězslav Švejdar, who was at that time working on his master thesis on interpretability; later, in 1982, he defended a dissertation 'Interpretability and modal logic' (see [30, 31]). As already remarked, interpretability was a key topic of Petr Hájek's research; Švejdar's work explored interpretability as a modality on arithmetical sentences, in a manner analogous to that of provability.

A mini-seminar on arithmetic was started in the Hájeks' flat around 1978, in which Marie also participated. Gradually a working group on arithmetic formed itself in the Institute of Mathematics; somewhat later on, this group would include Jan Krajíček (then a student of Pavel Pudlák). Shortly before 1980, a regular seminar was started in the Institute. It would meet weekly in long, lively sessions to discuss the group's own results or to present interesting papers; at the especially busy period when Hájek and Pudlák were working on *Metamathematics of First-Order Arithmetic*, reportedly two hours were not sufficient, so there were two sessions; often one was dedicated to what Petr was writing, the other occupied by topic of the attendees' choice. The seminar is still alive at the Institute of Mathematics; after Petr Hájek left, it has been run by Jan Krajíček and Pavel Pudlák for a long period of time; currently, it is run by the joint effort of Pavel Pudlák and Neil Thapen.

The arithmetic group (within the Department of Numerical Algebra, Graph Theory and Mathematical Logic, headed by Miroslav Fiedler) cooperated with other groups, especially set theorists and recursion theorists in Prague, organizing workshops in Alšovice in the Czech mountains of Jizerské hory. The workshops were quite popular, enjoying a warm, informal atmosphere; occasionally the Czech community would be able to welcome distinguished guests, such as Jeff Paris, Per Lindström, or Alex Wilkie. Otherwise, travelling options of Czech logicians, and hence also their chance of meeting researchers from abroad, were limited.

It was a great honour for logicians in Prague to be entrusted with organizing the Logic Colloquium 1980. Petr Vopěnka was appointed chair of the programme committee. Petr Hájek was chair of the organizing committee, and the whole working group in the Institute of Mathematics was involved in the preparations, alongside other Prague logicians. The preliminary list of participants counted nearly 400 heads from all over the world. Before the conference, in the spring of 1980, there was some deal of perplexity among the foreign researchers who were about to take part, regarding whether and how to express their

views on the totalitarian regime then in full swing in Czechoslovakia. Particular regard was paid to Václav Benda, a Czech mathematician, a Charter 77 signatory and the father of five small children, who was at that time imprisoned for political reasons (a so-called “prisoner of conscience”). His wife, Kamila Bendová, was a member of the logic group at the Institute of Mathematics, involved in the organization of the event. The general idea was that a focused effort of many mathematicians might help a fellow mathematician to lessen the pressure of authorities on himself. However, before these intentions were allowed to take a concrete direction, the State Security, in fear of any kind of trouble (the term “provocations” is used in their files), set things in motion so that the Colloquium had to be cancelled. Petr Hájek was obliged to personally send out letters of apology, stating a fictitious reason for cancellation. The affair hit him deeply; moreover, he was, for a time, prevented from travelling abroad.

Despite limitations in contact, Prague came to be considered an important member of the European arithmetic community; apart from the already mentioned researchers in Warsaw, the arithmetic group in the Institute of Mathematics enjoyed longterm, fruitful cooperation with Manchester (Peter Clote, Richard Kaye, Jeff Paris, Alex Wilkie), Amsterdam and Utrecht (Dick de Jongh, Rineke Verbrugge, Albert Visser), Siena (Franco Montagna) and other researchers; many people considered it worth their while to come and stay, even in modest circumstances (see Mathias Baaz’s account [2]). In the summer of 1991, Prague hosted a month-long workshop and an associated conference on proof theory, arithmetic and complexity, complementing a similar event in San Diego a year earlier; see [5] for papers from the meeting.

In arithmetic, Petr applied his craft especially to studying *conservativity* and *interpretability*: given that a consistent, recursively axiomatizable theory T containing arithmetic is incomplete, for each φ independent of T one may ask how conservative it is over T , and whether $T \cup \{\varphi\}$ has an interpretation in T . The notions are studied in the context of arithmetical hierarchy of formulas; particular attention is paid to fragments of arithmetic obtained by setting an upper bound on arithmetical complexity of formulas used in the induction schema. In Petr Hájek’s treatment, these notions became a rather neat way of capturing the strength of theories of arithmetic. These topics are extensively covered in Petr’s dissertation submitted in 1988 for the ‘doctor of sciences’ (DrSc) degree. The dissertation is called ‘Metamathematics of first-order arithmetic’ (see [11]), and it is a direct predecessor of Petr’s part of the famous book on arithmetic bearing the same title, written jointly with Pavel Pudlák a couple of years later. The dissertation is typewritten in lovely, docile Czech, with handwritten formulas and symbols. Based on this work, Petr became doctor of sciences in 1990.

Around 1990, the Ω -Group, through one of its members, Gert Müller, approached Petr Hájek with the question whether he would be willing to write a monograph on arithmetic. Petr agreed, inviting Pavel Pudlák as a coauthor. *Metamathematics of First-Order Arithmetic* was published by Springer in 1993, in the ‘Perspectives in Mathematical Logic’

series (see [19]). The book has three parts. The first one investigates fragments of Peano Arithmetic obtained by bounding the arithmetical complexity of formulas used in the induction axiom, showing them sufficient for some parts of mathematics (e.g., combinatorial principles) and developing some technical tools. The second part is devoted to the incompleteness phenomenon and the study of various notions of relative strength of theories, such as the above. The third part, written by Pavel Pudlák, studies bounded arithmetic, reflecting the tumultuous development of this area during the eighties.

4 Logic applied to computer science

A prevailing trait of Petr Hájek's personality is his strong desire to offer his service. This desire has many facets, and we shall not be exploring all of them; in this section, we shall look into Petr's efforts to offer the services of logic to other scientific disciplines, mainly computer science, and through it also to medicine, biology, humanities, etc. Characteristically, Petr was always keen to help and employ his skill in interdisciplinary research, but never willing to make one step down from the high standards on clarity and rigour that he maintained.

Very soon after he finished his postgraduate training, a challenge to apply a rather nice portion of logic presented itself. It was initiated by Metoděj Chytil from the Institute of Physiology of the Czechoslovak Academy of Sciences; he proposed some ideas that initiated the development of the GUHA (General Unary Hypotheses Automaton) method. The idea of GUHA rested in listing exhaustively all valid universally quantified implications about a given data matrix, where lines represent objects and columns represent their Boolean properties. A suggested usage was performing the exhaustive search for valid statements on a small sample of data, thus obtaining all valid statements within reasonable time; then conceiving the "most interesting" statements as hypotheses to be tested on a larger dataset.

The authors of the method were Petr Hájek (who contributed the element of logic), Ivan Havel (who implemented the algorithm) and Metoděj Chytil; it was first presented in 1965 and published in 1966 as [16]. The first implementation was running on a MINSK 22 machine.

This pioneering work grounded a new area of applied research in Prague, and much effort was devoted to enhancement of the GUHA method; part of the effort naturally went to implementing and applying GUHA, and to collaborating with intended users, mainly researchers in biology and social sciences. The word 'user' is perhaps too laden with recent connotations to convey what it was like to use the early implementation (or, one may say, any implementation) of GUHA; a small interdisciplinary team was usually needed, to collect and prepare the data, to correctly define the parameters of each run, to actually run the program, and to cope with the results.

However, GUHA also lent itself to theoretical endeavours. Obviously, if any operation on data is costly, then time can be saved with applying deduction wherever possible and refraining from testing the validity of deducible statements in the data. Petr Hájek spoke about ‘observational calculi’, and these form his main contribution to publications about the theoretical aspects of GUHA.

The GUHA team included Kamila Bendová from the Institute of Mathematics, Zdeněk Renc from the Faculty of Mathematics and Physics, Dan Pokorný from Mathematical centre of Biological Institutes of the Czech Academy of Sciences, and many other people.

The method benefited considerably from the arrival of Tomáš Havránek on the team. Havránek was a statistician, and under his guidance, statistical quantifiers were introduced to GUHA in addition to a logical implication: moreover, he supervised the employ of the statistical paradigm in the whole approach.

Petr Hájek and Tomáš Havránek wrote a very comprehensive book about GUHA—*Mechanizing Hypothesis Formation: Mathematical Foundations of a General Theory*, published by Springer in 1978 (see [17]). The book contained the full thitherto developed theory, and also many methodological and historical remarks. A Czech book about GUHA, targeting mainly its potential users, was published by Academia in Prague in 1983.

Petr gained two successful doctoral students in the GUHA line: Jiří Ivánek (defended 1984, see [22]) and Jan Rauch (defended 1986, see [27]). Both of them have retained an interest in the development of the method, and have continued their work on the method or related issues. The GUHA research continued naturally in the Institute of Computer Science, before and after Petr became its director (in 1992); perhaps we can say that this line of Petr’s research played a major role in eventually bringing him into the Institute. The research group there included Anna Sochorová, Dagmar Harmancová, Jana Zvárová, Martin Holeňa and David Coufal.

GUHA never enjoyed a large-scale application or the interest of software-developing companies. Its limitations are easy to grasp: it was designed at a time and place where any kind of commercial enterprise was hardly thinkable; its theoretical aspects were too formidable for a user from a different background; it only operated on binary data; there was little demand for exploratory data analysis. However, it remained an interesting subject of study, a tool for academic applications, and a ground for interdisciplinary cooperation.

Around 1980, Petr Hájek became interested in expert systems (the term ‘consulting systems’ appears in some of his works), then very popular artificial intelligence tools. Apart from viewing expert systems as a possible application of logic and a stimulation for its development, the interest was due to a practical need for such system, to complement the existing GUHA procedures. In particular, it was hoped that such a system might guide a nonexpert user through the advanced options offered by GUHA implementations, especially its many quantifiers; the ultimate target was a fully automated GUHA. This

target provided a name for the earliest version of the expert system—it was called G-QUANT ('G' for 'GUHA' and 'QUANT' for 'quantifiers').

Petr Hájek and his colleagues focused on *rule-based* systems, i.e., those using the architecture of a knowledge base and rules. A knowledge base is a set of propositions. Rules of the form $A \rightarrow S(w)$ express the fact that knowing A contributes to knowing S with some weight w . The weights are taken from a chosen set endowed with some mathematical structure, allowing for comparison and combining weights. Weights intuitively represent how *certain* the given individual is of validity of the given information. Moreover, uncertainty may be present in the form of missing information, inherent vagueness, imprecision, etc.

Dempster–Shafer theory of evidence assigns *intervals* within some chosen domain (with an ordered-group structure) to statements and their combinations. During the eighties, Petr acted as advisor to a graduate student from Cuba, Julio Valdés. Together they undertook an algebraic analysis of the system of weight assignments developed by Dempster and Shafer. The structure is that of the Dempster semigroup; their results are collected in Valdés's dissertation (see [33]). Also Milan Daniel, originally a student of Tomáš Havránek (who passed away in 1991) wrote his dissertation under the guidance of Petr Hájek (see [6]). David Harmanec, Petr Hájek's doctoral student, finished his studies in the United States under supervision of George Klir.

On a practical line, Petr and his colleagues, mostly based at the Institute of Computer Science—Marie Hájková, Milan Daniel, and Tomáš Havránek—developed and implemented an expert system shell, called EQUANT, in Prolog. 'E' stands for 'empty'—the system has no fixed knowledge base, but concerns itself with combining the assigned weights and the propagation of uncertainty. The system developed over time, and several implementations existed. However, the dream did not come quite true: GUHA never became fully automated.

Theoretical issues on processing uncertainty gave rise to a book, *Uncertain Information Processing in Expert Systems*, written by Petr Hájek, Tomáš Havránek, and Ivan Jiroušek, published in 1992 by CRC Press (see [18]). The issues discussed in the book attracted a wider community; Ivan Kramosil, previously at the Institute of Information Theory and Automation, joined the group in the Institute of Computer Science in 1992.

In the late sixties, Petr Hájek founded a seminar to pursue the GUHA issues; it is referred to as 'seminar of applied mathematical logic' or simply 'Hájek's seminar'. The seminar would meet weekly, at first at the Faculty of Mathematics and Physics in Karlín, then in a Czech Technical University building at Albertov, later also at the Institute of Mathematics. As time passed, the scope of the seminar widened, and it attracted many people from the mathematical logic and computer science communities in Prague. It later moved with Petr to the Institute of Computer Science, and changed contents according to the shift of Petr's interests—recently, a lot of time has been devoted to fuzzy logic. The

seminar is still being run by the joint effort of Petr Hájek and Petr Cintula.

The difficulty in travelling abroad and maintaining contact with researchers from other countries perhaps contributed to bringing local and regional conferences to rather high standards. There was a lot of meetings and workshops, on regular and irregular basis; some of them grew into a tradition and are still continued nowadays. Distinguished speakers from abroad were invited where possible, and the possibility to meet them was regarded as a treat. Let us recall two of the regular events.

MFCS (Mathematical Foundations of Computer Science) is an annual conference started in 1972. The conference is organized in turns in Czech Republic, Slovakia, and Poland, in summertime; it remains a major regional event in theoretical computer science in each of these countries. Petr Hájek would be frequently a member of the programme committee of MFCS, and also a speaker there.

SOFSEM (Software Seminar) is held annually since 1974; intended for the Czechoslovak computer science community, it usually took place in the mountains in wintertime, and until 1994, a meeting would last two weeks, resembling a school more than a conference. The SOFSEM meetings had a warm, lively atmosphere and were extremely popular; at the height of their glory, they were so crowded that it was difficult to secure a place there. As time passed, the SOFSEMs grew more and more international, now being a regular international conference, held in Czech Republic or in Slovakia. Petr Hájek was invited as a speaker there several times, contributing topics discussed in this section.

In the beginning of the 1990's, big changes were in order both for Petr Hájek and for his homeland, Czechoslovakia. The country had just seen the Velvet Revolution, and the fall of the totalitarian regime had splashed away a lot of repression. Many people who had been barely tolerated by the regime, for their political stands, class origin, religious beliefs, or family ties, and consequently had been prevented from developing their careers, travelling abroad, and doing many other things that human spirit longs to do, were free at last. Petr Hájek was, to a considerable degree, such a person.

In 1991, Tomáš Havránek, director of the Institute of Computer Science and Petr Hájek's coauthor and friend, passed away at the bloom of his scientific powers. Soon after, it was proposed to Petr to consider himself a candidate for the position of director. The link to Petr consisted in his longterm engagement in the scientific agenda of the Institute. It was felt that Petr was able to contribute not only his scientific excellence on an international scale, but also an unblemished personal record; at the particular time at the particular place, the second quality was to be appreciated as much as the first one. Petr considered and accepted the idea, he was elected and appointed director of the Institute, and assumed office in March 1992.

The Institute of Computer Science⁵ has an interesting history. It was established in

⁵The name 'Institute of Computer Science' was established in 1997, but for simplicity we use it also for the earlier period.

1975 as a General Computing Centre of the Czechoslovak Academy of Sciences, relatively well equipped to provide computing services on demand of the institutes of the Academy. During the 1980's, it was transformed into a scientific institute in its own right. At that time, and especially in the 1990's, the Institute strove to establish itself as a fully fledged academic organization. By being appointed its director, Petr Hájek became an important partaker in the effort.

With the change of political regime, it was also possible for Petr to extend his activities by starting teaching students on a regular basis. In 1993, he became associate professor at the Faculty of Mathematics and Physics, Charles University in Prague; in 1997, he was appointed full professor of mathematics there. He taught a comprehensive course in first-order logic. At the Faculty of Nuclear Sciences and Physical Engineering, he later taught fuzzy logic⁶. He also taught logic at the Vienna University of Technology, being fluent in German, and was appointed honorary professor there in 1994.

5 Fuzzy Logic

The monograph *Metamathematics of First-Order Arithmetic* brought both its authors worldwide recognition, not to say stardom. Arithmetic was a subject well in the mainstream of mathematical logic. On the other hand, fuzzy logic, even now, after a continued effort of many researchers spanning more than two decades, still seems to stand slightly in need of defence, or at least, an explanation. Petr has always been a person capable of providing very convincing explanations. We will try to retrace his path, exploring the intercourse between Petr and fuzzy logic, tracing the shift of meaning of the phrase over time.

Fuzzy logic is based on the conviction that the truth of a proposition is a matter of degree, that the truth degrees of propositions can be compared, and that the truth degree of a compound proposition can be computed from those of its constituents. This leads to the concept of an algebra of truth degrees; key examples of fuzzy logics have emerged as formal deductive counterparts of some desirable algebraic semantics.

In 1965 Lotfi Zadeh introduced fuzziness in his keynote paper [38], dealing with fuzzy sets. A fuzzy set was an object of classical set theory, being modelled by its characteristic function on a fixed universe, taking values in some algebra of truth degrees (typically the real unit interval). The concept turned out to be extremely helpful in applications and also intrigued many theoretical researchers, spreading rapidly and giving rise to a fast-growing research area, perhaps best labelled 'theory of fuzzy sets' (though, quite often, the terms 'fuzzy set theory' or even 'fuzzy logic' are used to denote it).

One of the persons who pursued Zadeh's ideas on fuzziness was his doctoral student,

⁶It was there that he met Petr Cintula.

Joseph Goguen. His paper [8] remains a source of inspiration for generations of readers; among other things, he distinguishes various kinds of imprecision (e.g., vagueness or ambiguity), he points out the difference between fuzziness and probability, he implicitly introduces a residuated product algebra, and he also sets the challenge to develop a formal deductive system for partially true propositions.

Zadeh's and Goguen's works on fuzziness did not pass unnoticed in the Czech Republic. First one must mention the work of Aleš Pultr (see [26]), analyzing the concept of fuzziness mainly from a categorical point of view (as Goguen also did). Pultr's doctoral student Jan Pavelka, in his thesis defended in 1976, developed a formal deductive system of fuzzy logic introducing truth constants for elements of the algebra in the language. Pavelka was intrigued by the challenge posed by Goguen; most researchers in fuzzy logic will have heard about Pavelka's logic, as a propositional system conservatively expanding Łukasiewicz logic, allowing for inference among partially true statements, using the values from the standard Łukasiewicz algebra as labels. In fact Pavelka's work is much more comprehensive (see [24]).

Petr Hájek was the reviewer of Pavelka's thesis; thus he had, quite early on, a direct contact with results obtained in our country and the works they referred to. Many years later, in his monograph [13], Petr Hájek continued the ideas of Pavelka and designed what he called 'rational Pavelka logic', a system expanding Łukasiewicz logic with constants for rationals within $[0, 1]$ (thus in a countable language).

A bit later, in 1988, a somewhat similar situation recurred: Petr Hájek was the reviewer of the thesis of Vilém Novák, who, like Jan Pavelka many years before him, was a student of Aleš Pultr working on fuzzy logic in language expanded with constants. It was his endeavour to extend Pavelka's results to the first-order case.

In 1991, Gaisi Takeuti visited Prague to attend the already mentioned workshop on proof theory, arithmetic and computational complexity. It was just then that Takeuti had finished a joint paper with Satoko Titani, called *Fuzzy logic and fuzzy set theory* (see [32]). In this comprehensive piece of work, the terms 'fuzzy logic' and 'fuzzy set theory' acquired a new meaning: the paper contains an axiomatization (with an infinitary rule) of a Gödel logic enriched with Łukasiewicz connectives and the product conjunction, and the constant $1/2$ (a predecessor of the logic $\text{LIII}1/2$). The appeal of this system is plain to see: it is a semantically rich logic, subsuming several other already existing systems (such as Łukasiewicz logic or Gödel logic), and it has standard completeness (at the cost of decidability). However it may be argued that the real beauty of the paper lies in the set theory developed in this logic; a first-order theory, the axioms mimicking the Zermelo-Fraenkel ones, governed by the laws of fuzzy logic. The paper leans back on well-established results on set theory in intuitionistic logic, exploiting the fact that Gödel logic is a semilinear extension of intuitionistic logic. Petr Hájek must have been captivated by the paper, because he later contributed both to the logic, rephrasing it in his monograph [13], and to the set theory, recasting the ZF-style theory into the setting of his basic logic

([15]).

In the early 1990's, learning from others, Petr clarified to himself the traits that distinguished fuzzy logic among dozens of other approaches that could be labelled 'reasoning under uncertainty'; he gradually started to clarify the distinction to others, and did so with the unrelenting determination of a true missionary. He argued that fuzzy logic, like many-valued logic, has a purely formal deductive facet; he stressed the distinction between *degrees of truth* (involving vague notions, such as 'beautiful') *degrees of belief* (involving the subject's views on potentially crisp notions), and *probability* (see [12]); he ventured to seek the ties of fuzziness to natural language semantics, and to philosophical treatment of the vagueness phenomenon.

Quite importantly, Petr was not alone in his efforts: he was able to pursue some previously made bonds and acquaintances, since many researchers shared his interest in fuzzy logic. At the time, our country's boundaries were open, so it was possible to go abroad and receive guests. Petr knew Franco Montagna, Matthias Baaz, and Jeff Paris from his arithmetic years. He also enjoyed a longterm cooperation with Francesc Esteva and Lluís Godo, initiated by an earlier project on managing uncertainty in medicine. He also knew Siegfried Gottwald. He knew, and was on visiting terms with, researchers in Italy pioneering many-valued and fuzzy logic, such as Daniele Mundici, Antonio Di Nola and Giangiacomo Gerla. He was aware of Ulrich Höhle's work. Moreover, fuzzy logic had had a continuing tradition in Prague.

In mid 1990's, a group of researchers from fourteen European countries applied successfully for a COST (European Cooperation in Science and Technology) project. The project *Many Valued Logics for Computer Science Applications* was approved and initiated in 1995. The countries (managers) involved in the project were Austria (Matthias Baaz and Erich Peter Klement), Belgium (Etienne Kerre and Marc Roubens), the Czech Republic (Petr Hájek), Finland (Esko Turunen), France (Luisa Itturioz and Guy Tassart), Germany (Peter H. Schmitt and Siegfried Gottwald), Greece (Costas Drossos), Italy (Daniele Mundici and Antonio Di Nola), Poland (Ewa Orłowska and Janusz Kacprzyk), Portugal (Isabel M. A. Ferreira), Slovakia (Radko Mesiar), Spain (Ventura Verdú Solans and Immaculada P. de Guzmán Molina), Sweden (Patrik Eklund), Turkey (Aydan M. Erkmen and Ismet Erkmen) and the United Kingdom (Dov Gabbay and Hans Jürgen Ohlbach). The scope of the grant was rather broad; however, among other things, for the five years of its duration, it continued to promote cooperation among European researchers who focused on fuzzy logic as a rigorous mathematical discipline. This grant was a milestone in that it established the fuzzy logic community in Europe (however vaguely defined and subject to change in time); in analogously broad terms, the agenda of this group of researchers can be (and is, nowadays) labelled *mathematical fuzzy logic*. Within the community, many loose ends were tied together, many different perspectives united, and fuzzy logic saw a rapid development, with close ties to already existing many-valued logics, residuated lattices, intuitionistic theories, philosophy of vagueness, and other areas.

Starting in 1992, Petr Hájek served two four-year terms as director of the Institute of Computer Science. He did not mitigate his research during the period of his appointment; quite on the contrary. After an initial phase of searching and sorting the territory, the mid nineties saw him developing a new formal system, intended to capture the logic of continuous t-norms and their residua. This system, since it was a common fragment of some already existing logics describing particular examples of continuous t-norms, was named the Basic Logic (abbreviated BL). At the time, it may have indeed seemed basic and rather weak; nowadays, when both Petr and his peers have delved much deeper and brought to light many weaker systems, the term ‘basic logic’ (even ‘basic fuzzy logic’) seems a bit awkward, so many people choose to call it ‘Hájek’s basic logic’.

The monograph *Metamathematics of Fuzzy Logic* was published in 1998, the fourth volume of the ‘Trends of Logic’ series of Kluwer Academic Publishers (see [13]). It offers a thorough development of the basic logic BL (propositional and first-order), which provided the subject of fuzzy logic with a much needed formal treatment meeting the standards of a subarea of mathematical logic. The book also offers an explanation of how these results project back to applications, and some neighbouring areas. The monograph was a product of several years’ continued effort, evolving from lecture notes for tutorials given on the new and captivating topic. It roughly marks the end of an era that can be viewed as pioneering work in mathematical fuzzy logic for Petr Hájek. The next decade would see mathematical fuzzy logic in full bloom.

Other books on closely related topics emerged at about the same time as Petr’s monograph. To start with, Siegfried Gottwald published the English translation [9] of his earlier monograph in German. Roberto Cignoli, Itala M. L. D’Ottaviano, and Daniele Mundici wrote a book on MV-algebras [3]. Vilém Novák, Irina Perfilieva, and Jiří Močkoř prepared a book covering the evaluated-syntax approach of the group [23].

In 2000 Petr’s term in office as director of the Institute of Computer Science elapsed; his successor was Jiří Wiedermann. Petr was appointed head of the Department of Theoretical Computer Science, a position he held for several years. Currently, he holds the position of a senior researcher.

A publication of a monograph is a good step in spreading the knowledge and involving other people in the topic. With the publication of Hájek’s book and some of the above, more people became involved in fuzzy logic: Petr was active in evangelizing people, gaining the attention of some of his former colleagues in arithmetic for example. Jeff Paris joined efforts with Petr in several papers about fuzzy logic, and Franco Montagna made fuzzy logic his primary research topic. Moreover, a group of students gradually formed around Petr: these included Petr Cintula, Rostislav Horčík, Libor Běhounek and myself; a working group on fuzzy logic was formed. I wrote a dissertation under Petr’s supervision in 2003 (see [21]) and so did Petr Cintula in 2005 (see [4]). Together with people already working with Petr, such as Ivan Kramosil, Dagmar Harmanová, Peter Vojtáš, Martin Holeňa, Milan Daniel, and some regular visitors, such as Mirko Navara, we saw some very active

years, meeting at the seminar of applied mathematical logic, going to conferences, reading papers, and broadening our perspective. Importantly, we were also more and more able to recognize the role of fuzzy logic among other nonclassical logics, in the philosophy of vagueness, as a ground for developing fuzzy mathematics, etc.

The first decade of the new millennium has also been a marked success for mathematical fuzzy logic on an international scale. Though still not quite accepted by the mainstream of mathematical logic, the discipline attracted the attention of more and more researchers, including those who did not work in it, but saw it as relevant for or related to their research. Many young people became involved. In particular, Prague continued the fruitful cooperation with the Barcelona group and with the Vienna group, and with many researchers in Italy. A *MathFuzzLog* working group of EUSFLAT has been established in 2007. The amount of results gathered by the community through the decade called for a new book that would encompass all the new material. In fact, several books were published, but, from the point of view of Petr Hájek, a key moment was the decision to prepare not another monograph, but a handbook with chapters written by people who closely pursued the particular subareas. The *Handbook of Mathematical Fuzzy Logic* was edited by Petr Cintula, Petr Hájek and Carles Noguera; eleven chapters were agreed upon, roughly covering the main areas, and authors started writing their chapters around the middle of 2009. The book was published in 2011, comprising nearly one thousand pages. Apart from editorship, Petr coauthored the introductory chapter and the chapter on arithmetical complexity of fuzzy logics. The main import of this book is that it collects current knowledge in key areas of mathematical fuzzy logic, offering it to interested readers.

6 Sources and acknowledgements

Some of Petr's older papers may be available online through Czech Digital Mathematics Library at www.dml.cz. His full bibliography is maintained by the library of Institute of Computer Science, and is also available online.

In 2009, the volume *Witnessed Years: Essays in Honour of Petr Hájek*, dedicated to Petr Hájek on the occasion of his 70th birthday, was edited by Petr Cintula, Vítězslav Švejdar and myself and published by College Publications. Many of Petr's friends and colleagues contributed and the book contains a lot of information about Petr and his scientific interests.

I am indebted to a number of people for their willingness to share their recollections with me, and for finding time to actually do so; without them, writing this biography would not have been possible. They include (in no particular order) Dagmar Harmanová, Pavel Pudlák, Vítězslav Švejdar, Kamila Bendová, Milan Daniel, Petr Cintula, Aleš Pultr, Miroslav Tůma, Petr Vopěnka, Jiří Ivánek, Marie Hájková, Daniele Mundici, and Franco Montagna. Moreover, a few people read drafts of this text and suggested many im-

provements; these include Jirka Hanika, Vítězslav Švejdar, Milan Daniel, Miroslav Tůma, and Daniele Mundici. Our librarian, Ludmila Nývltová, has been miraculous in retrieving literature (especially various people's dissertations) and other information. Last but not least, Petr Hájek has borne the fact that his biography is being written, and my repeated questioning him, with a degree of patience usually only found in saints, and he was so kind as to read a draft of the biography as well. Shortcomings in the text remain, of course, my own.

The preparation of this text was supported by grant P202/10/1826 of the Czech Science Foundation and by RVO: 67985807.

References

- [1] Zofia Adamowicz. My cooperation with Petr Hájek. In P. Cintula, Z. Haniková, and V. Švejdar, editors, *Witnessed Years: Essays in Honour of Petr Hájek*, volume 10 of *Tributes*, pages 53–55. College Publications, 2009.
- [2] Matthias Baaz. Recollections of a non-contradictory logician. In P. Cintula, Z. Haniková, and V. Švejdar, editors, *Witnessed Years: Essays in Honour of Petr Hájek*, volume 10 of *Tributes*, pages 57–60. College Publications, 2009.
- [3] Roberto Cignoli, Itala M. L. D'Ottaviano, and Daniele Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordrecht, 1999.
- [4] Petr Cintula. *From Fuzzy Logic to Fuzzy Mathematics*. Doctoral dissertation, Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, 2005.
- [5] Peter G. Clote and Jan Krajíček, editors. *Arithmetic, Proof Theory, and Computational Complexity*. Number 23 in Oxford Logic Guides. Clarendon Press, 1993.
- [6] Milan Daniel. *Dempsterova plogrupa a práce s nejistotou v pravidlově orientovaných expertních systémech*. Kandidátská disertační práce, ÚIVT AV ČR, 1993.
- [7] Solomon Feferman. Arithmetization of metamathematics in a general setting. *Fundamenta Mathematicae*, 49:35–92, 1960.
- [8] Joseph Amadee Goguen. The logic of inexact concepts. *Synthese*, 19(3–4):325–373, 1969.
- [9] Siegfried Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

- [10] Petr Hájek. *Modely teorie množin s individuuy*. Kandidátská práce, Matematický ústav ČSAV, 1965.
- [11] Petr Hájek. *Metamatematika aritmetiky prvního řádu*. Doktorská disertační práce, Matematický ústav ČSAV, 1990.
- [12] Petr Hájek. On Logics of Approximate Reasoning. In M. Masuch and L. Pólos, editors, *Knowledge Representation and Reasoning Under Uncertainty*, volume 808 of *Lecture Notes in Computer Science*, pages 17–29. Springer, 1994.
- [13] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.
- [14] Petr Hájek. Zemřel profesor Karel Čulík. *Pokroky matematiky, fyziky a astronomie*, 4:344–348, 2002.
- [15] Petr Hájek and Zuzana Haniková. A development of set theory in fuzzy logic. In Melvin Chris Fitting and Ewa Orłowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 273–285. Physica-Verlag, Heidelberg, 2003.
- [16] Petr Hájek, Ivan Havel, and Metoděj Chytil. Guha — metoda automatického zjišťování hypotéz. *Kybernetika*, 2(1):31–47, 1966.
- [17] Petr Hájek and Tomáš Havránek. *Mechanizing Hypothesis Formation: Mathematical Foundations of a General Theory*. Springer, Berlin, 1978.
- [18] Petr Hájek, Tomáš Havránek, and Radim Jiroušek. *Uncertain Information Processing in Expert Systems*. CRC Press, Boca Raton, 1992.
- [19] Petr Hájek and Pavel Pudlák. *Metamathematics of First-Order Arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, 1993.
- [20] Marie Hájková. My life with Petr, Petr’s life with me. In P. Cintula, Z. Haniková, and V. Švejdar, editors, *Witnessed Years: Essays in Honour of Petr Hájek*, volume 10 of *Tributes*, pages 9–12. College Publications, 2009.
- [21] Zuzana Haniková. *Mathematical and Metamathematical Properties of Fuzzy Logic*. Ph.D. thesis, Charles University in Prague, Faculty of Mathematics and Physics, 2004.
- [22] Jiří Ivánek. *Příspěvky k matematické teorii generování hypotéz*. Kandidátská disertační práce, Matematický ústav ČSAV, 1984.

- [23] Vilém Novák, Irina Perfilieva, and Jiří Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer, Dordrecht, 2000.
- [24] Jan Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.
- [25] Pavel Pudlák. Arithmetic in Prague in the 1970–80s. In P. Cintula, Z. Haniková, and V. Švejdar, editors, *Witnessed Years: Essays in Honour of Petr Hájek*, volume 10 of *Tributes*, pages 27–31. College Publications, 2009.
- [26] Aleš Pultr. Fuzzy mappings and fuzzy sets. *Commentationes Mathematicae Universitatis Carolinae*, 17(3):441–459, 1976.
- [27] Jan Rauch. *Logické základy automatické tvorby hypotéz z databází*. Kandidátská disertace, Matematický ústav ČSAV, 1986.
- [28] Ladislav S. Rieger. Problém tzv. absolutně nerozhodnutelných vět teorie čísel. *Časopis pro pěstování matematiky*, 85(1):1–13, 1960.
- [29] Antonín Sochor. Petr Vopěnka (* 16.5.1935). *Annals of Pure and Applied Logic*, 109:1–8, 2001.
- [30] Vítězslav Švejdar. *Modální logika a interpretovatelnost*. Kandidátská disertační práce, 1982.
- [31] Vítězslav Švejdar. Modal analysis of generalized Rosser sentences. *Journal of Symbolic Logic*, 48(4):986–999, 1983.
- [32] Gaisi Takeuti and Satoko Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32(1):1–32, 1992.
- [33] Julio J. Valdés. *Algebraic and Logical Foundations of Uncertainty Processing in Rule-Based Expert Systems of Artificial Intelligence*. Kandidátská disertační práce, Matematický ústav ČSAV, 1987.
- [34] Petr Vopěnka. *Úvod do matematiky v alternativní teorii množin*. Alfa, 1979.
- [35] Petr Vopěnka. Prague set theory seminar. In P. Cintula, Z. Haniková, and V. Švejdar, editors, *Witnessed Years: Essays in Honour of Petr Hájek*, volume 10 of *Tributes*, pages 5–8. College Publications, 2009.
- [36] Petr Vopěnka. The problem of actualizability of the classical infinity. In P. Cintula, Z. Haniková, and V. Švejdar, editors, *Witnessed Years: Essays in Honour of Petr Hájek*, volume 10 of *Tributes*, pages 13–26. College Publications, 2009.

- [37] Petr Vopěnka and Petr Hájek. *The Theory of Semisets*. Academia Praha/North-Holland Publishing Company, 1972.
- [38] Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.