Standard algebras for fuzzy propositional calculi

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1. Introduction

Each continuous t-norm determines uniquely a structure for the language of fuzzy PC; these structures are called *standard algebras*. By the representation theorem for continuous t-norms there are 2^ω pairwise non-isomorphic standard algebras.

Fuzzy PC is axiomatized by a set of propositional formulas commonly referred to as basic fuzzy logic, or shortly *basic logic* (BL). Completeness of BL w.r.t. the class of standard algebras has recently been proved in [2] (note that standard algebras are a subclass of algebras of truth values of BL).

[6] investigates three outstanding schematic extensions of BL: Łukasiewicz's logic, Gödel's logic, and product logic, given by Łukasiewicz's, Gödel's, and product t-norm respectively, and proves standard completeness for each of these systems. The aim of this paper is to study logics given by other continuous t-norms.

The following section is an overview of known facts. New results are contained in section three. After some preliminary work we show how sets of tautologies of standard algebras are partially ordered by inclusion; for example, the tautologies of any standard algebra are included in tautologies of either Łukasiewicz's or Gödel's or product standard algebra. We derive some consequences of this ordering. We show that BL is complete w.r.t. particular standard algebras. To show that the hierarchy induced by the ordering is non-trivial, we exhibit an infinite chain of weaker and weaker logics.The appendix offers a couple of open problems and questions, possibly as hints for future work in this area, and also a sonnet for repose.

Notation and conventions

Closed intervals (mostly of reals) are denoted by square brackets—like $[0,1]$; open intervals by round brackets—like $(0,1)$.

A 'formula' means a propositional formula (the language of fuzzy PC is defined in the following section). Formulas are denoted with Greek letters. We sometimes omit parentheses for readability's sake. Precedence of connectives: negation binds stronger than any binary connective and conjunctions and disjunction bind stronger than implication and equivalence: $\varphi \mathbin{\&} \neg\psi \to \chi$ should be parsed as $(\varphi \mathbin{\&} (\neg\psi)) \to \chi$.

We say at times that 'intervals are isomorphic', which should be read as 'isomorphic w.r.t. all operations defined'. By 't-norms are isomorphic on $[x,y]$' we mean that the underlying interval(s) are isomorphic w.r.t. the t-norms; we refrain from such abbreviations where confusion could arise.

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2. A Reader's Digest of Fuzzy PC

This section contains selected facts about fuzzy PC and is intended for reference. A detailed treatment will be found in [6]; [5] provides important results about BL, and the standard completeness result for BL comes from [2].

2.1 Definition. *A t-norm is a binary operation $*$ on $[0,1]$, satisfying these conditions:*

(i) *$*$ is commutative and associative*
(ii) *$*$ is non-decreasing in both arguments*
(iii) *$1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.*

Continuous t-norms are considered as truth functions of the (strong) conjunction &.

For each continuous t-norm $*$ we define its *residuum* $\Rightarrow$ by $x \Rightarrow y = \max\{z; x * z \leq y\}$; this is used as the truth function of the implication $\rightarrow$. Note that by the above definition, $x \Rightarrow y = 1$ iff $x \leq y$.

2.2 Definition. *The standard algebra given by a continuous t-norm $*$ is the structure*

$$[0,1]_* = ([0,1], 0, *, \Rightarrow)$$

We define three outstanding continuous t-norms:

Lukasiewicz's t-norm: $x * y = \max(0, x + y - 1)$
Gödel's t-norm: $x * y = \min(x, y)$
Product t-norm: $x * y = x.y$ (multiplication)

The standard algebras given by Lukasiewicz's, Gödel's, and product t-norm will be denoted $[0,1]_{\mathrm{L}}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_{\Pi}$ respectively.

The following is the representation (or decomposition) theorem for continuous t-norms. Let $*$ be a continuous t-norm; an element $x \in [0,1]_*$ is *idempotent* (w.r.t. $*$) iff $x * x = x$. The set of all idempotents of $*$ is a closed subset of $[0,1]$. Its complement is a union of (countably many) non-overlapping open intervals; denote this set of intervals $\mathcal{I}_\circ$. Let $\mathcal{I}$ be the set of closures of intervals in $\mathcal{I}_\circ$.

2.3 Representation theorem for continuous t-norms.

(i) *For each $[a,b] \in \mathcal{I}$, $*$ on $[a,b]$ is isomorphic either to the product t-norm (on $[0,1]$) or to the Lukasiewicz t-norm (on $[0,1]$).*
(ii) *If for $x, y \in [0,1]$ there is no $[a,b] \in \mathcal{I}$ such that $x, y \in [a,b]$, then $x * y = \min(x,y)$.*

The language and syntax of fuzzy PC are almost the same as in classical propositional logic: the language consists of propositional variables, basic connectives &, $\rightarrow$ and a truth constant $\overline{0}$, and other definable connectives $\wedge$, $\vee$, $\equiv$, $\neg$ and a truth constant $\overline{1}$. Formulas are defined as usual. Propositional connectives are defined from &, $\rightarrow$, and $\overline{0}$ as follows:

$$\begin{array}{rcl}
\varphi \wedge \psi & \text{is} & \varphi \,\&\, (\varphi \rightarrow \psi) \\
\varphi \vee \psi & \text{is} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\
\neg\varphi & \text{is} & \varphi \rightarrow \overline{0} \\
\varphi \equiv \psi & \text{is} & (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi) \\
\overline{1} & \text{is} & \overline{0} \rightarrow \overline{0}
\end{array}$$

Fix a continuous t-norm $*$; this determines uniquely a standard algebra $[0,1]_*$ and the corresponding propositional calculus PC($*$), where evaluations of propositional variables extend to formulas as follows:

$$\begin{aligned}
e(\overline{0}) &= 0 \\
e(\varphi \,\&\, \psi) &= e(\varphi) * e(\psi) \\
e(\varphi \rightarrow \psi) &= e(\varphi) \Rightarrow e(\psi)
\end{aligned}$$

The operations $\min(x, y)$, $\max(x, y)$, and $x \Rightarrow 0$ are used as shortcuts for evaluating $\wedge$, $\vee$, and $\neg$, respectively. Note that formulas translate to terms, i.e., for each formula φ there is a term τ in the language of standard algebras, such that in any standard algebra $[0, 1]_*$ $\tau = 1$ holds iff $[0, 1]_* \models \varphi$.

A formula φ is a 1-tautology of a standard algebra $[0, 1]_*$ iff it evaluates to 1 under any evaluation in $[0, 1]_*$. φ is a *t-tautology* iff it is a 1-tautology of $[0, 1]_*$ for all standard algebras.

Now we define the *basic logic*; its axioms are t-tautologies. Proofs are defined as usual.

2.4 Definition. *The following formulas are the axioms of the basic logic (denoted* BL*).*

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \,\&\, \psi) \to \varphi$
(A3) $(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$
(A4) $(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$
(A5b) $((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\overline{0} \to \varphi$

The inference rule of BL *is modus ponens.*

BL may be extended by additional axioms (or, schemas of axioms); below are listed those that yield the calculi given by Łukasiewicz's, Gödel's, and product t-norm, respectively:

Łukasiewicz's logic:	$\neg\neg\varphi \to \varphi$	(Ł)
Gödel's logic:	$\varphi \to (\varphi \,\&\, \varphi)$	(G)
Product logic:	$\neg\neg\chi \to [((\varphi \,\&\, \chi) \to (\psi \,\&\, \chi)) \to (\varphi \to \psi)]$	(Π1)
and	$\varphi \wedge \neg\varphi \to \overline{0}$	(Π2)

We define algebras of truth values for BL (and its schematic extensions, under additional assumptions). Standard algebras, with the language expanded by the (definable) operations 1, max, min, are a subclass of these.

2.5 Definition. *A residuated lattice is an algebra* $\mathbf{L} = (L, \cup, \cap, *, \Rightarrow, 0, 1)$ *with four binary operations and two constants such that*

(i) $(L, \cup, \cap, 0, 1)$ *is a lattice with the greatest element* 1 *and the least element* 0 *(with respect to the lattice ordering* $\leq$*)*
(ii) $(L, *, 1)$ *is a commutative semigroup with the unit element* 1*, i.e.,* $*$ *is commutative, associative,* $1 * x = x$ *for all* $x \in L$
(iii) $*$ *and* $\Rightarrow$ *form an adjoint pair, i.e.,* $z \leq (x \Rightarrow y)$ *iff* $x * z \leq y$ *for all* $x, y, z \in L$.

A residuated lattice is a BL-algebra iff the following identities hold for all $x, y \in L$:

(i) $x \cap y = x * (x \Rightarrow y)$
(ii) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$

A BL-algebra is linearly ordered iff its lattice ordering is linear, i.e., for any x,$y \in L$, $x \cap y = x$ *or* $x \cap y = y$.

Let $\mathbf{L} = (L, \cup, \cap, *, \Rightarrow, 0, 1)$ be a *BL*-algebra. An $\mathbf{L}$-evaluation of propositional variables is a mapping e, assigning to each propositional variable an element of L; evaluation of formulas is obtained using 0, $*$, and $\Rightarrow$ to evaluate $\overline{0}$, $\&$, and $\to$. A formula φ is an $\mathbf{L}$-tautology iff $e(\varphi) = 1$ for all $\mathbf{L}$-evaluations e.

Let $\mathcal{C}$ be a schematic extension of BL (a set of propositional formulas containing the axioms of BL). A *BL*-algebra $\mathbf{L}$ is a $\mathcal{C}$-algebra iff all $\varphi \in \mathcal{C}$ are $\mathbf{L}$-tautologies.

2.6 Theorem (Completeness). *Let* $\mathcal{C}$ *be a schematic extension of* BL *and* φ *a formula. The following three conditions are equivalent:*

(i) φ *is provable in* $\mathcal{C}$

(ii) *for each $\mathcal{C}$-algebra* $\mathbf{L}$, φ *is an* $\mathbf{L}$*-tautology*
(iii) *for each linearly ordered $\mathcal{C}$-algebra* $\mathbf{L}$, φ *is an* $\mathbf{L}$*-tautology.*

This gives completeness of BL w.r.t. (linearly ordered) BL-algebras if $\mathcal{C}$ is BL. Also, Łukasiewicz's logic is complete w.r.t. the subclass of BL-algebras in which $\neg\neg\varphi \to \varphi$ holds, these are called MV-algebras; Gödel's logic is complete w.r.t. BL-algebras satisfying the axiom $\varphi \to \varphi \,\&\, \varphi$, which are called regular Heyting algebras; product logic is complete w.r.t. BL-algebras satisfying the two axioms $\neg\neg\chi \to [((\varphi \,\&\, \chi) \to (\psi \,\&\, \chi)) \to (\varphi \to \psi)]$ and $\varphi \wedge \neg\varphi \to \overline{0}$, these are called product algebras.

It has long been an open problem whether BL is complete w.r.t. the class of standard algebras. In [5], this problem is reduced to the question of whether two additional axioms B1 and B2 are redundant (provable in BL) or not. An affirmative answer to this question was recently presented in [2].

2.7 Theorem (Standard Completeness). BL *is complete w.r.t. t-tautologies, i.e., a formula* φ *is a theorem of* BL *iff it is a t-tautology.*

Besides, [6] shows Łukasiewicz's, Gödel's, and product logic to be complete w.r.t. the corresponding standard algebras $[0,1]_{\mathrm{L}}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_{\Pi}$, respectively. We rely on these completeness results in the rest of this paper.

3. Standard algebras

3.1. Isomorphisms of standard algebras

Obviously there are 2^ω continuous t-norms in all. This is exactly the cardinality of mappings of ω into 2^ω (since continuous functions are determined by their values on rational numbers). Some of them are isomorphic; later in this section we show that isomorphisms w.r.t. the operation $*$ also preserve the operation $\Rightarrow$. Thus the equivalence relation defined by existence of isomorphisms between continuous t-norms extends to standard algebras. To complete the picture, we exhibit 2^ω pairwise non-isomorphic continuous t-norms.

3.1.1 Definition. *Two t-norms $*_1$ and $*_2$ are isomorphic via $f : [0,1] \longrightarrow [0,1]$ iff f is bijective and $\forall a, b \in [0,1]\; f(a *_1 b) = f(a) *_2 f(b)$. We say that $*_1$ and $*_2$ are isomorphic iff some isomorphism exists.*

3.1.2 Lemma. *Let f be a continuous, increasing, bijective mapping of $[0,1]$ onto itself (an increasing homeomorphism). Let $*$ be a continuous t-norm. For each $a, b \in [0,1]$, define $a *_1 b = f(f^{-1}(a) * f^{-1}(b))$; then $*_1$ is a continuous t-norm (isomorphic to $*$).*

The following lemma determines the cardinalities of equivalence classes (of isomorphic t-norms).

3.1.3 Lemma. *Let $*$ be a continuous t-norm. If $a, b \in [0,1]$ exist so that $a * b < \min(a,b)$, then there are 2^ω distinct continuous t-norms isomorphic to $*$; otherwise there are no t-norms isomorphic to $*$ (distinct from $*$).*

Proof. Take the two elements a and b for which $a * b < \min(a,b)$; suppose $a \le b$ (observe that a is nonzero). Define a homeomorphism f on $[0,1]$ as follows: first choose $0 < f(a) \le f(b) \le 1$; then choose $f(a * b)$ arbitrarily in $(0, f(a))$. There is a continuum of values to choose from. Then define the rest of f—as an arbitrary homeomorphism, e.g., four linear functions connecting 0, $f(a*b)$, $f(a)$, $f(b)$, 1.
This defines 2^ω distinct homeomorphisms f, and these in turn define 2^ω distinct continuous t-norms $*'$, isomorphic to $*$ (the t-norms differ (at least) in the value of $f(a) *' f(b)$).
If no such a and b exist, then $*$ is Gödel's t-norm, and the above construction fails. It is easy to check that there are indeed no t-norms isomorphic to Gödel's t-norm. $\mathcal{QED}$

Hence the cardinality of each cluster of isomorphic t-norms is 2^ω, but for Gödel's t-norm, the equivalence class of which is a singleton.

How many equivalence classes are there? We give a recipe[1] for construing 2^ω pairwise non-isomorphic t-norms, using the representation theorem.

3.1.4 Lemma. *There are 2^ω pairwise non-isomorphic t-norms.*

Proof. Take an infinite, monotonous sequence of reals within $[0,1]$, converging to 1; for example, $1-1/2^i, i \in N$. These will be the idempotents of the t-norms. The $*$ on each of the intervals $[1-1/2^i, 1-1/2^{i+1}], i \in N$ will be isomorphic either to $*_{\mathrm{L}}$ or to $*_\Pi$. Any such $*$ is therefore encoded by an infinite sequence of L's and Π's, and any such sequence defines a continuous t-norm. There are 2^ω such sequences. The resulting t-norms are pairwise non-isomorphic. $\mathcal{QED}$

Next we show that if two continuous t-norms are isomorphic via f, so are their residua. Thus the standard algebras given by two isomorphic t-norms are isomorphic and have the same sets of tautologies.

3.1.5 Theorem. *Let $*_a$, $*_b$ be two continuous t-norms, let $0 \le a_1 < a_2 \le 1$, where a_1 and a_2 are idempotents of $*_a$, let $0 \le b_1 < b_2 \le 1$ and let f be a bijective mapping of $[a_1,a_2]$ onto $[b_1,b_2]$, such that $(\forall x,y \in [a_1,a_2])\, f(x *_a y) = f(x) *_b f(y)$. Then*

(i) *f is increasing (and hence continuous) on $[a_1,a_2]$, and*
(ii) *$(\forall x,y \in [a_1,a_2])$, $x > y$ implies $f(x \Rightarrow_a y) = f(x) \Rightarrow_b f(y)$.*

Proof.
(i) Suppose $x < y$ where $x,y \in [a_1,a_2]$. Then $\exists z(z *_a y = x)$. In this case $x \le z \le a_2$, because if $z > a_2$, then $z *_a y = y$, but this can't happen because $y > x$. Thus $f(z) *_b f(y) = f(x)$ and therefore $f(x) \le f(y)$. Because $f(x) = f(y)$ is impossible, $f(x) < f(y)$.
(ii) We want to prove $f(x \Rightarrow_a y) = f(x) \Rightarrow_b f(y)$ for $x > y$, $x,y \in [a_1,a_2]$.
By definition, $f(x) \Rightarrow_b f(y) = \max z : z *_b f(x) \le f(y)$. Because f is increasing, this is equal to $f(\max u : u *_a x \le y)$, i.e., $f(x \Rightarrow_a y)$. $\mathcal{QED}$

This theorem considers subintervals of $[0,1]$ and states precisely in what sense they can be isomorphic. For the sake of brevity, we shall henceforward say simply that two intervals are isomorphic, or that an interval is an isomorphic copy of a particular standard algebra; this means that the isomorphism concerns the operations $x * y$ for arbitrary arguments, and $x \Rightarrow y$ for $x > y$.

Note, however, that an analogous statement cannot be formed for homomorphisms. For example, take an algebra A given by a continuous t-norm with one non-extremal idempotent element $\frac{1}{2}$. Define a mapping $f : A \longrightarrow [0,1]$ such that $f(x) = 0$ for $x \in [0,\frac{1}{2}]$, and $f(x) = 2x-1$ otherwise. Then f is a homomorphism with respect to $*$'s, but not to $\Rightarrow$'s.

3.2. Representation of t-norms

By the representation theorem, each continuous t-norm is determined by its idempotents delimiting the copies of $[0,1]_{\mathrm{L}}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_\Pi$ (we call these idempotents *cutpoints*), and by the distribution of these copies between consecutive cutpoints (the distribution is called the *layout* of a t-norm). We define ordered sums of algebras and develop some notation. The definition of ordered sum is a borrowing from [5]; we shall only need its special case for standard algebras.

3.2.1 Definition. *Let $\mathcal{C}$ be a linearly ordered set with least element 0 and largest element 1. For $c \in \mathcal{C}$, let c^+ be the upper neighbour of c in $\mathcal{C}$ if it exists, otherwise $c^+ = c$. For each $c \in \mathcal{C}$, let A_c be an isomorphic copy of a linearly ordered BL-algebra with the least element c and the largest element c^+. Assume that the non-extremal elements of $A_c; c \in \mathcal{C}$ are not elements of any $A_{c'}, c' \in \mathcal{C}, c \ne c'$.*

[1] due to Petr Hájek

The ordered sum $\bigoplus_{c\in\mathcal{C}} A_c$ is defined as follows:

(i) *the domain is $\bigcup_{c\in\mathcal{C}} A_c$*
(ii) *for $x \in A_{c_1}$, $y \in A_{c_2}$ define $x \leq y$ iff either $c_1 = c_2$ and $x \leq_{c_1} y$, or $c_1 < c_2$, $c_1, c_2 \in \mathcal{C}$*
(iii) *$x * y = x *_c y$ for $x, y \in A_c, c \in \mathcal{C}$, otherwise $x * y = \min(x, y)$*
(iv) *$x \Rightarrow y = 1$ iff $x \leq y$*
(v) *$x \Rightarrow y = x \Rightarrow_c y$ for $x, y \in A_c$, $x > y$, $c \in \mathcal{C}$*
(vi) *$x \Rightarrow y = y$ for $x \in A_{c_1}$, $y \in A_{c_2}$, $c_1, c_2 \in \mathcal{C}$, and $c_1 > c_2$.*

3.2.2 Lemma. *In the above notation, $\mathbf{A} = \bigoplus_{c\in\mathcal{C}} A_c$ is a linearly ordered BL-algebra.*

Each standard algebra is an ordered sum of isomorphic copies of $[0,1]_{\mathrm{L}}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_{\Pi}$ in the sense of the above definition. Indeed, if $[u,v]$, $u < v$, is an interval of idempotents, then it is isomorphic to $[0,1]_{\mathrm{G}}$ (as is easily verified). In the representation, we take each *maximal* interval of idempotents (w.r.t. inclusion) as an isomorphic copy of $[0,1]_{\mathrm{G}}$. As stated in 3.1.5, each isomorphism w.r.t. $*$ also preserves $\Rightarrow$.

3.2.3 Definition. *Let x be an idempotent element of $[0,1]_*$. We say that x is a cutpoint (of $*$, or of $[0,1]_*$) iff there is no interval $[a,b]$, $a < b$, of idempotent elements such that $x \in (a,b)$.*
A cutpoint x is a proper cutpoint iff it is non-extremal (i.e., distinct from 0 and 1).

Thus we dismiss idempotents lying sharply within an interval of idempotents: in the representation, we do not allow two consecutive copies of $[0,1]_{\mathrm{G}}$. There are but three standard algebras without proper cutpoints: $[0,1]_{\mathrm{L}}$, $[0,1]_{\mathrm{G}}$, $[0,1]_{\Pi}$. For any other continuous t-norm, the set of its cutpoints is a countable closed subset of $[0,1]$.

The converse also holds: an arbitrary countable closed subset $\mathcal{C}$ of $[0,1]$, such that $0 \in \mathcal{C}$, $1 \in \mathcal{C}$, is the set of cutpoints of some continuous t-norm (in fact, of uncountably many continuous t-norms). To define a particular continuous t-norm $*$, it is sufficient to define $*$ on each $[c, c^+]$, $c < c^+$, $c \in \mathcal{C}$ as isomorphic to Łukasiewicz's, Gödel's, or product t-norm on $[0,1]$ (in accord with our understanding of the representation, we avoid two consecutive copies of $[0,1]_{\mathrm{G}}$ when defining $*$).

3.2.4 Definition. *Let $\mathcal{X}$, $\mathcal{Y}$ be countable closed subsets of $[0,1]$. We say that $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic (via f) iff there is a bijective mapping f of $\mathcal{X}$ onto $\mathcal{Y}$ and $\forall u, v \in \mathcal{X}$, $u \leq v$ iff $f(u) \leq f(v)$.*

Observe that *if* two t-norms are isomorphic, *then* their sets of cutpoints are isomorphic likewise. Between two countable closed sets of reals, there may of course be more than one bijective mapping preserving the ordering.

3.2.5 Definition. *Let $\mathcal{X}$ be a countable closed subset of $[0,1]$, such that $0 \in \mathcal{X}$, $1 \in \mathcal{X}$. A layout for $\mathcal{X}$ is an arbitrary mapping λ, assigning to each $[x, x^+]$, $x \in \mathcal{X}$, $x \neq x^+$, one of the symbols Ł, G, or Π, and the symbol U (for 'undefined'[2]) if $x = x^+$.*
Let $*$ be a continuous t-norm and $\mathcal{C}$ the set of its cutpoints. We say that λ is the layout of $*$ if for each $c \in \mathcal{C}$, $\lambda([c, c^+])$ is:

Ł iff A_c is an isomorphic copy of $[0,1]_{\mathrm{L}}$,
G iff A_c is an isomorphic copy of $[0,1]_{\mathrm{G}}$,
Π iff A_c is an isomorphic copy of $[0,1]_{\Pi}$,
U iff A_c is trivial.

3.2.6 Lemma. *Let $*$ be a continuous t-norm, $\mathcal{C}$ the set of its cutpoints, and λ its layout. Then*

(i) *$\mathcal{C}$ and λ determine $*$ up to isomorphisms*
(ii) *if $\mathcal{D}$ is a countable closed subset of $[0,1]$, such that $0 \in \mathcal{D}$, $1 \in \mathcal{D}$, λ_1 is a layout for $\mathcal{D}$, and $*_1$ is the continuous t-norm defined by $\mathcal{D}$ and λ_1, then $[0,1]_*$ and $[0,1]_{*_1}$ are isomorphic*

[2] technically, we prefer to have λ total

iff there is an $f : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic via f, and for each $c \in \mathcal{C}$, $\lambda(A_c) = \lambda_1(A_{f(c)})$ (we say that f preserves the layout).

Proof.
(i) If $*_1$ and $*_2$ are two continuous t-norms with cutpoints $\mathcal{C}$ and layout λ, then their isomorphism is an identity on $\mathcal{C}$, and on each $[c, c^+]$, $c < c^+$, $C \in \mathcal{C}$ it is a composition of the two isomorphisms mapping $[c, c^+]$ onto either $[0,1]_{\text{Ł}}$ or $[0,1]_{\text{G}}$ or $[0,1]_{\Pi}$.
(ii) Clearly $\mathcal{D}$ and λ_1 define a continuous t-norm (up to an isomorphism). If $[0,1]_*$ and $[0,1]_{*_1}$ are isomorphic via f, then f is an isomorphism of $\mathcal{C}$ onto $\mathcal{D}$; obviously the layouts match on each $[c, c^+]$, $c \in \mathcal{C}$ and $[f(c), f(c^+)]$. Conversely, if we have an isomorphism between $\mathcal{C}$ and $\mathcal{D}$, and the layouts match on each of the intervals, the isomorphism of the standard algebras is a union of isomorphisms on each pair of intervals $[c, c^+]$, $[f(c), f(c)^+]$, which is obtained by composing the isomorphisms mapping these intervals onto one of the standard algebras. $\mathcal{QED}$

For t-norms with finitely many cutpoints we may conveniently encompass both the exact number of cutpoints and the layout by an expression which is an "ordered sum" of symbols Ł, G, Π. For example, a t-norm with three cutpoints, one of them proper (called h), and $\lambda([0,h]) = \text{Ł}$, $\lambda([h,1]) = \Pi$, will be denoted by an expression Ł⊕Π. The corresponding standard algebra will be denoted $[0,1]_{\text{Ł}\oplus\Pi}$. Similarly for other finitely cut t-norms. The ordering is expressed in the infix notation.

This notation may of course be introduced even for a few infinitely cut t-norms: for example, $[0,1]_{\text{Ł}\oplus\text{Ł}\oplus\text{Ł}\oplus\ldots}$ is a standard algebra with cutpoints $\{0, c_1, c_2, c_3, \ldots\}$, where $c_i < c_j$ for $i < j$ and $\lim_{i\to\infty} c_i = 1$, and $\lambda([c_i, c_{i+1}]) = \text{Ł}$ for any i. In most cases however, there is little hope of using the infix notation with infinitely cut t-norms.

Several times in this paper we use variables for ordered sums of symbols. Variables are denoted with uppercase letters like X, Y; we use X's (possibly indexed) as variables for single symbols (one of the symbols Ł, G, or Π), and Y's for sums of symbols, indexed by a closed subset of $[0,1]$. Note that the evaluation of X's and Y's is arbitrary but fixed when we use them. Example:

3.2.7 Definition. *Let $[0,1]_{\text{Y}_1}$ and $[0,1]_{\text{Y}_2}$ be two standard algebras. Then $[0,1]_{\text{Y}_1\oplus\text{Y}_2}$ denotes a standard algebra (up to an isomorphism) with an idempotent element h such that $[0,h]$ is isomorphic to $[0,1]_{\text{Y}_1}$ and $[h,1]$ is isomorphic to $[0,1]_{\text{Y}_2}$.*

3.3. Shrinking intervals

We introduce an operation called *shrinking* on continuous t-norms and give its important properties. Shrinking takes two arguments: a continuous t-norm $*$, and an interval $[u,v]$, $u, v \in [0,1]$, u and v being idempotents of $*$, of which at least one is non-extremal. The output of shrinking is a continuous t-norm; informally, it is the t-norm which results from $*$ by shrinking the interval $[u,v]$ into one point z, the rest of the layout remaining unchanged. The formal definition is as follows.

3.3.1 Definition. *Let $*$ and $*_1$ be two continuous t-norms, let $0 \le u < v \le 1$ be two idempotents of $*$ of which at least one is non-extremal, and let z be an idempotent of $*_1$. We say that $*_1$ results from $*$ by shrinking of $[u,v]$ into z, and write $*_1 = shrink(*, [u,v], z)$ iff there is a mapping f which*

(i) *maps $[0,1]$ onto itself*
(ii) *is a homomorphism w.r.t. $*$ and $*_1$*
(iii) *is a 1-1 mapping of $[0,u]$ (on)to $[0,z]$ and of $[v,1]$ (on)to $[z,1]$.*

Condition (ii) implies that f is non-decreasing: take $x < y$, then there is a z such that $y * z = x$; then also $f(y) *_1 f(z) = f(x)$, hence $f(x) \le f(y)$.

The function f maps $[u,v]$ to z. Note that z is an idempotent of $*_1$, but it is not a cutpoint of $*_1$ iff there exist $u_1 < u$, $v_1 > v$ such that $[u_1, u]$ and $[v, v_1]$ are intervals of idempotents of $*$.

Moreover, denoting $\mathcal{D}$ the cutpoints of $*_1$, the sets $\mathcal{C}\cap([0,u)\cup(v,1])$ and $\mathcal{D}\cap([0,z)\cup(z,1])$ are isomorphic (as ordered sets) and, denoting λ_1 the layout of $*_1$, we have for each $d \in \mathcal{D}$, $\lambda_1([d,d^+]) = \lambda([f^{-1}(d),(f^{-1}(d))^+])$; if $z \in \mathcal{D}$ and $v \notin \mathcal{C}$, then $\lambda_1([z,z^+]) = \mathrm{G}$.

The operation of shrinking determines the resulting continuous t-norm up to an isomorphism.

Using shrinking we now show how the sets of tautologies of individual standard algebras are ordered by inclusion.

3.3.2 Lemma. *Let $*$ be a continuous t-norm, $0 \le c_1 \le c_2 \le 1$ two of its idempotents. Then:*

(i) *if $x,y \in [c_1,c_2]$, then $x*y \in [c_1,c_2]$ and evaluates to c_2 iff $x=y=c_2$*
(ii) *if $x,y \in [c_1,c_2]$, $x>y$, then $x \Rightarrow y \in [y,c_2)$.*

Proof.
(i) $c_1 = c_1 * y \le x*y \le y \le c_2$; if $y<c_2$, then $x*y<c_2$.
(ii) $x \Rightarrow y = \max z(z*x \le y)$; since $c_2 * x = x > y$, $x \Rightarrow y < c_2$. Besides, $y*x \le y$, therefore $x \Rightarrow y \ge y$. $\mathcal{QED}$

3.3.3 Lemma. *Let $*$ be a continuous t-norm, $0<c_1<c_2 \le 1$ two of its idempotents, λ its layout. Then:*
(i) *$([0,c_1)\cup[c_2,1],0,*,\Rightarrow)$ is a subalgebra of $([0,1],0,*,\Rightarrow)$*
(ii) *$(\{0\}\cup(c_1,1],0,*,\Rightarrow)$ is a subalgebra of $([0,1],0,*,\Rightarrow)$ iff no element d exists such that $c_1<d\le 1$ and $\lambda[c_1,d]=\text{Ł}$.*

Proof.
(i) If $x,y \in [0,c_1)$ or $x,y \in [c_2,1]$, use the preceding lemma. If $x \in [0,c_1)$ and $y \in [c_2,1]$, then $x*y=x$, and since $\forall z \in [0,c_1] z*y=z$, also $y \Rightarrow x = x$.
(ii) Take $x,y \in (c_1,1]$, $x>y$. By the preceding lemma $x \Rightarrow y \ge y$. If a d exists so that $\lambda[c_1,d]=\text{Ł}$, then for $x,y \in (c_1,d]$, $x*y$ will, for sufficiently small x and y, evaluate to c_1. Otherwise $x*y>c_1$ for $x,y \in (c_1,1]$. $\mathcal{QED}$

3.3.4 Theorem. *Let $*$ be a continuous t-norm, let $0<u<v\le 1$ be two idempotents of $*$. Let $*_1 = shrink(*,[u,v],z)$. Then the structures $[0,1]_{*_1}$ and $([0,u)\cup[v,1],0,*,\Rightarrow)$ are isomorphic.*

Proof. By 3.3.2, $[0,u)\cup[v,1]$ is closed w.r.t. $*$. The mapping f from the definition of shrinking is an isomorphism of $*$ on $[0,u)\cup[v,1]$ and $*_1$ on $[0,1]$ by definition. Besides, f is an isomorphism w.r.t. $\Rightarrow$ and $\Rightarrow_1$ by 3.1.5 (and regarding that for $x<u$ and $y \ge v$ we have $f(y \Rightarrow x) = f(x) = f(y) \Rightarrow_1 f(x)$). $\mathcal{QED}$

3.3.5 Theorem. *Let $*$ be a continuous t-norm, $\mathcal{C}$ its cutpoints, λ its layout. Let $0=u<v<1$ be two idempotents of $*$. Let $*_1 = shrink(*,[u,v],z)$. If there is no w such that $\lambda([v,w])=\text{Ł}$, then the structures $[0,1]_{*_1}$ and $(\{0\}\cup(v,1],0,*,\Rightarrow)$ are isomorphic.*

Proof. The interval $(v,1]$ is closed w.r.t. $*$ and $\Rightarrow$ provided there is no segment $[v,w]$ isomorphic to $[0,1]_{\text{Ł}}$. Again $*$ on $\{0\}\cup(v,1]$ and $*_1$ on $[0,1]$ are isomorphic by definition and the corresponding residua are isomorphic by 3.1.5. $\mathcal{QED}$

3.3.6 Corollary. *Let $*$ be a continuous t-norm, $\mathcal{C}$ its cutpoints, λ its layout, and $0 \le u < v \le 1$ two idempotents of $*$, of which at least one is non-extremal. Suppose that if $u=0$, then there is no $c \in \mathcal{C}$ such that $\lambda([v,c])=\text{Ł}$. Let $*_1 = shrink(*,[u,v],z)$. Then, for any formula φ, if $[0,1]_* \models \varphi$, then $[0,1]_{*_1} \models \varphi$.*

Proof. If $[0,1]_{*_1} \not\models \varphi$ then $[0,1]_* \not\models \varphi$ either, because any counterexample in $[0,1]_{*_1}$ can be isomorphically embedded in $[0,1]_*$. $\mathcal{QED}$

3.4. The hierarchy

The last statement establishes a hierarchy on the sets of tautologies of individual standard algebras. In the notation induced by the representation theorem, it can be rewritten as follows:

3.4.1 Theorem. *Let* Y, Y_1 *and* Y_2 *be arbitrary (fixed) nonempty ordered sums of symbols. Then the following inclusions hold:*

(i) $\mathrm{Taut}([0,1]_{\mathrm{Y}_1\oplus\mathrm{Y}\oplus\mathrm{Y}_2}) \subseteq \mathrm{Taut}([0,1]_{\mathrm{Y}_1\oplus\mathrm{Y}_2})$

(ii) $\mathrm{Taut}([0,1]_{\mathrm{Y}_1\oplus\mathrm{Y}} \subseteq \mathrm{Taut}([0,1]_{\mathrm{Y}_1})$;

if Y_2 *does not start with an* Ł, *then also*

(iii) $\mathrm{Taut}([0,1]_{\mathrm{Y}\oplus\mathrm{Y}_2}) \subseteq \mathrm{Taut}([0,1]_{\mathrm{Y}_2})$.

As a consequence of this theorem, if a continuous t-norm $*$ has a G (or a Π) in its layout, then the tautologies of $[0,1]_*$ are included among the tautologies of $[0,1]_{\mathrm{G}}$ (or $[0,1]_\Pi$). The result for Ł must be stated somewhat more carefully (note that, for example, $\Pi 2$ holds in$[0,1]_{\Pi\oplus \text{Ł}}$ but not in $[0,1]_{\text{Ł}}$).Thus there are no standard algebras whose 1-tautologies would lie outside the three systems Ł, G, and Π.

The theorem has various consequences, and some questions arise of it. We start with a simple one:

3.4.2 Theorem. *There is no formula expressing that a continuous t-norm has a proper cutpoint.*

Proof. Let X stand for either Ł or G or Π. If a formula is valid in a standard algebra $[0,1]_{\mathrm{X}\oplus\mathrm{Y}}$, then it is also valid in $[0,1]_{\mathrm{X}}$, which is an algebra without proper cutpoints. $\mathcal{QED}$

On the other hand, there are examples of formulas stating that the t-norm has *no* proper cutpoints—i.e., formulas that hold only in $[0,1]_{\text{Ł}}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_\Pi$. One such formula is the disjunction $(\theta \to \theta \mathbin{\&} \theta) \vee (\neg\neg\chi \to [(\varphi \mathbin{\&} \chi \to \psi \mathbin{\&} \chi) \to (\varphi \to \psi)])$; another appears in [2], extending BL to a complete axiomatization of the intersection of $\mathrm{Taut}([0,1]_{\text{Ł}})$, $\mathrm{Taut}([0,1]_{\mathrm{G}})$ and $\mathrm{Taut}([0,1]_\Pi)$.

The next result shows that there exists a non-collapsing chain within the hierarchy, i.e., an infinite chain of sharp inclusions. For any k we give a formula which says that a t-norm (consisting of Π's only) has *at most* k proper cutpoints.

3.4.3 Theorem. *Let* $Id(\varphi)$ *stand for* $\varphi \to \varphi \mathbin{\&} \varphi$; *let* $\Pi 1(\chi,\varphi,\psi)$ *stand for*

$$\neg\neg\chi \to ((\varphi \mathbin{\&} \chi \to \psi \mathbin{\&} \chi) \to (\varphi \to \psi))$$

and let $\Pi!(\alpha,\beta,\gamma)$ *stand for*

$$\Pi 1(\alpha,\beta,\gamma) \wedge \Pi 1(\alpha,\gamma,\beta) \wedge \Pi 1(\beta,\alpha,\gamma) \wedge \Pi 1(\beta,\gamma,\alpha) \wedge \Pi 1(\gamma,\alpha,\beta) \wedge \Pi 1(\gamma,\beta,\alpha)$$

Let k *be fixed; take* $k+2$ *propositional variables* $\{p_i\}_{i=1}^{k+2}$ *and a formula*

$$\left(\bigvee_{i=1}^{k+2} Id(p_i)\right) \vee \left(\bigvee_{i=1}^{k+2} \bigvee_{j=i+1}^{k+2} \bigvee_{m=j+1}^{k+2} \Pi!(p_i,p_j,p_m)\right) \tag{Φ_k}$$

Suppose $*$ *is a continuous t-norm with cutpoints* $\mathcal{C}$ *and a layout* λ, *such that for all* $c \in \mathcal{C}$, $c \neq c^+$, $\lambda([c,c^+]) = \Pi$; *then* Φ_k *holds in* $[0,1]_*$ *iff the cardinality of* $\mathcal{C}$ *is at most* $k+1$, *i.e., if* $*$ *has at most* $k-1$ *proper cutpoints.*

Proof. Suppose $*$ consists of Π's only. First observe that if $c \neq c^+$, $c \in \mathcal{C}$ and $e(p_i)$, $e(p_j)$, $e(p_m) \in (c,c^+)$, then $e(\Pi!(p_i,p_j,p_m)) = 1$ (evaluation in (c,c^+) is as in $[0,1]_\Pi$).

Also, if $c \neq c^+$, $c \in \mathcal{C}$, and $e(p_i), e(p_j) \in (c, c^+)$, $e(p_m) \in (d, d^+)$ for some $d > c$, $d \in \mathcal{C}$, then $e(\Pi!(p_i, p_j, p_m)) = 1$.Ve verify this in detail, simplifying the notation: suppose $\pi 1$ is the term corresponding to $\Pi 1$, denote $e(p_i)$ by i, $e(p_j)$ by j and $e(p_m)$ by m, and suppose $i \leq j$.Observe that $(- - x) = 1$ for any $x > 0$. We check that all six cases of permutations of arguments in $\pi 1$ evaluate to 1:

(i) $\pi 1(i, j, m)$ evaluates to 1 because the last term $j \Rightarrow m$ evaluates to 1; similarly for $\pi 1(j, i, m)$ and $\pi 1(m, i, j)$

(ii) $\pi 1(i, m, j)$: here $m * i = i$, $((m * i) \Rightarrow (j * i)) = (i \Rightarrow (j * i)) = j$, and $m \Rightarrow j = j$, hence we get $1 \Rightarrow (j \Rightarrow j) = 1$.

(iii) $\pi 1(m, j, i)$: here $((j * m) \Rightarrow (i * m)) \Rightarrow (j \Rightarrow i) = (j \Rightarrow i) \Rightarrow (j \Rightarrow i) = 1$.

(iv) $\pi 1(j, m, i)$: here $((m * j) \Rightarrow (i * j)) = (j \Rightarrow i * j) = i$ and $m \Rightarrow i = i$, which gives $i \Rightarrow i = 1$.

Next, if $e(p_i) < c$ and $e(p_j) > e(p_m) > c$, $c \in \mathcal{C}$, then $e(\Pi 1(p_i, p_j, p_m)) < 1$ (the second implication evaluates to 1, but the third is sharply less than 1) and hence also $e(\Pi!(p_i, p_j, p_m)) < 1$.

If $*$ has at most $k-1$ proper cutpoints, e is any evaluation in $[0,1]_*$, then either at least one p_i is evaluated by an idempotent (then the left disjunction of Φ_k evaluates to 1), or there is a $c \in \mathcal{C}$ such that (at least) three p_i's are evaluated in (c, c^+), or there exist $c, d \in \mathcal{C}$, $c < d$, such that $e(p_i), e(p_j) \in (c, c^+)$ and $e(p_m) \in (d, d^+)$ (which is the case investigated above); thus at least one member in the right disjunction evaluates to 1. Therefore Φ_k holds under any evaluation in $[0,1]_*$.

Suppose $*$ has k proper cutpoints $\{c_1 < c_2 < \ldots < c_k\}$; we define an evaluation e such that $e(\Phi_k) < 1$. Put $e(p_1) \in (0, c_1)$, $e(p_2) \in (c_1, c_2)$,..., $e(p_{k+1}), e(p_{k+2}) \in (c_k, 1)$. Then none of the $e(p_i)$'s is idempotent and each member of the right disjunction in Φ_k contains a subformula $\Pi 1(p_{i_1}, p_{i_2}, p_{i_3})$ such that for some $c \in \mathcal{C}$, $e(p_{i_1}) < c$, $e(p_{i_2}, p_{i_3}) > c$. Therefore no member evaluates to 1, and the whole disjunction evaluates to a certain $n < 1$. $\mathcal{QED}$

Thus $\mathrm{Taut}([0,1]_\Pi) \supset \mathrm{Taut}([0,1]_{\Pi\oplus\Pi}) \supset \mathrm{Taut}([0,1]_{\Pi\oplus\Pi\oplus\Pi}) \supset \ldots$ etc.

This theorem may be generalized also to t-norms with both Π's and G's in layouts, since a layout with k Π's can accommodate at most $k+1$ G's (we do not allow G$\oplus$G). The formula Φ_k holds for standard algebras without Ł's iff the number of proper cutpoints is at most $2k$ (the layout would be then G$\oplus\Pi\oplus$G$\oplus\Pi\oplus\ldots\oplus$G, with k occurrences of Π).

The construction of the formula Φ_k used in the previous statement is inspired by the one used in [4].

3.5. Completeness results for BL

We rely upon the fact that BL is complete w.r.t. t-tautologies. We shall look out for standard algebras for which the sets of tautologies are exactly BL-provable formulas; uncountably many pairwise non-isomorphic algebras can be presented, though we are at present unable to give any *characterization* of the class of such algebras. In addition, we show that BL is complete w.r.t. the class of standard algebras the layout of which starts with an Ł.

We outline some prerequisites now.

3.5.1 Lemma. *If φ is not a t-tautology, then there is a continuous t-norm $*$ with finitely many cutpoints such that φ is not a 1-tautology in $[0,1]_*$.*

Proof. A straightforward way to prove this is as follows. Suppose $\varphi(p_1, \ldots, p_n)$ is not a t-tautology, $[0,1]_{*_1}$ is a standard algebra and e_1 an evaluation in $[0,1]_{*_1}$ such that $e_1(\varphi(p_1, \ldots, p_n)) < 1$. Let $a_i = e_1(p_i)$, $i = 1, \ldots, n$, and suppose $a_{i_1} \leq a_{i_2} \leq \ldots \leq a_{i_n}$. Let $c_j, j = 1, \ldots, n$ be the cutpoints of $*_1$ for which $a_{i_j} \in [c_j, c_j^+]$. By 3.3.2, $\{0\} \cup [c_1, c_1^+] \cup \ldots \cup [c_n, c_n^+] \cup \{1\}$ is a subalgebra of $[0,1]_{*_1}$. Define $*_2$: its cutpoints are $\{0, c_1, c_1^+, c_2, \ldots, c_n^+, 1\}$, its layout is the same as in $*_1$ on each $[c_j, c_j^+], j = 1, \ldots, n$ and is either Ł or G or Π inbetween (these intervals do not affect the evaluation). Let e_2 be an evaluation in $[0,1]_{*_2}$, such that $e_2(p_i) = a_i$; then $e_2(\varphi) < 1$. By construction $*_2$ has finitely many cutpoints. $\mathcal{QED}$

We are now looking for those standard algebras in which any formula φ which is not a t-tautology will have an evaluation e such that $e(\varphi) < 1$. An easy way to achieve this is to use the preceding lemma and demand that the *layout* of each finitely cut t-norm, that is, any sum of symbols $\mathrm{X}_1 \oplus \mathrm{X}_2 \oplus \ldots \oplus \mathrm{X}_\mathrm{n}$, $X_i \in \{\mathrm{L}, \mathrm{G}, \Pi\}$, be embeddable into the layout of the requested t-norm. It is obviously sufficient if a t-norm $*$, with cutpoints $\mathcal{C}$ and layout λ, satisfies the following: there is an increasing (or decreasing) sequence $\{c_i\}_{i=0}^{\infty}$, $c_i \in \mathcal{C}$, such that for each i, if $\lambda[c_i, c_i^+] = \mathrm{L}$, then there are $j > i$ and $k > i$ such that $\lambda[c_j, c_j^+] = \mathrm{G}$ and $\lambda[c_k, c_k^+] = \Pi$, and similarly for G and Π. We will call such t-norms $\{\mathrm{L}, \mathrm{G}, \Pi\}$-*periodical*[2]. Of course a t-norm is $\{\mathrm{L}, \mathrm{G}, \Pi\}$-periodical iff it is $\{\Pi, \mathrm{G}, \mathrm{L}\}$-periodical, etc. A similar definition could be produced for $\{\mathrm{L}, \Pi\}$-periodical t-norms, etc.

3.5.2 Theorem. *Let $*$ be a continuous t-norm, $\mathcal{C}$ its cutpoints and λ its layout. If $\lambda[0, 0^+] =$ L and $*$ is $\{\mathrm{L}, \mathrm{G}, \Pi\}$-periodical, then* $\mathrm{Taut}([0,1]_*)$=$\mathrm{Thm(BL)}$.

Proof. First note that $*$ must start with a copy of $[0,1]_\mathrm{L}$, otherwise the formula $\Pi2$ would hold in it.
If φ is not a t-tautology, it has, by the preceding lemma, a finite counterexample, i.e., a t-norm $*_1$ with finitely many cutpoints $\mathcal{D} = \{0 = d_0 < d_1 < \ldots < d_n = 1\}$ and layout λ_1. We shall show that $*_1$ is obtained from $*$ by finitely many shrink operations which satisfy the conditions of 3.3.6, and consequently, if a formula is not valid in $[0,1]_{*_1}$, it is not valid in $[0,1]_*$.

To each $d_i \in \mathcal{D}$ we assign c_i in $\mathcal{C}$ so that:

(i) $c_0 < c_1 < c_2 < \ldots < c_n = 1$
(ii) if $\lambda_1([d_0, d_0^+]) = \mathrm{L}$, then $c_0 = 0$
(iii) for each $i = 0 \ldots n$, $\lambda([c_i, c_i^+]) = \lambda_1([d_i, d_i^+])$.

Obviously (iii) is possible due to the periodicity of $*$. Now we shrink each of the intervals $[0, c_0]$, and $[c_{i-1}^+, c_i]$, $i = 1, \ldots, n$, if of nonzero length; this amounts to finitely many shrinkings. The resulting t-norm is isomorphic to $*_1$. Let us verify the conditions of 3.3.6: we always shrink an interval between two idempotents; if $[0, c_0]$ is of nonzero length, then $\lambda([c_0, c_0^+]) \neq \mathrm{L}$ (otherwise the interval $[d_0, d_0^+]$ maps to the initial L in $*$ and $0 = c_0$).
$\mathcal{QED}$

Moreover, a slight improvement may be done to the lemma about finite counterexamples; we may even request that the continuous t-norm have finitely many *idempotents*. The same result is presented in [1]. The resulting theorem is proved in exactly the same way as 3.5.2.

3.5.3 Lemma. *If $\varphi(p_1, \ldots, p_n)$ is not a t-tautology, then there is a continuous t-norm with finitely many idempotents such that φ is not a 1-tautology in $[0,1]_*$.*

Proof. Like 3.5.1; from the original set of idempotents we retain

(a) each two cutpoints c_i and c_i^+ such that there there is an i for which $e(p_i) \in [c_i, c_i^+]$ and $\lambda[c_i, c_i^+]$ is L or Π, and
(b) each $e(p_i)$ which is idempotent.

We preserve the original layouts on each $[c_i, c_i^+]$, and fill in arbitrarily L's and Π's inbetween.
$\mathcal{QED}$

3.5.4 Theorem. *Let $*$ be a continuous t-norm, $\mathcal{C}$ its cutpoints and λ its layout. If $\lambda[0, 0^+] =$ L and $*$ is $\{\mathrm{L}, \Pi\}$-periodical, then* $\mathrm{Taut}([0,1]_*) = \mathrm{Thm(BL)}$.

The other promised result in this section claims the completeness of BL w.r.t. standard algebras starting with an L. In [3], the authors give a completeness result for the complementary class of standard algebras, i.e., algebras *not* starting with an L; these are called SBL-algebras and axiomatized by BL$\cup\{\Pi2\}$.

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[2] note that the word 'periodical' does not imply that L, G and Π take turns in a regular pattern

3.5.5 Theorem. BL *is complete with respect to the class of standard algebras whose layout starts with an* Ł.

Proof. Suppose φ is not a theorem of BL. Take a formula φ and a standard algebra $[0,1]_{\mathrm{Y}_0}$ in which φ does not hold. Our aim is to construct a Y such that in the standard algebra $[0,1]_{\text{Ł}\oplus\mathrm{Y}}$, φ is not a 1-tautology.

If $\mathrm{Y}_0 =$Ł$\oplus\mathrm{Y}_1$ for some Y_1 then we have finished since Y_1 is the desired Y. Suppose not; take the standard algebra $[0,1]_{\text{Ł}\oplus\mathrm{Y}_0}$ (a sum of $[0,1]_{\text{Ł}}$ and the above counterexample), and its cutpoint c such that $[0,c]$ is isomorphic to $[0,1]_{\text{Ł}}$. Then since $[0,1]_{\mathrm{Y}_0}$ results by shrinking of the interval $[0,c]$ from $[0,1]_{\text{Ł}\oplus\mathrm{Y}_0}$ (satisfying the conditions of 3.3.6 since the initial segment of Y_0 is not Ł), if φ is not a 1-tautology of $[0,1]_{\mathrm{Y}_0}$, it is not a 1-tautology of $[0,1]_{\text{Ł}\oplus\mathrm{Y}_0}$ either and Y_0 is the desired Y. $\mathcal{QED}$

3.6. Other results

This section contains results which do not fit elsewhere. They give answers to some questions which happened to occur; other problems of similar nature, but without answers, appear in the Appendix.

Let Ł$G(\varphi)$ stand for the formula $(\neg\neg\varphi \to \varphi) \vee (\varphi \to \varphi \,\&\, \varphi)$ and let $\Pi1(\chi,\varphi,\psi)$ stand for $\neg\neg\chi \to ((\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi))$.

3.6.1 Theorem.

(i) *The axiom* $\Pi1$ *holds only in* $[0,1]_{\text{Ł}}$ *and in* $[0,1]_{\Pi}$.
(ii) *The axiom* $\Pi2$ *holds in* $[0,1]_*$ *iff the layout of* $*$ *does not start with an Ł.*
(iii) $\mathrm{BL} \subset (\text{Ł} \cap \mathrm{G} \cap \Pi)$

Proof.
(i) We show that $\Pi1$ is provable from the axioms of Łukasiewicz's logic. Since Ł states $\neg\neg\chi \equiv \chi$, it is sufficient to prove the formula $\chi \to ((\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi))$ in BL.

$\mathrm{BL} \vdash ((\varphi \,\&\, \chi) \,\&\, (\varphi \,\&\, \chi \to \psi \,\&\, \chi)) \to \psi$
$\mathrm{BL} \vdash (\chi \,\&\, (\varphi \,\&\, \chi \to \psi \,\&\, \chi)) \to (\varphi \to \psi)$
$\mathrm{BL} \vdash \chi \to ((\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi))$.

As for other t-norms, if $*$ is a continuous t-norm with a non-extremal idempotent h, evaluate $e(\chi) = h$, $h < e(\psi) < e(\varphi) < 1$. Then $e(\neg\neg\chi) = 1$, $e(\varphi \,\&\, \chi) = h$ and $e(\psi \,\&\, \chi) = h$, hence $e(\varphi \,\&\, \chi \to \psi \,\&\, \chi) = 1$, but $e(\varphi \to \psi) = c < 1$, and we get $1 \Rightarrow (1 \Rightarrow c)$, which is $c < 1$.
(ii) See [3].
(iii) To prove the sharp inclusion we just verify the remark after 3.4.2, stating that the formula $(\theta \to \theta \,\&\, \theta) \vee (\neg\neg\chi \to [(\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi)])$ holds only in $[0,1]_{\text{Ł}}$ and $[0,1]_{\Pi}$ (due to the $\Pi1$-part) and in $[0,1]_{\mathrm{G}}$ (due to the G-part), but not elsewhere: a counterexample evaluation e in a standard algebra with a nonextremal idempotent h will have $e(\chi) = h$, $h < e(\psi) < e(\varphi) < 1$, and $e(\theta)$ a non-idempotent. Then $e(\theta \to \theta \,\&\, \theta) < 1$, and the $\Pi1$ part does not evaluate to 1 as shown above. $\mathcal{QED}$

3.6.2 Lemma. BL *is not complete w.r.t.* $[0,1]_{\text{Ł}\oplus\Pi}$.

Proof. The formula

$$\text{Ł}G(\varphi) \vee \text{Ł}G(\psi) \vee \text{Ł}G(\chi) \vee \Pi1(\chi,\varphi,\psi)$$

is a 1-tautology of $[0,1]_{\text{Ł}\oplus\Pi}$, but not, e.g., of $[0,1]_{\text{Ł}\oplus\text{Ł}}$. Indeed, if in $[0,1]_{\text{Ł}\oplus\Pi}$ none of the ŁG's evaluates to 1, then the values for φ, ψ and χ must be from the interval $(h,1)$; but then the Π-part evaluates to 1, because the evaluation copies evaluations in $[0,1]_{\Pi}$.

Now take $[0,1]_{\text{Ł}\oplus\text{Ł}}$, suppose h is the proper cutpoint in it. Observe that we may assign to φ, ψ and χ values from $(h,1)$ in such a way that $e(\varphi \,\&\, \chi) = e(\psi \,\&\, \chi) = h$ and $e(\varphi) > e(\psi)$. Then the evaluation yields $1 \Rightarrow (1 \Rightarrow (e(\varphi) \Rightarrow e(\psi)))$, which is certainly not 1. $\mathcal{QED}$

The tautologies of $[0,1]_{Ł\oplus\Pi}$ are sharply within the intersection of tautologies of $[0,1]_Ł$ and $[0,1]_\Pi$ (for example, $\Pi 1$ does not hold in $[0,1]_{Ł\oplus\Pi}$ although it holds in $[0,1]_Ł$ and in $[0,1]_\Pi$). This follows in general from the existence of formulas stating that a standard algebra has no proper cutpoints (see the paragraph after 3.4.2).

We present an example of axiomatizing a simple algebra with proper cutpoints.

3.6.3 Lemma. *The formula $ŁG(\varphi)$ is a 1-tautology of $[0,1]_*$ iff $*$ is $*_Ł$ or $*_G$ or $*_{Ł\oplus G}$.*

Proof. Obviously, in $[0,1]_Ł$, $[0,1]_G$, and $[0,1]_{Ł\oplus G}$ $ŁG(\varphi)$ holds. Conversely, if the layout contains a Π or a non-initial Ł, then $ŁG(\varphi)$ does not hold. Therefore any Ł in the layout must be initial and hence there can be at most one; hence there can be at most one G (since Ł$\oplus$G$\oplus$G is not allowed). $\mathcal{QED}$

3.6.4 Corollary. *The logic* $\mathrm{BL} \cup ŁG(\varphi)$ *is complete w.r.t.* $[0,1]_{Ł\oplus G}$.

Proof. These axioms define the smallest variety containing all MV-algebras, G-algebras, and algebras which are an ordered sum of an MV-algebra and a G-algebra (i.e., no other types of ordered sums occur; proof as in the preceding lemma). By theorem 2.6, if a formula α is unprovable in $\mathrm{BL} \cup ŁG(\varphi)$ then it has a counterexample in a linearly ordered BL-algebra which is (as just observed) either an MV-algebra or a G-algebra or an ordered sum of an MV-algebra and a G-algebra. In all cases the counterexample evaluation for α can be embedded in the standard algebra with the corresponding type; but then α cannot hold in $[0,1]_{Ł\oplus G}$. $\mathcal{QED}$

3.6.5 Theorem. *For $i = 1, 2$ let $*_i$ be a continuous t-norm with an infinite set of cutpoints $\mathcal{C}_i$ and a layout λ_i such that for each $c \in \mathcal{C}_i$, $c \neq c^+$, $\lambda_i[c, c^+] = \Pi$. Then* $\mathrm{Taut}([0,1]_{*_1}) = \mathrm{Taut}([0,1]_{*_2})$.

Proof. We show $\mathrm{Taut}([0,1]_{*_2}) \subseteq \mathrm{Taut}([0,1]_{*_1})$. Suppose φ is not a tautology of $[0,1]_{*_1}$. Then there is a standard algebra A with a finite number of cutpoints and only Π's in its layout, such that φ is not a tautology of A (see 3.5.1). Since the tautologies of $[0,1]_{*_2}$ are a subset of the tautologies of A, φ does not hold in $[0,1]_{*_2}$ either. $\mathcal{QED}$

Note that a direct analogy for Ł's is false.

Appendix

Unsolved problems

(i) Find a single, simple axiom schema, preferably with one propositional variable, that could be used as an axiom of product logic instead of $\Pi 1$ and $\Pi 2$.
(ii) Find a complete axiomatics for $[0,1]_{Ł\oplus\Pi}$.
(iii) Characterize the class of standard algebras for which $\mathrm{Taut}([0,1]_*)$ are exactly BL-provable formulas.
(iv) Find an analogy of 3.4.3 for sums of Ł's, or prove that the inclusions are not sharp (that the chain collapses).
(v) Characterize standard algebras for which a recursive axiomatics exists.

These problems are of course not independent. For example, (v) is based on the fact that there can be only countably many recursive axiomatics; but, if the "majority" of standard algebras has BL-provable formulas for tautologies, then (v) is solved.

That things be black and white, and nought between,
No mortal thinks. That it is either day,
Or night; or fire, or frost; our sense is keen
Enough to wipe these boundaries away.
Great poets shed their blood and strain their voice
To count the many hues 'twixt dusk and dawn,
And words, those self-willed slaves, rejoice
To smooth all edges; thence new words are born.
Aye, tongue doth allow blacks to seem all whites;
Base vices for the heights of virtues reach
And by a claim most wrong assume their rights:
For all is possible, within our speech.

And yet each thing or black or white must be;
But which is which is not for us to see.

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