Complexity of some language fragments of fuzzy logics[*]

Zuzana Haniková
Institute of Computer Science
Academy of Sciences of the Czech Republic
182 07 Prague, Czech Republic
hanikova@cs.cas.cz

Abstract

Computational complexity of the semigroup fragment (of the algebraic semantics) and the implicational fragment of some fuzzy logics is studied, from the perspective of the complexity of the full logic. The available results appear to confirm the key role of the implicational fragments. Some other language fragments, as well as the notion of language fragment itself, are discussed.

1 Introduction

This paper is a contribution to studies of language fragments of propositional substructural logics extending[1] $\mathrm{FL_{ew}}$ or of their equivalent algebraic semantics. Computational properties of the semigroup fragment and the implicational fragment of some fuzzy logics are studied, with remarks on other language fragments. A *language fragment* of a set of terms in some language is a subset thereof, given by restricting the set of function symbols (connectives) in that language.

Studies of language fragments (alongside the respective logics) try to clarify the interplay of connectives, their role, and their expressive power. There are many facets, such as interdefinability of connectives, subvariety structure, computational properties, etc. Often one or more language fragments determine the behaviour of the full logic: for example, the chain of extensions of Gödel logics is isomorphic to the chain of (implicational) extensions of its implicational fragment. Moreover, for each of the extensions, its tautologies can be easily retrieved from its implicational tautologies.

Decision procedures are studied, using *reductions* as key tools. In particular, many-one polynomial-time reductions are considered, where reducing one propositional logic to another amounts to finding a translation function f operating (in polynomial time) on terms that faithfully preserves validity, i.e., φ is valid in the source logic iff $f(\varphi)$ is valid in the target logic. Such reductions yield a "library of translations" between logics and their fragments. Polynomial-time reductions, used in this paper, can be viewed as incurring a reasonable translation overhead; in fact, for all translations explicitly considered here, the overhead is negligible.

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[1]An *extension* is understood to be an *axiomatic extension* in the same language.

It is a matter of course that polynomial-time translations exist between logics or theories known to be polynomially equivalent (e.g., coNP-complete). We provide simple, explicit reductions of some fuzzy logics already known to be coNP-complete (e.g., Łukasiewicz logic and its extensions) to their implicational fragments, that are moreover one-one functions invertible in polynomial time. We thus establish a lower bound on complexity for the implicational fragments in terms of the complexity of the full logic. This new, if expected, coNP-completeness result for the implicational fragments is ultimately the stronger statement.

On the other hand, some fragments are potentially much easier to decide; moreover, the language fragment does not have the distinguishing power of the full language, so the corresponding subvarieties do not present a rich structure. As an example, we analyze identities of the semigroup language fragment.

Studies in language fragments of substructural logics are too numerous to be mentioned here. We point out a few previous works that are directly relevant. The paper [11] presents a systematic treatment of language fragments of fuzzy logics (axiomatic extensions of MTL) that contain the implication, starting from the logic FBCK, the implicational fragment of MTL, and presenting other logics and their language fragments as axiomatic expansions; the paper does not study algorithmic problems but does address the problem of (non)coincidence of language fragments of the logics. [1] studies *basic hoops*, i.e., 0-free subreducts of BL-algebras, and their implicative subreducts.[2] Moreover, [13] studies 0-free fragments of important fuzzy logics, as logics of basic hoops and semihoops: among other results, it shows that 0-free fragments of MTL, SMTL, and IMTL coincide, that 0-free fragments of BL and SBL coincide, and that each of the logics MTL, BL, Ł, G, Π is poly-time reducible to its 0-free fragment. A poly-time reduction of intuitionistic propositional logic to its implicational fragment is presented in [29]; the former, and hence the latter, is shown to be PSPACE-complete. Regarding decision problems in fuzzy logics studied in this paper, [25] gives coNP-completeness of Łukasiewicz logic, [17] proves the same for Gödel and product logics, [3] shows coNP-completeness for Hájek's BL, and [10] shows coNP-completeness for extensions of Łukasiewicz logic. In contrast, there is no known upper bound on the complexity of MTL (which is known to be decidable by [5]).

2 Preliminaries

This paper does not introduce and define all of its notions and background theory; owing to a rather special topic, it is assumed that an interested reader is already familiar with the agenda and definitions of substructural and fuzzy logics. Comprehensive works include [14, 27, 17, 12]. Likewise, standard notions of decidability and computational complexity theory are taken for granted. In particular, P, NP, coNP, LOGSPACE, PSPACE denote complexity classes, and $\preceq_{\mathrm{P}}$ denotes polynomial-time many-one (*poly-time*) reducibility.

This paper limits its attention to propositional logics extending the logic $\mathrm{FL_{ew}}$ and their algebraic semantics; therefore, a *logic* is always *propositional*.[3] While logics are now generally understood to be substitution-invariant consequence relations on a set of terms, we opt for a simpler (and more traditional) view that takes logics to be just sets of terms that are closed

[2] It follows from that paper that the implicational fragment of product logic is contained in the implicational fragment of Łukasiewicz logic; this also follows from the fact that the standard MV-algebra is isomorphic to the cut product algebra (cf. [17]).

[3] A propositional logic can itself be regarded as a syntactic fragment of its (putative) predicate expansions.

under substitution and deduction. This choice is pragmatic; this paper predominantly works with sets of terms/equational theories.

A *language* is a set of function symbols, each with a given arity. The language $\mathcal{L}$ of $\mathrm{FL_{ew}}$ has binary symbols $\cdot$ (*multiplication*), $\to$ (*implication*), $\wedge$ and $\vee$ (*lattice conjunction* and *disjunction*), and two constants 0 and 1. A countably infinite set of variables is considered. $\mathrm{FL_{ew}}$-terms are defined inductively as usual, and denoted with lowercase Greek letter such as φ, ψ, χ. No distinction is made between function symbols and propositional connectives, or between algebraic terms and propositional formulas. The set of all $\mathcal{L}$-terms is denoted $Tm^{\mathcal{L}}$ (or just Tm if no confusion can arise).

The unary symbol $\neg$ (negation) is introduced by writing $\neg\varphi$ for $\varphi \to 0$ for any term φ; moreover, $\varphi \equiv \psi$ stands for $(\varphi \to \psi) \cdot (\psi \to \varphi)$. For a term φ, we write φ^n for $\varphi \cdot \varphi \cdot \dots \cdot \varphi$ (n times).

An interpretation of a function/predicate symbol f in an algebra $\mathcal{A}$ is denoted $f^{\mathcal{A}}$; $\approx$ is the identity symbol and $=$ denotes equality of elements of an algebra $\mathcal{A}$. The superscripts may be omitted if no confusion can arise.

Definition 1. *An algebra* $\mathcal{A} = \langle A, \cdot^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, 0^{\mathcal{A}}, 1^{\mathcal{A}}\rangle$ *is an* $\mathrm{FL_{ew}}$*-algebra if*

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, 0^{\mathcal{A}}, 1^{\mathcal{A}}\rangle$ *is a* $\{0,1\}$*-bounded lattice; we use* $\leq^{\mathcal{A}}$ *for the lattice order*
2. $\langle A, \cdot^{\mathcal{A}}, 1^{\mathcal{A}}\rangle$ *is a commutative monoid with the unit element* $1^{\mathcal{A}}$
3. $\cdot^{\mathcal{A}}$ *and* $\to^{\mathcal{A}}$ *form a residuated pair, i.e.,* $x \cdot^{\mathcal{A}} y \leq^{\mathcal{A}} z$ *iff* $x \leq^{\mathcal{A}} y \to^{\mathcal{A}} z$

Important examples of $\mathrm{FL_{ew}}$ algebras include MTL algebras (semilinear $\mathrm{FL_{ew}}$-algebras), BL-algebras (divisible MTL-algebras), the standard MV-algebra $[0,1]_{\text{Ł}}$, the standard Gödel algebra $[0,1]_{\mathrm{G}}$, the standard product algebra $[0,1]_{\Pi}$, Heyting algebras, Boolean algebras including the two-element Boolean algebra $\{0,1\}_{\mathrm{B}}$. Weakly contractive $\mathrm{FL_{ew}}$-algebras are defined by the identity $x \wedge \neg x \approx 0$, while distributive ones are defined by the lattice-theoretic identity delimiting distributive lattices.

The logic $\mathrm{FL_{ew}}$ is algebraizable and the class $\mathbb{FL}_{\mathbb{ew}}$ of $\mathrm{FL_{ew}}$-algebras forms its equivalent algebraic semantics. We take the logic $\mathrm{FL_{ew}}$ to be just the set of terms that are valid in all $\mathrm{FL_{ew}}$-algebras.

Lemma 2. *Let* $\mathcal{A}$ *be a* $\mathrm{FL_{ew}}$*-algebra,* $a \in A$*. Then* $\mathcal{A}_a = \langle [a,1], \cdot_a, \to, \wedge, \vee, a, 1\rangle$ *is a* $\mathrm{FL_{ew}}$*-algebra, where* $[a,1]$ *is the upper cone of* a *and* $x \cdot_a y = a \vee (x \cdot y)$*.*

Proof. Using weakening, the operation $\to$ is total function on $[a,1]$. Moreover:
1. $\langle [a,1], \wedge, \vee, a, 1\rangle$ is a bounded lattice;
2. $\langle [a,1], \cdot_a, 1\rangle$ is a commutative monoid: for $a \leq x, y, z$, we have $1 \cdot_a x = a \vee (1 \cdot x) = a \vee x = x$; commutativity is clear; associativity: $(x \cdot_a y) \cdot_a z = a \vee [(a \vee (x \cdot y)) \cdot z] = a \vee [a \cdot z \vee x \cdot y \cdot z] = a \vee x \cdot y \cdot z$, and the latter is equal to $x \cdot_a (y \cdot_a z)$ by similar reasoning.
3. residuation follows from the fact that $x \cdot_a z \leq y$ iff $x \cdot z \leq y$. □

It is obvious that if, in addition, $\mathcal{A}$ is a Heyting algebra, then so is A_a. If $\mathcal{A}$ is a chain, then so is $\mathcal{A}_a$, and if $\mathcal{A}$ is semilinear, then so is $\mathcal{A}_a$. If $\mathcal{A}$ is an MTL-algebra, then so is $\mathcal{A}_a$; this was proved in [13], Lemma 3.13.; as pointed out therein, divisibility is preserved by the construction, so the same is true with respect to BL-algebras; moreover, if $\mathcal{A}$ is an MV-algebra, so is $\mathcal{A}_a$ (to see that the latter is involutive, it suffices to recall that for each $x \in A$, $(x \to a) \to a = x \vee a$). If $\mathcal{A}$ is $[0,1]_{\Pi}$, then $\mathcal{A}_a$ is isomorphic to $[0,1]_{\text{Ł}}$; this was proved in [17].

3 Language fragments

Let $\mathcal{F} \subseteq \mathcal{L}$. The $\mathcal{F}$-fragment of Tm, denoted $Tm^{\mathcal{F}}$, consists of those terms in Tm that only employ symbols from $\mathcal{F}$. If L is a logic in $\mathcal{L}$ (i.a., $\mathrm{L} \subseteq Tm$), the $\mathcal{F}$-fragment of L is $\mathrm{L}^{\mathcal{F}} = \mathrm{L} \cap Tm^{\mathcal{F}}$; note that $\mathrm{L}^{\mathcal{F}}$ is a logic (in the language $\mathcal{F}$). For language fragments of algebraic theories, the notation is analogous.

Two remarks seem to be due. (1) Some connectives (such as $\neg$) are not included in $\mathcal{L}$ on the grounds of their term-definability, and are understood as shortcuts. However, their term definability may fail in a fragment $\mathcal{F} \subset \mathcal{L}$. Therefore, a comprehensive study of language fragments would need to take into account at least the most common examples of definable connectives (obtaining, for example, the $\{\cdot, \wedge, \vee, \neg\}$-fragment). Formally this could be achieved by making $\mathcal{L}$ broader, not insisting on connective independence, and considering subsets of this broader set. (2) Apart from restricting the set of connectives, no further conditions are posed on term syntax; more syntactic fragments would occur by imposing additional rules for term formation, such as demanding that negations only occur next to atoms.

Observation 3. *Let* L *be a logic,* $\emptyset \neq \mathcal{F} \subseteq \mathcal{L}$. *Then* $\mathrm{L}^{\mathcal{F}} \preceq_{\mathrm{P}} \mathrm{L}$.

Proof. On input $\varphi \in Tm$, the decision procedure for $\mathrm{L}^{\mathcal{F}}$ first checks that $\varphi \in Tm^{\mathcal{F}}$; if so, it calls the decision procedure for L on φ. □

This easy statement is spelled out here for several reasons. First, it is a natural and ubiquitous method for providing upper/lower bounds on complexity in the following manner:

- If $\mathrm{L}^{\mathcal{F}}$ is hard for a complexity class under $\preceq_{\mathrm{P}}$, then so is L.
- If L belongs to a complexity class closed under $\preceq_{\mathrm{P}}$ reductions, then so does $\mathrm{L}^{\mathcal{F}}$.

Second, it highlights some freedom in choosing the type of reduction. Checking membership in the language fragment (on an existing promise that the term is well-formed), for fragments considered here, can be done in linear time and logarithmic space. Third, it prompts the question whether the two problems are actually *equivalent* under $\preceq_{\mathrm{P}}$ (or another reduction under consideration).

Naturally, language fragments are potentially easier to decide; for example, the logic BCK is in NP[4], while $\mathrm{FL_{ew}}$ is PSPACE-complete (see [19] for a proof of hardness). It remains to be seen whether fuzzy logics (here, MTL and its extensions) can be reduced to their implicational fragments.

Let us review some facts on available decision procedures for fuzzy logics, taking Łukasiewicz logic and MV-algebras as an example. Upper bounds on complexity, where available, rest mainly on structural knowledge of their algebraic semantics. As shown in [18], such knowledge can provide coNP-containment not just for identities, but for the universal theory of the standard algebra for the logic. For example, MV-algebras have coNP-complete equational theory due to the results of [25] concerning the standard MV-algebra $[0,1]_{\text{Ł}}$; also the universal theory of $[0,1]_{\text{Ł}}$ is coNP-complete ([18]). *A fortiori*, the quasi-equational theory of $[0,1]_{\text{Ł}}$ is coNP-complete; by finite strong standard completeness of Łukasiewicz logic, the latter corresponds to the finite consequence relation of Ł. Thus by general strong finite completeness, quasi-equational theory of MV-algebras is coNP-complete, and it is not difficult to show that

[4] Proofs are of polynomial size in formula size; the same is true about, e.g., BCI and indeed MLL, cf. [7, 20].

this extends to their universal theory. (Note that the universal theory of MV-algebras does *not* coincide with the universal theory of $[0,1]_{\text{Ł}}$). The lower bound for the implicational fragment, shown in this paper, applies to the implicational fragments of each of these theories.

Let L be a nontrivial logic extending $\mathrm{FL_{ew}}$. Consider a language $\emptyset \neq \mathcal{F} \subseteq \mathcal{L}$. If $\mathcal{F}$ does not contain $\to$, then $\mathrm{L}^{\mathcal{F}}$ yields uninteresting theorems: in particular, for $\mathcal{F} \subseteq \{\cdot, \wedge, \vee, 0\}$, $\mathrm{L}^{\mathcal{F}}$ has *no* theorems, for $\mathcal{F} \subseteq \{\cdot, \wedge, 0, 1\}$, the only theorems are combinations of 1, and for the remaining fragments, theorems are the same for all nontrivial $\mathrm{FL_{ew}}$-extensions. Therefore, the $\mathcal{F}$-reduct of the equational theory for the class $\mathbb{K}$ of algebras corresponding to L, (or equivalently, the theory of the $\mathcal{F}$-reduct of $\mathbb{K}$) is considered instead.

If $\mathcal{F}$ contains $\to$, then for all logics considered here $\mathrm{L}^{\mathcal{F}}$ and ${\mathrm{Th_{Eq}}}^{\mathcal{F}}(\mathbb{K})$ (where $\mathrm{Th_{Eq}}(\mathbb{K})$ denotes the equational theory of $\mathbb{K}$) are polynomially equivalent: clearly for $\varphi, \psi \in Tm^{\mathcal{F}}$, $\varphi \in \mathrm{L}^{\mathcal{F}}$ iff $\varphi \approx (x \to x) \in {\mathrm{Th_{Eq}}}^{\mathcal{F}}(\mathbb{K})$; on the other hand, $\varphi \approx \psi \in {\mathrm{Th_{Eq}}}^{\mathcal{F}}(\mathbb{K})$ iff $(\varphi \to \psi) \cdot (\psi \to \varphi) \in \mathrm{L}^{\mathcal{F}}$, and in case $\cdot$ is not in $\mathcal{F}$, then two separate questions to $\mathrm{L}^{\mathcal{F}}$ can be used, and the result is obtained by combining them in conjunction; the classes P, coNP and PSPACE are closed under this extended type of reduction.

4 The $\{\cdot\}$-fragment

This section investigates identities in the semigroup language in $\mathrm{FL_{ew}}$-algebras. The language of semigroups has a single binary symbol $\{\cdot\}$; the algebras in question are commutative, which allows for a simpler notation. The abbreviations CSG is used for 'commutative semigroup'.

The equational theory of CSGs has two axioms, namely,

$$x \cdot y \approx y \cdot x$$
$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$$

These axioms enable a *normal form* for terms and identities; each identity in the language of CSGs can be written as

$$x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \approx x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$$

where a_i, b_i are nonnegative integers with $a_i + b_i > 0$ for each i. An identity in normal form is called *trivial* iff $a_i = b_i$ for all i, otherwise it is *nontrivial.*

[28] shows that each variety of CSGs is finitely based, hence the lattice of varieties of CSGs (and monoids) is countable; its structure was studies further, e.g., in [26, 21]. We shall look at the equational theories of semigroup reducts of $\mathrm{FL_{ew}}$-algebras, which form a simpler pattern.

Remark 4. *(1) For any class $\mathbb{K}$ of CSGs, and for any nonnegative integer t, if $\mathbb{K} \models x^t \approx x^{t+1}$, then $\mathbb{K} \models x^m \approx x^n$ for $t \leq m, n$.*

(2) For any class $\mathbb{K}$ of $\mathrm{FL_{ew}}$-algebras, if $\mathcal{K} \models x^m \approx x^n$ for $0 \leq m < n$, then $\mathbb{K} \models x^p \approx x^n$ for each $m \leq p \leq n$.

Theorem 5. *Let $\mathbb{K}$ be a class of $\mathrm{FL_{ew}}$-algebras. Consider an arbitrary nontrivial semigroup identity in normal form, i.e., $x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \approx x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$. The following are equivalent:*

(1) $\mathbb{K} \models x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \approx x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$

(2) $\mathbb{K} \models x^k \approx x^{k+1}$, where $k = \min\{a_1, \ldots, a_n, b_1, \ldots, b_n \mid a_i \neq b_i\}$.

Proof. Let $\mathbb{K} \models x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \approx x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$. Then, in particular, for each $1 \le i \le n$ and each $\mathcal{A} \in \mathbb{K}$, the identity holds for assignments that assign the element $1^{\mathcal{A}}$ to all variables except x_i; since $1^{\mathcal{A}}$ is the neutral element of $\cdot^{\mathcal{A}}$, it follows that $\mathcal{A} \models x_i^{a_i} \approx x_i^{b_i}$. Suppose, for a given i, that $a_i > b_i$; then $\mathcal{A} \models x^{b_i} \approx x^{b_i+1}$. It follows that $\mathbb{K} \models x^k \approx x^{k+1}$ for $k = \min\{a_1, \ldots, a_n, b_1, \ldots, b_n \mid a_i \neq b_i\}$.

On the other hand, if $\mathbb{K}$ validates $x^k \approx x^{k+1}$ for k as above, then it validates $x^{a_i} \approx x^{b_i}$ for any $1 \le i \le n$, and consequently also $x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \approx x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$. □

Subvarieties of $\mathrm{FL_{ew}}$-algebras given by the identities

$$x^k \approx x^{k+1} \tag{E_k}$$

were studied in some detail; the paper [27] gives a summary. It refers to the algebras belonging to the varieties given by E_k as *k-potent*. We shall denote, for $k \ge 1$,

$$\mathbf{Var}(E_k) = \{\mathcal{A} \in \mathbb{FL}_{\mathsf{ew}} \mid \mathcal{A} \models E_k\}$$

Clearly, $\mathbf{Var}(E_1)$ is the variety of Heyting algebras. It is remarked in [27] that the $\mathrm{FL_{ew}}$-term expressing E_k is $p^k \to p^{k+1}$ (of course, this is no longer in the semigroup language). Moreover, the paper discusses the chain (ordered by inclusion) of $\mathbf{Var}(E_k)$: we have $\mathbf{Var}(E_1) \subseteq \mathbf{Var}(E_2) \subseteq \ldots \subseteq \mathbb{FL}_{\mathsf{ew}}$. It gives, for each $k \in \mathbb{N}$, examples of algebras in $\mathbf{Var}(E_k) \setminus \mathbf{Var}(E_{k-1})$, whose logics are moreover *neighbors* (direct predecessors) of classical logic in the inclusion order ([27], Theorem 7.3). We remark here that the finite $k+1$-valued MV-chain is in $\mathbf{Var}(E_{k+1}) \setminus \mathbf{Var}(E_k)$; the standard MV-algebra $[0,1]_{\text{Ł}}$ is not in $\bigcup_k \mathbf{Var}(E_k)$. Finally, it follows from Theorem 5 that any algebra in $\mathbb{FL}_{\mathsf{ew}} \setminus \bigcup_k \mathbf{Var}(E_k)$ satisfies only trivial CSG identities.

The paper [15] shows that the semigroup fragment of the equational theory of $[0,1]_{\text{Ł}}$ and of $[0,1]_\Pi$ coincide; this is a particular example of the general phenomenon discussed here, i.e., no nontrivial CSG identities hold in either. There are many more natural examples of $\mathrm{FL_{ew}}$-algebras that do not belong to E_k for any k; a sufficient condition is that the algebra contain an infinite descending chain $1 > a > a^2 > a^3 > \ldots$, or descending chains of every finite cardinality. Examples include the Chang algebra, any of the Komori algebras, and any continuous t-norm algebra whose ordinal sum contains an MV- or a product component.

Summing up, a nontrivial CSG identity is $\mathrm{FL_{ew}}$-equivalent to E_k for some $k \ge 1$. We have provided examples of $\mathrm{FL_{ew}}$-algebras that satisfy only trivial CSG identities; *a fortiori*, only trivial CSG identities hold in the CSG variety. Among the $\mathrm{FL_{ew}}$ subvarieties that satisfy only trivial CSG identities, there are minimal subvarieties of $\mathrm{FL_{ew}}$; for example, the variety given by the Chang algebra $\mathcal{K}_2$.

Theorem 6. *The following problems are in* LOGSPACE*.*

(1) given an arbitrary CSG identity, is its normal form trivial?

(2) given $k \in \mathbb{N}$ and an arbitrary CSG identity, is the identity $\mathrm{FL_{ew}}$*-equivalent to E_k?*

For each CSG identity, either (1) is the case, or (2) is the case for a unique k. Note that normal form for the input identity $s \approx t$ is not assumed; a variable may have multiple occurrences in each of the two terms, and may occur in powers with exponent greater than 1; for x^k, k is referred to as *multiplicity* of the occurrence.

Proof. All numerical values are assumed to be represented in binary. Any brackets in the terms are ignored. Without going into details of representation, it is assumed that the procedure is able to scan a CSG term for occurrences of x_i and their multiplicities, and retrieve individual digits of these numbers.

The input is an identity $s \approx t$, where s and t are CSG terms. The term s is of the form $x_{i_1}^{p_1} x_{i_2}^{p_2} \dots x_{i_m}^{p_m}$, where each x_{i_j}, $j \leq m$, is one of $x_1, \dots, x_n$, and m denotes the number of occurrences of these variables in s. For each occurrence of a variable, we need $O(\log n)$ space to identify its index (1 to n). Thus the length of s, denoted by $|s|$, is $O(m \log n + \Sigma_{i=1}^{m} |p_i|)$. Analogously for t; the input size is $|s| + |t|$.

There are n distinct variables $x_1, \dots x_n$ in s and t. Let us denote by S_i the sum of multiplicities of the occurrences of x_i in s, and analogously for T_i, for $1 \leq i \leq n$. The procedure needs to check the following:
(1) $S_i = T_i$ for each $i \in \{1, \dots, n\}$;
(2) there is an $i \in \{1, \dots, n\}$ s.t. $S_i \neq T_i$ and $S_i = k$ or $T_i = k$, and moreover for each $i \in \{1, \dots, n\}$, if $S_i \neq T_i$, then $S_i \geq k$ and $T_i \geq k$.
Clearly, the conditions given by either (1) or (2) can be evaluated by subsequently processing each i, and the latter consists in performing several comparisons.

The decision procedure can use (apart from the read-only input) a space that is proportional to the logarithm of the input size. In particular, it can store a fixed number of counters or pointers to the input string. One of the things the procedure cannot generally afford to do is search for occurrences of x_i in s and store a counter for summing up their multiplicities. In fact, it cannot even store the multiplicity of a single occurrence of x_i, because each of those is, in general, linear in $|s|$. (Were it assumed that the multiplicity of each variable occurrence was 1, the counter would be of logarithmic size in $|s|$).

However, it is easy to compute any *digit* of S_i (or T_i) in logarithmic space. Let us compute the k-th lowermost (i.e., rightmost) digit of S_i. This is done in k steps. As first step the lowest digits of multiplicities of all occurrences of x_i in s are summed; this yields a number r that is logarithmic in the number of occurrences, and hence in the size of s. If k is 1, the procedure outputs the lowest digit of r; otherwise, it forgets the lowest digit of r, and proceeds to the next step. At step $j \leq k$, it adds to r the j-th lowermost digits of multiplicities of occurrences of x_i in s (this amounts to addition of two numbers of logarithmic length); if $j < k$, the lowest digit of the sum is forgotten, and j increased. Finally, when $j = k$, the rightmost digit of r is put out.

Let us spell out how to perform a comparison of two binary numbers $a = a_q a_{q-1} a_{q-2} \dots a_1$ and $b = b_r b_{r-1} b_{r-2} \dots b_1$. The numbers are streamed on demand: for any $i \geq 1$, the procedure can retrieve the digits a_i and b_i or conclude that one or both are undefined. To compare a and b, the procedure first determines the smallest i such that the i-th digit of one or both numbers are undefined. If only one of a and b has a valid digit at position i, that number is greater. If both numbers fail to have a valid digit at position i, the procedure (i) decrements i by one, and if $i > 0$, (ii) retrieves a_i and b_i (technically, this can be done by setting a separate counter to i and subtracting from it until 0 is reached). If a_i and b_i differ, the number with the greater value is greater; otherwise, (i) and (ii) are repeated. If i reaches 0, the numbers are equal.

□

Theorem 5 shows that, for $\mathrm{FL_{ew}}$-algebras, one can restrict one's attention to identities with one variable, of the form E_k, when considering subvarieties given by the semigroup fragment $\{\cdot\}$. We do not know whether the same restriction can be applied to CSG *quasi-identities*. In this regard, we remark that there are no nontrivial cancellative $\mathrm{FL_{ew}}$-algebras: if $xy \approx xz \to y \approx z$ is valid in $\mathcal{A}$, setting $x = 0$ gives $y \approx z$ as special case.

By [23], the quasi-equational theory CSGs is EXPSPACE-complete. This shows that it is distinct from the quasi-equational theory of the $\{\cdot\}$-reduct of any standard BL-algebra for which coNP-completeness of its universal theory is a known fact. Thus there are quasi-identities that hold in such standard BL-algebras but not in all CSGs.

5 The $\{\to\}$-fragment

This section studies the $\{\to\}$-fragment of Łukasiewicz, Gödel and product logics and their extensions, and of Hájek's basic logic BL. These logics are known to be coNP-complete; the same is shown for their implicational fragments.

The situation is familiar from classical propositional logic (CL). Note that within $\mathrm{FL_{ew}}$-extensions, $\mathrm{CL}^{\to}$ as an implicational fragment is unique to CL. Within this section, let us consider classical propositional logic in the language $\{\to, 0\}$.

Definition 7. *Let φ be a term and p be a new variable. Define $\varphi^\star$ by the following translation of propositional atoms:*

$$0 \mapsto p$$
$$x \mapsto (x \to p) \to p$$

where x is any variable in φ.

If φ is a $\{\to, 0\}$-term then $\varphi^\star$ is an implicational term. Clearly CL is poly-time reducible to $\mathrm{CL}^{\to}$: φ is a classical tautology iff $\varphi^\star$ is.

It is well known that in a BL-chain $\mathcal{A}$, the map $a \mapsto \neg\neg a$ is a homomorphism of $\mathcal{A}$ onto the first Wajsberg hoop in the ordinal sum of $\mathcal{A}$. If $0 < c < 1$ in A, then $[c, 1]$ with the operations modified as in Lemma 2, is again a BL-chain, even though $a \mapsto (a \to c) \to c$ is no longer a homomorphism. In the following, the weaker statement will be used.

Łukasiewicz logic Ł, like classical logic, can be presented in the language $\{\to, 0\}$, and this feature extends to all extensions. Recall the following result of Komori:

Theorem 8. *([22]) Let a logic* L *be an axiomatic extension of* Ł*. Then $\mathrm{L}^{\{\to\}} \neq \text{Ł}^{\{\to\}}$ iff* L *is given by a finite MV-chain. Moreover, for two finite MV-chains Ł_n and Ł_m, we have $\text{Ł}_n^{\{\to\}} \subset \text{Ł}_m^{\{\to\}}$ iff $m < n$.*

Theorem 9. *Let a logic* L *be an axiomatic extension of* Ł*. Then $\mathrm{L}^{\{\to\}}$ is coNP-complete.*

Proof. We rely on completeness theorems of Ł with respect to $[0,1]_{\text{Ł}}$ and of Ł_n with respect to the n-element MV-chain. Let $\mathcal{A}$ be the standard, or a finite, MV-chain. We show $\mathrm{TAUT}(\mathcal{A}) \preceq_{\mathrm{P}} \mathrm{TAUT}(\mathcal{A})^{\to}$, thereby proving coNP-hardness for all $\{\to\}$-fragments in view of Theorem 8. Containment in coNP follows from Observation 3.

Let $\varphi^\star$ be a translation of a term φ as in Definition 7. Claim: φ is a tautology of $\mathcal{A}$ iff $\varphi^\star$ is. Indeed, if φ is not true in $\mathcal{A}$ under some e, define e' by extending e with $e'(p) = 0$; this entails $e'(x) = e'((x \to p) \to p))$; then $e'(\varphi^\star) < 1$. On the other hand, assume $\varphi^\star$ is not true in $\mathcal{A}$ under some $e_{\mathcal{A}}$. If $e(p) = 0$, this yields an assignment that does not validate φ; $e(p) = 1$

contradicts the assumption. Hence assume $0 < e(p) < 1$. By Lemma 2 and subsequent remarks, the interval $[e(p), 1]$ in $\mathcal{A}$, endowed with $\to^{\mathcal{A}}$ and $e(p)$ as the interpretation of 0, forms an MV-chain: if $\mathcal{A}$ is the standard MV-chain, then $[e(p), 1]$ is isomorphic to it, and if $\mathcal{A}$ is $Ł_n$ for some n, then $[e(p), 1]$ is $Ł_m$ for some $m < n$. The values $e(p)$ and $(e(x) \to e(p)) \to e(p)$ for each variable x give an assignment in the new MV-chain that does not validate φ. Depending on what $\mathcal{A}$ is, it may be concluded either φ is not valid in $[0,1]_Ł$ or not valid in $Ł_m$, and hence not valid in $Ł_n$; i.e., φ is not a tautology of $\mathcal{A}$. □

For Gödel logic, the situation is even simpler. The structure of axiomatic extensions of G ordered by inclusion is well known:

$$\mathrm{CL} = \mathrm{G}_2 \supset \mathrm{G}_3 \supset \ldots \supset \mathrm{G}$$

It can be shown that the logics $\mathrm{G}_i^{\to}$ keep this structure.

Definition 10. *([24]) Let P_2 denote the* Peirce formula, $((p_2 \to p_1) \to p_2) \to p_2$*; let, for each $i \geq 3$, P_n denote*

$$((p_n \to P_{n-1}) \to p_n) \to p_n$$

for p_n a new variable.

Theorem 11. *([24])*

- *The formula P_n is valid in the finite Gödel chain G_i iff $i \leq n$.*
- *The formula P_n axiomatizes G_n within Gödel logic.*

Two variants of proving coNP-hardness for implicational fragments of Gödel logic G and its extensions G_n ($n \geq 3$) are presented below. The fact that G is complete w.r.t. the standard Gödel algebra $[0,1]_\mathrm{G}$, as is G_n w.r.t. the n-element Gödel chain, is employed.

Lemma 12. $\mathrm{CL}^{\{\to,0\}} \preceq_\mathrm{P} \mathrm{G}^{\to}$ *and* $\mathrm{CL}^{\{\to,0\}} \preceq_\mathrm{P} \mathrm{G}_n^{\to}$ *for $n \geq 3$.*

Proof. Let $\mathcal{A}$ be the standard, or a finite, Gödel chain. For $a, b \in \mathcal{A}$,

$$(a \to b) \to b = \begin{cases} b & \text{if } a \leq b \\ 1 & \text{otherwise.} \end{cases}$$

Claim: $\mathrm{TAUT}(\{0,1\}_\mathrm{B})^{\{\to,0\}} \preceq_\mathrm{P} \mathrm{TAUT}(\mathcal{A})^{\to}$; in particular, $\varphi \in \mathrm{TAUT}(\{0,1\}_\mathrm{B})^{\{\to,0\}}$ iff $\varphi^\star \in \mathrm{TAUT}(\mathcal{A})^{\to}$.

If φ is not a classical tautology under some e, then set $e'(p) = 0$ and $e'(x) = e(x)$ in $\mathcal{A}$; then e' does not satisfy $\varphi^\star$ in $\mathcal{A}$. On the other hand, if $\varphi^\star$ is not true in $\mathcal{A}$ under e', then assuming $e'(p) = 0$ yields a classical counterexample, $e'(p) = 1$ contradicts the assumption, and if $0 < e'(p) < 1$, then $[e'(p), 1]$ is a Gödel chain where $e'(p), e'((x \to p) \to p)$ yields a classical evaluation not satisfying φ, so φ is not a classical tautology. □

Lemma 13. $\mathrm{G} \preceq_\mathrm{P} \mathrm{G}^{\to}$ *and* $\mathrm{G}_n \preceq_\mathrm{P} \mathrm{G}_n^{\to}$ *for $n \geq 3$.*

Proof. The reduction employed in [29] to reduce the intuitionistic logic to its implicational fragment works here in a simpler form, as there is no need to consider $\vee$; the ground language of G and its extensions is $\wedge, \to, 0$.

For a given formula $\varphi(x_1, \ldots, x_n)$, consider the set Sub of its subformulas and for each subformula ψ introduce a new variable y_ψ. Let Def be the union of sets
$\{x_i \to y_{x_i}, y_{x_i} \to x_i\}$ for $i = 1, \ldots, n$;
$\{y_0 \to y_\psi\}$ for $\psi \in$ Sub (y_0 denoting the new variable for 0);
$\{y_\psi \to y_\chi, y_\psi \to y_\rho, y_\chi \to (y_\rho \to y_\psi)\}$ for $\psi = \chi \wedge \rho$ in Sub;
$\{y_\psi \to (y_\chi \to y_\rho), (y_\chi \to y_\rho) \to y_\psi\}$ for $\psi = \chi \to \rho$ in Sub.
Since Def is finite, one can write $\bigwedge$Def for the conjunction of its elements.

By an argument similar to those in preceding proofs, we get $\models_{[0,1]_{\mathrm{G}}} \varphi$ iff $\models_{[0,1]_{\mathrm{G}}} \bigwedge \mathrm{Def} \to y_\varphi$ iff $\bigwedge \mathrm{Def} \models_{[0,1]_{\mathrm{G}}} y_\varphi$, and analogously for the finite chains. To conclude, we note that $\bigwedge \mathrm{Def} \to y_\varphi$ can be equivalently written in pure implicational form, using residuation. □

Corollary 14. $\mathrm{G}^{\to}$ *and* $\mathrm{G}_n^{\to}$ *for each* $n \geq 3$ *are* coNP*-complete.*

Proof. Gödel logic and extensions are known to be coNP-complete, their implicational fragments inherit the complexity class. Hardness for the implicational fragment follows from either of Lemma 12 or Lemma 13. □

For product logic Π, the only consistent axiomatic extension is classical logic.

Theorem 15. $\Pi^{\to}$ *is* coNP*-complete.*

Proof. We show $\text{Ł}^{\to} \preceq_{\mathrm{P}} \Pi^{\to}$, relying on standard completeness for both logics.

Lemma: for $a, b \in [0,1]_\Pi$, we have

$$(a \to b) \to b = \begin{cases} 1 & \text{if } a > b = 0 \\ \max(a,b) & \text{otherwise.} \end{cases}$$

Proof: $a \leq b$ gives $a \to b = 1$ and $(a \to b) \to b = b = \max(a,b)$. If $a > b = 0$, then $a \to b = 0$ and $(a \to b) \to b = 1$. If $a > b > 0$, then $a \to b = b/a$; an assumption that $(a \to b) \leq b$, combined with weakening, gives $b = a \to b$, i.e., $b = b/a$, whence $a = 1 = \max(a,b)$; assuming $b < (a \to b)$, we get $(a \to b) \to b = b/(b/a) = a = \max(a,b)$. This concludes the lemma proof.

For each two (implicational) terms φ and ψ, denote by $M(\varphi,\psi)$ the (implicational) term $(\varphi \to \psi) \to \psi$. Define the reduction as assigning, to each term φ, a term $M((p \to q), \varphi^\circ)$, where φ° occurs from φ by replacing each occurrence of a variable x with a term $(x \to p) \to p$; the atoms p and q are new variables, not occurring in φ. Since φ is implicational, so is $M((p \to q), \varphi^\circ)$.

Recall that for any element $0 < c < 1$, the algebra $[c,1]$ in $[0,1]_\Pi$, with the cut product, residuum, and c as bottom, is isomorphic to $[0,1]_{\text{Ł}}$; in fact the $\to$-reduct of $[c,1]$ is isomorphic to the $\to$-reduct of $[0,1]_{\text{Ł}}$ (as $[c,1]$ is closed under product residuum). The isomorphism between $[0,1]_{\text{Ł}}$ and cut product is unique for each c (as $[0,1]_{\text{Ł}}$ has no nontrivial automorphisms); denote it g. Of course, $g(1) = 1$.

If φ is a tautology of $[0,1]_{\text{Ł}}$, let us show $M((p \to q), \varphi^\circ)$ is a tautology of $[0,1]_\Pi$. Consider assignments in $[0,1]_\Pi$; suppose e is such that $e(p) = 0$; then $e(p \to q) = 1$ no matter what

$e(q)$ is; this entails $e(M((p \to q), \varphi^\circ)) = 1$. Suppose $e(p) > 0$, then $e((x \to p) \to p) = \max(e(x), e(p)) \in [e(p), 1]$; therefore,

$$e(\varphi^\circ) = \varphi^{[0,1]_\Pi}(\max(e(x_1), e(p)), \dots, \max(e(x_n), e(p))) = \\ \varphi^{[0,1]_Ł}(g^{-1}(\max(e(x_1), e(p))), \dots, g^{-1}(\max(e(x_n), e(p))))$$

The latter is always 1 in $[0,1]_Ł$, so $M((p \to q), \varphi^\circ)$ holds in $[0,1]_\Pi$.

On the other hand, if φ is not valid in $[0,1]_Ł$ under some assignment e, then one easily constructs an assignment e' in $[0,1]_\Pi$ such that $e'(M((p \to q), \varphi^\circ))$ is not true, by setting $e'(p) > 0$, $e'(q) < e'(p)$, and $e'(x) = g(e(x))$; note that $e'((x \to p) \to p) = \max(e'(x), e'(p)) = e'(x)$ (since $e'(p) > 0$). □

The following theorem addresses the implicational fragment of BL. By [13], $\mathrm{BL}^{\to}$ coincides with $\mathrm{SBL}^{\to}$.

Theorem 16. $\mathrm{BL}^{\to}$ *is* coNP-*complete.*

Proof. We consider the algebra $\omega Ł$, the infinite ordinal sum ordered by ω of standard MV-components, as an algebra such that $\mathrm{TAUT}(\omega Ł) = \mathrm{BL}$ (this was shown in [2]). Consequently, $\mathrm{BL}^{\to} = \mathrm{TAUT}(\omega Ł)^{\to}$.

We show $\mathrm{TAUT}([0,1]_Ł) \preceq_{\mathrm{P}} \mathrm{TAUT}(\omega Ł)^{\to}$, using the translation $\varphi^\star$ as in Definition 7.

Consider $c \in \omega Ł$; since this is a sum of Wajsberg hoops whose only common element is 1, either $c = 1$ or c belongs to a unique component. For the latter case,

$$(x \to c) \to c = \begin{cases} c & \text{if } x \leq c \\ x & \text{if } x > c \text{ and } x \text{ is in the same Wajsberg component as } c \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, if d is the smallest idempotent such that $c < d$, then $[c, d) \cup 1$, endowed with $\to$ (as in $\omega Ł$) and c interpreting 0 is an MV-chain, isomorphic to the standard MV-chain.

If φ is not a tautology of $[0,1]_Ł$ under some e in $[0,1]_Ł$, then define e' in $\omega Ł$ by setting $e'(p) = 0$ and $e'(x) = f(e(x))$ for f being an isomorphism of $[0,1]_Ł$ onto the first Wajsberg component in the sum. Clearly $e'(\varphi) < 1$.

If $\varphi^\star$ is not a tautology of $\omega Ł$, fix e' such that $e'(\varphi) < 1$. This entails $e'(p) < 1$; then, by the above, $[e(p), d) \cup 1$ (for d the least idempotent strictly above $e(p)$) is an isomorphic copy of the standard MV-algebra, and moreover, $e'((x \to p) \to p) \in [e'(p), d) \cup 1$. It follows that φ is not a tautology of $[0,1]_Ł$. □

6 Remarks on other language fragments

Fuzzy logics (MTL and extensions) are *distributive*, i.e., the lattice reduct of each MTL-algebra is a distributive lattice. Therefore, any lattice identity that is true in all distributive lattices (or equivalently, in $\{0,1\}_{\mathrm{B}}$) is true in any MTL-algebra. The converse also holds: any lattice identity true in a nontrivial MTL-algebra holds in all distributive lattices, since the variety of distributive lattices has no proper subvarieties beside the trivial one. The problem of deciding the validity of any lattice identity in a nontrivial MTL-algebra is thus coNP-complete by [6].

The $\{\cdot, \vee\}$-fragment of $\mathrm{FL_{ew}}$ has not been studied in this paper; in particular, we do not know the structure of the lattice of $\{\cdot, \vee\}$-generated subvarieties of the $\mathrm{FL_{ew}}$-variety. (In [4], axiomatization by $\{\cdot, \vee\}$-canonical formulas is given for all extensions of k-potent $\mathrm{FL_{ew}}$, for each k.) We remark that validity for $\{\cdot, \vee\}$-identities is coNP-hard in any nontrivial $\mathrm{FL_{ew}}$-algebra, by the results of [6].

Glivenko-like theorems provide lower bounds for $\{\to, 0\}$-fragments of some logics. Two logics K and L in the same language $\mathcal{L}$ are *Glivenko equivalent* iff for each $\mathcal{L}$-term φ we have $\neg\varphi \in \mathrm{K}$ iff $\neg\varphi \in \mathrm{L}$. It is not difficult to show that if K is Glivenko equivalent to L, $\mathrm{K} \subseteq \mathrm{L}$, and L is involutive, then $\varphi \in \mathrm{L}$ iff $\neg\neg\varphi \in \mathrm{K}$ for each φ. Logics that allow for this reduction (in the role of L and K respectively) include classical logic and intuitionistic logic (or weakly contractive logic), or Łukasiewicz logic and Hájek's basic logic BL ([8]); thus (as is well known), one can read classical tautologies from intuitionistic (or weakly contractive) ones, and one can read MV-tautologies from BL-tautologies. Moreover, since only $\to$ and 0 are needed for the reduction, the same holds for the $\{\to, 0\}$ fragments of the logics involved. Unfortunately, this is not applicable to MTL and its involutive extension IMTL, because these two logics are not Glivenko equivalent (cf. [9]).

7 Concluding remarks

This paper has studied the (equational theories of) the semigroup and the implicational fragment of some propositional fuzzy logics, predominantly extensions of BL, from a computational perspective. A basic complexity classification is an interesting but secondary pursuit; the main target is to gain more information about the logics in relation to their fragments. The results given here confirm our intuitions about these fragments:

- the semigroup reducts can generate a chain of subvarieties of CSGs; due to strong additional characteristics, such as the existence of a neutral element, a partial order, and monotonicity of powers, the subvariety structure is much simpler than the full CSG subvariety lattice. Valid identities in any such subvariety of CSG are easy to recognize.
- the implicational fragments of BL and some of its extensions are polynomially equivalent to the full logic.

One should adopt a humble perspective in interpreting these results. Glivenko theorem [16] and other negative translations of classical to intuitionistic logic are sometimes interpreted as saying that "classical logic is *contained in/a fragment of* intuitionistic logic"; this is true insofar as consistency strength or existence of decision procedures are concerned, but it is not the case that one loses nothing by working in intuitionistic rather than classical logic. The reductions provided here show, in exactly the same sense, that Lukasiewicz logic (or classical logic for that matter) is "contained in" its implicational fragment; the translation provided by Definition 7 is easily invertible, so from any formula in the range of $\star$, one can retrieve a unique source formula. Moreover, the range of $\star$ is itself a syntactic fragment of (implicational fragment of) Tm. Still, much was arguably lost in accomplishing such a translation; among other things, one starts with the language $\{\to, 0\}$, which may already seem awkward in many contexts. Moreover, at present there seem to be no outstanding decision procedures tailored to the implicational fragment in particular, whose features would be preserved by the reduction. On the other hand, $\text{Ł} \preceq_{\mathrm{P}} \text{Ł}^{\to}$ as shown in the proof of Theorem 9, and $\text{Ł}^{\to} \preceq_{\mathrm{P}} \Pi^{\to}$ by the proof of Theorem 15; this shows that a reduction between Ł and Π is due to a reduction between respective implicational fragments, which supports the perspective of their key role for the logic.

Many questions are left open by this paper. Some questions are implicit to the material presented here; concerning the semigroup fragment, it remains to be determined whether each quasi-identity is equivalent to a quasi-identity in one variable, and the subquasivariety structure of semigroup reducts of $\mathrm{FL}_{\mathrm{ew}}$ remains to be determined, and the complexity of the corresponding theories. For the implicational fragment, an immediate question concerns the structure of the lattice of implicational fragments (as logics), e.g., which logics share their implicational fragment, and which logics are reducible to their implicational fragment. In particular, we would like to know this for MTL, whose complexity is a long-standing open problem within propositional fuzzy logics.

On a broader scale, language fragments form only a small area of syntactic fragments in general (see Section 3 for a discussion). Finite-variable fragments, or fragments of the universal algebraic theory, are further noteworthy theories.

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