I agree that this work be lent for study purposes.

Souhlasím s tím, aby tato práce byla půjčována ke studijním účelům.

Zuzana Honzíková
Some contributions to axiomatizations of fuzzy propositional calculi

Zuzana Honzíková

April 1999

supervisor:
prof. RNDr. Petr Hájek, DrSc.
Institute of Computer Science
Academy of Sciences of the Czech Republic
I hereby declare
that the contents of this volume
are the output of my own brain
unless stated otherwise

Likewise
that all sources used
are acknowledged
in the Appendix
to this volume

I would like to express my gratitude to prof. Petr Hájek for his advice, friendly approach, patience and encouragement. He spent many hours of his precious time with me, discussing, explaining again and again, correcting errors (including typographical errors) within the manuscript, and never tiring of the topic. Besides, my sincere thanks go to all the people who have been affected, one way or another, with the fact of my working on this thesis: to my family, friends, teachers and fellow students who walked on tiptoe around me and did all the work I was supposed to do; who listened to lectures; who assisted in printing this work. Last but not least,

Deo GRATIAS.
That things be black and white, and nought between,
No mortal thinks. That it is either Day,
Or Night; or Fire, or Frost; our sense is keen
Enough to wipe these boundaries away.
Great Poets shed their blood and strain their voice
To count the many hues 'twixt Dusk and Dawn,
And Words, those self-willed slaves, rejoice
To smooth all edges; thence new words are born.
Aye, Tongue doth allow Blacks to seem all Whites;
Base Vices for the heights of Virtues reach
And by a claim most Wrong assume their Rights:
For all is possible, within our Speech.
     And yet each thing or Black or White must be;
     But which is which is not for us to see.

“It seems very pretty”, she said when she had finished it, “but it’s rather hard to understand!” (You see she didn’t like to confess, even to herself, that she couldn’t make it out at all.) “Somehow it seems to fill my head with ideas—only I don’t exactly know what they are! However, somebody killed something; that’s clear, at any rate—”

Lewis Carroll, *Through the Looking Glass*
# Table of contents

1. **Introduction** ................................................................. 6  
   Notation and conventions ............................................. 7
2. **A Reader’s Digest of Fuzzy PC** ..................................... 8  
   2.1. t-norms and their residua ........................................ 8  
   2.2. Basic logic and its extensions ................................. 9  
   2.3. BL-algebras and completeness theorems .................... 11  
   2.4. Standard completeness ............................................ 12
3. **L⊕Π-norms** .................................................................. 13  
   3.1. L⊕Π-norms .............................................................. 13  
   3.2. Axioms of L⊕Π ....................................................... 13  
   3.3. L⊕Π-algebras ........................................................ 15  
   3.4. Standard completeness for L⊕Π ............................... 16  
   3.5. Reducing the axioms of L⊕Π .................................. 18
4. **Axiomatizations of Standard Algebras** ............................... 21  
   4.1. Isomorphisms of standard algebras .......................... 21  
   4.2. Intersection logics .................................................. 23  
   4.3. Representation of t-norms ..................................... 25  
   4.4. Shrinking intervals ................................................. 28  
   4.5. The hierarchy ........................................................ 30  
   4.6. Completeness results for BL ................................ 32  
   4.7. Other completeness results ................................... 34
**Appendix** ........................................................................ 36  
   Unsolved problems ...................................................... 36  
   References ................................................................. 36
1. Introduction

Fuzzy logic has been of late, and still is perhaps, quite a hit of the season. The scientists’ folklore tells us that each year has its “key words” which should be included in all grant proposals, if the grant is to be obtained. Apparently the term ‘fuzzy logic’ is such a key, which opened the treasury to many; maybe, when annals of mathematical logic are written, this transitory period will be denoted “fuzzy years”.

The fuss about fuzziness provided it with an air of something revolutionary, possibly because the halo reaches far beyond the scope of logic itself. It is very likely that the attention which fuzziness receives now springs from its manifold extra-logical applications, rather than from bringing any revolution into logic itself.

I regard fuzzy logic as the natural next step in the evolution of many-valued logics, maybe the ultimate step; and I think its real value for a logician is not in its being extraordinary, but in its being ordinary—that is, a member of the family of calculi of mathematical logic; simply because the results of one branch of the system are never of more interest than when compared to the achievements in other branches. Fuzziness was subdued to the laws of a logical system by virtue of the strict treatment applied to it in (doubtless many) publications now available, of which I relied fully on [6]. Fuzzy logic as presented is a thoroughly developed logical system, which may, e.g., be taken as a logical foundation of a proper mathematical theory (e.g., set theory).

Though the historical and philosophical aspects of fuzzy logic (as a branch of many-valued logic) are charming, they will not be touched here. The topic is treated in a purely synchronous manner, and the scope of this work is very narrow; its aim is to shift forward investigation in a certain area of fuzzy PC\(^1\). Nor is this work exhaustive in the mathematical aspects of the matter (as the reader will soon find out).

I have decided to stick to briefness in the review of what is already known on the subject. A systematic treatment of fuzzy PC, as well as numerous references, will be found in [6]; the second chapter of this thesis, despite its bold title, is but a list of definitions and theorems that should provide a necessary background for my own contributions, but cannot be taken as a substitute for a textbook on fuzzy PC. My own work follows the approach of [6], which has been used throughout. Besides, I have been provided with several recent papers dealing with related topics. A list of references can be found at the end of this volume.

This work investigates fuzzy propositional calculi given by continuous t-norms\(^2\). Each continuous t-norm determines uniquely a structure (on the real interval \([0, 1]\)) for the language of fuzzy PC; these structures are called standard algebras. By the representation theorem for continuous t-norms, uncountably many pairwise non-isomorphic standard algebras exist.

Fuzzy PC is axiomatized by a set of propositional formulas, commonly referred to as basic logic (BL); these formulas are tautologies of all standard algebras. It has long been an open problem whether BL is complete with respect to the standard algebras.

\(^1\) propositional calculus
\(^2\) see chapter two for definition
The completeness result has only appeared recently (see [1]); the paper contains some additional results related to this work.

In [6], three outstanding schematic extensions of BL have been investigated: Łukasiewicz’s logic, Gödel’s logic, and product logic; each corresponds to a particular standard algebra, given by Łukasiewicz’s, Gödel’s, and product t-norm respectively. Completeness has been proved for each of these systems. The aim of this thesis is to study logics given by other continuous t-norms. Prior to the commencement of this work, no results upon this subject were known to the author.

As a warming-up exercise, a particular example of a standard algebra is investigated very minutely in chapter three. Its axioms are assembled and proved complete. For that purpose, the language of fuzzy PC had been enriched with an additional truth constant, which eased the task considerably.

In chapter four, a hierarchy is established on the sets of tautologies of standard algebras, and some of its consequences are derived. We include remarks on intersection logics (investigated also in [1]; borrowings from that source are explicitly stated). Among others, we show that BL is the logic of a particular standard algebra, and give some hints to and examples of axiomatizations of standard algebras.

Unless stated otherwise, the results of chapters three and four are original.

The appendix gives a couple of open problems and questions, possibly as hints for future work in this area. It also provides a list of references used.

**Notation and Conventions**

Closed intervals (mostly of reals) are denoted by use of square brackets (like [0, 1]); open intervals by round brackets (like (0, 1)).

By ‘formula’ we mean a propositional formula in the corresponding language (in chapter three, we use an additional constant). Formulas are always denoted with Greek letters.

Precedence of connectives: negation binds stronger than any binary connective. Besides, we sometimes omit parentheses around conjunctions, for readability’s sake. \( \varphi \land \psi \rightarrow \chi \) should be parsed as \((\varphi \land \psi) \rightarrow \chi\).

We say at times that ‘intervals are isomorphic’, which should be read as ‘isomorphic w.r.t. all operations defined’; or that ‘t-norms are isomorphic on \([x, y]\)’, which in turn means that the underlying interval(s) are isomorphic w.r.t. the t-norms; we refrain from such abbreviations where confusion could arise.
2. A Reader’s Digest of Fuzzy PC

This chapter gives a brief overview of basic notions and statements of fuzzy propositional calculus, following the approach of [6]. It gives no explanations (e.g., to the choice of truth functions for propositional connectives), no proofs, and omits everything that can possibly be omitted. Hence it may be skipped by anyone who is familiar with fuzzy PC, or may be used only for reference, which is indeed its main purpose.

2.1. t-norms and their residua

In this chapter we define truth functions for all propositional connectives of fuzzy PC, namely $&, \rightarrow, \neg, \land, \lor$, and $\equiv$. We start with the strong conjunction $&$; the choice of the truth function for $&$ determines the whole calculus uniquely.

2.1.1 Definition. A \textit{t-norm} is a binary operation $*$ on $[0, 1]$, satisfying these conditions:

(i) $*$ is commutative and associative
(ii) $*$ is non-decreasing in both arguments
(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0, 1]$.

We shall only be interested in \textit{continuous} t-norms (i.e., continuous mappings of $[0, 1]$ onto $[0, 1]$) as possible truth functions for $&$.

For each continuous t-norm $*$ there is a unique operation $\Rightarrow$, defined as $x \Rightarrow y = \max\{z; x * z \leq y\}$; this operation is called the \textit{residuum} of the t-norm $*$, and is used as the truth function for the implication $\rightarrow$. Note that by the above definition, $x \Rightarrow y = 1$ iff $x \leq y$.

There are three outstanding examples of t-norms (their importance is justified by a theorem included further on): Lukasiewicz’s t-norm $*_{L}$, Gödel’s t-norm $*_{G}$, and product t-norm $*_{\Pi}$; their definitions, including the respective residua for $x > y$, are listed below:

\[ L : \quad x * y = \max(0, x + y - 1) \quad x \Rightarrow y = 1 - x + y \]
\[ G : \quad x * y = \min(x, y) \quad x \Rightarrow y = y \]
\[ \Pi : \quad x * y = x \cdot y \quad x \Rightarrow y = \frac{y}{x} \]

A continuous t-norm $*$ determines an algebra on $[0, 1]$, with operations $*$ and $\Rightarrow$, and possibly with other operations definable by open formulas, esp. \textit{precomplement} (the truth function of negation $\neg$), defined as $-x = x \Rightarrow 0$, and the operations $\min(x, y)$ and $\max(x, y)$ (truth functions of $\land$ and $\lor$).

Structures (on $[0, 1]$) given by continuous t-norms are referred to as \textit{standard al-gebras} for fuzzy PC. Standard algebras will be denoted $[0, 1]_{*}$, where $*$ stands for a particular t-norm. A standard algebra $[0, 1]_{*}$ is thus the structure $([0, 1], 0, 1, *, \Rightarrow)$ (1 is definable).

\[ 1 \] Wherever t-norms appear in this work without the attribute ‘continuous’, we always mean continuous t-norms.
Now we introduce a theorem characterizing all continuous t-norms, and explaining the importance of the three examples of t-norms above. Let $*$ be a continuous t-norm. An element $x \in [0,1]$ is idempotent (w.r.t. $*$) iff $x * x = x$. The set of all idempotents of $*$ is a closed subset of $[0,1]$. Its complement is a union of (countably many) non-overlapping open intervals; denote this set of intervals $I_o$. Let $I$ be the set of closures of intervals in $I_o$.

2.1.2 Theorem.

(i) For each $[a,b] \in I$, $* | [a,b]$ is isomorphic either to the product t-norm (on $[0,1]$) or to Lukasiewicz t-norm (on $[0,1]$).

(ii) If for $x, y \in [0,1]$ there is no $[a,b] \in I$ such that $x, y \in [a,b]$, then $x * y = \min(x, y)$.

Implications of this theorem are used throughout this work; basic considerations will be found in 4.1.

To close this section, I include a handful of useful statements on the behaviour of operations of $[0,1]_*$.

2.1.3 Lemma. For $x, u, y \in [0,1]$, for any t-norm $*$:

(i) if $y \leq x$, then $\exists z \in [0,1] (y = z * x)$

(ii) $1 \Rightarrow y = y$

2.2. Basic logic and its extensions

The language and syntax of fuzzy PC are almost the same as in classical propositional logic: precisely speaking, the language has in addition a truth constant $0$ (which is not definable here) and the connective $\&$ (or $\land$; this just says that there are two distinct conjunctions). Formulas are defined as usual. Propositional connectives of fuzzy PC are defined from $\&$, $\to$, and $0$ as follows:

$\varphi \land \psi$ is $\varphi \& (\varphi \to \psi)$

$\varphi \lor \psi$ is $((\varphi \to \psi) \to \psi) \land (\psi \to \varphi)$

$\neg \varphi$ is $\varphi \to \overline{0}$

$\varphi \equiv \psi$ is $(\varphi \to \psi) \& (\psi \to \varphi)$

Fix a continuous t-norm $*$; this determines uniquely a standard algebra $[0,1]_*$ and the corresponding propositional calculus PC($*$), where evaluations of propositional variables extend to formulas as follows:

$e(\overline{0}) = 0$

$e(\varphi \& \psi) = e(\varphi) * e(\psi)$

$e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$

The operations $\min(x, y)$, $\max(x, y)$, and $x \Rightarrow 0$ are used as shortcuts for evaluating $\land$, $\lor$, and $\neg$, respectively.

A formula $\varphi$ is a 1-tautology of a standard algebra $[0,1]_*$ iff it evaluates to 1 under any evaluation in $[0,1]_*$. $\varphi$ is a $t$-tautology iff it is a 1-tautology of $[0,1]_*$ for any
continuous t-norm \(\ast\). Note that formulas translate to terms, i.e., for each formula \(\varphi\) there is a term \(\tau\) in the language of standard algebras, such that for every standard algebra \([0, 1]_\ast\), \([0, 1]_\ast\models (\tau = 1) \iff [0, 1]_\ast\models \varphi\).

Now we define the basic logic; its axioms are t-tautologies.

**2.2.1 Definition.** The following formulas are the axioms of basic logic (denoted BL).

\[\begin{align*}
\text{(A1)} & \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(A2)} & \quad (\varphi \& \psi) \to \varphi \\
\text{(A3)} & \quad (\varphi \& \psi) \to (\psi \& \varphi) \\
\text{(A4)} & \quad (\varphi \& (\psi \& \chi)) \to ((\varphi \& \psi) \& \chi) \\
\text{(A5a)} & \quad (\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi) \\
\text{(A5b)} & \quad ((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \\
\text{(A6)} & \quad ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \\
\text{(A7)} & \quad \overline{0} \to \varphi
\end{align*}\]

The inference rule of BL is modus ponens. Proofs are defined as usual.

For example and for convenience, we list formulas provable in BL that will be used further in this work; for proofs, as well as for more examples of BL-provable formulas, see [6].

**2.2.2 Lemma.** BL proves these formulas:

\[\begin{align*}
\text{(i)} & \quad \varphi \to (\psi \to \varphi) \\
\text{(ii)} & \quad (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \\
\text{(iii)} & \quad \varphi \to (\psi \to (\varphi \& \psi)) \\
\text{(iv)} & \quad (\varphi \& (\varphi \to \psi)) \to \psi \\
\text{(v)} & \quad ((\alpha \to \beta) \& (\gamma \to \delta)) \to ((\alpha \& \gamma) \to (\beta \& \delta)) \\
\text{(vi)} & \quad \varphi \& \psi \to \varphi, \varphi \& \psi \to \psi, \varphi \& \psi \to \psi \& \psi \\
\text{(vii)} & \quad ((\varphi \to \psi) \& (\varphi \to \chi)) \to (\varphi \to (\psi \& \chi)) \\
\text{(viii)} & \quad \varphi \to (\varphi \& \psi), \psi \to (\varphi \lor \psi), (\varphi \& \psi) \to (\psi \lor \varphi) \\
\text{(ix)} & \quad (\varphi \& (\psi \lor \chi)) \equiv ((\varphi \& \psi) \lor (\varphi \& \chi))
\end{align*}\]

BL may be extended by additional axioms (or, schemas of axioms); below are listed those that yield the calculi given by Łukasiewicz’s, Gödel’s, and product t-norm, respectively:

- **Łukasiewicz’s logic:** \(\neg \neg \varphi \to \varphi\) (L)
- **Gödel’s logic:** \(\varphi \to (\varphi \& \varphi)\) (G)
- **Product logic:** \(\neg \neg \chi \to [(\varphi \& \chi) \to (\psi \& \chi)] \to (\varphi \to \psi)\) (I1)
- and \(\varphi \& \neg \varphi \to \overline{0}\) (I2)
2.3. BL-algebras and completeness theorems

In this section we are going to define the algebraic structures that serve as models of BL (and its extensions). Standard algebras are a special subclass of these.

2.3.1 Definition. A residuated lattice is an algebra \( L = (L, \cup, \cap, *, \Rightarrow, 0, 1) \) with four binary operations and two constants such that

(i) \( (L, \cup, \cap, 0, 1) \) is a lattice with the greatest element \( 1 \) and the least element \( 0 \) (with respect to the lattice ordering \( \leq \))

(ii) \( (L, *, 1) \) is a commutative semigroup with the unit element \( 1 \), i.e., \( * \) is commutative, associative, \( 1 * x = x \) for all \( x \)

(iii) \( * \) and \( \Rightarrow \) form an adjoint pair, i.e., \( z \leq (x \Rightarrow y) \) iff \( x * z \leq y \) for all \( x, y, z \).

A residuated lattice is a BL-algebra iff the following three identities hold for all \( x, y \in L \):

(i) \( x \cap y = x * (x \Rightarrow y) \)

(ii) \( (x \Rightarrow y) \cup (y \Rightarrow x) = 1 \)

A BL-algebra is linearly ordered iff its lattice ordering is linear, i.e., for any \( x, y \in L \), \( x \cap y = x \).

Note that the class of BL-algebras is a variety of algebras.

2.3.2 Lemma. Let \( L = (L, \cup, \cap, *, \Rightarrow, 0, 1) \) be a BL-algebra and \( x, y, u \in L \). Then

(i) if \( x \leq u \leq y \) and \( u \) is idempotent then \( x * y = x \)

(ii) if \( x < u \leq y \) and \( u \) is idempotent then \( y \Rightarrow x = x \)

For proof, see [4].

Let \( L \) be a BL-algebra. An \( L \)-evaluation of propositional variables is a mapping \( e \), assigning to each propositional variable an element of \( L \); evaluation of formulas is obtained using \( 0, *, \Rightarrow \) (in \( L \)) to evaluate \( \overline{0}, \& \), and \( \rightarrow \). A formula \( \varphi \) is an \( L \)-tautology iff \( e(\varphi) = 1 \) for all \( L \)-evaluations \( e \).

Let \( C \) be a schematic extension of BL (i.e., a set of propositional formulas including the axioms of BL). A BL-algebra \( L \) is a C-lattice iff any \( \varphi \in C \) is a \( L \)-tautology, i.e., \( e(\varphi) = 1 \) for all \( L \)-evaluations \( e \).

2.3.3 Theorem. BL is sound w.r.t. BL-algebras, i.e., if \( \varphi \) is provable in BL, it is an \( L \)-tautology for each BL-algebra. Generally, if a schematic extension \( C \) proves \( \varphi \), then \( \varphi \) is an \( L \)-tautology for each \( C \)-lattice \( L \).

2.3.4 Theorem (Completeness). BL is complete; the following three conditions are equivalent:

(i) \( \varphi \) is provable in BL

(ii) for each BL-algebra \( L \), \( \varphi \) is an \( L \)-tautology

(iii) for each linearly ordered BL-algebra \( L \), \( \varphi \) is an \( L \)-tautology.

The proof of this statement also shows that a schematic extension \( C \) of BL proves \( \varphi \) iff \( \varphi \) is an \( L \)-tautology for each \( C \)-lattice \( L \) iff \( \varphi \) is an \( L \)-tautology for each linearly ordered \( C \)-lattice \( L \).
Note that this yields completeness for the three calculi $L$, $G$, and $\Pi$. In particular, $BL$-algebras $L$ for which $\neg\neg\varphi \rightarrow \varphi$ is an $L$-tautology are called $MV$-algebras ($MV$ standing for “many-valued”), and the completeness theorem states that Lukasiewicz’s logic is complete w.r.t. $MV$-tautologies. $BL$-algebras satisfying the axiom $G$ are called regular Heyting algebras; $BL$-algebras satisfying the two axioms of $\Pi$ are called $\Pi$-algebras.

2.4. Standard completeness

The above completeness results are analogous to the completeness theorem of classical logic, namely, that a formula is provable iff it is valid in all models. In this section we mention completeness results concerning special subclasses of models, namely, (various classes of) standard algebras.

It has long been an open problem whether $BL$ is complete w.r.t. the class of standard algebras. In [4], this problem is reduced to the question of whether two additional axioms $B_1$ and $B_2$ are redundant (provable in $BL$) or not. An affirmative answer to this question was recently presented in [1].

2.4.1 Theorem. $BL$ is complete w.r.t. standard algebras; i.e., a formula $\varphi$ is a theorem of $BL$ iff it is a 1-tautology of any $[0,1]_*$, $*$ being a continuous t-norm.

Besides, $L$, $G$, and $\Pi$ are each of them complete w.r.t. the corresponding standard algebra $[0,1]_L$, $[0,1]_G$, and $[0,1]_\Pi$.

2.4.2 Theorem. Each of the three extensions of $BL$ ($L$, $G$, $\Pi$) is complete with respect to its standard algebra.

The standard completeness proofs are carried out in slightly different fashion in each case. In all three cases, they rely heavily on the following:

2.4.3 Definition. A linearly ordered Abelian group $G$ is partially embeddable into $\mathbb{R}$ iff for each finite $X \subseteq G$ there is a finite $Y \subseteq \mathbb{R}$ and a 1-1 mapping $f$ of $X$ onto $Y$ which is a partial isomorphism (with respect to the addition operation and the ordering).

2.4.4 Theorem. Each linearly ordered Abelian group is partially embeddable into $\mathbb{R}$. Moreover, if $G$ is a l.o.A.g. with additional operations definable by open formulas (from $+$ and $\leq$), then a partial isomorphism may be found in such a way that it preserves all the operations.

How is this going to help in the standard completeness proofs? To prove that, say, Lukasiewicz logic is complete with respect to its standard algebra, we associate a linearly ordered Abelian group to every $MV$-algebra; we define the operations of the $MV$-algebra from the operation and the ordering of the group (let’s call this “translation”) in such a way that any formula in the language of $MV$-algebras is valid in the $MV$-algebra iff its translation is valid in the group. Thus if a formula $\varphi$ is not valid in an $MV$-algebra $A$, its translation $\varphi'$ is not valid in the associated Abelian group either, and, by the above theorem, $\varphi'$ is not valid in the standard algebra (with its operations defined by open formulas in the additive group of reals).
This chapter investigates a particular example of a continuous t-norm. To do this, a new truth constant has been added to the language. The same t-norm is further mentioned in chapter four, where some hints concerning axiomatization without the constant are investigated.

3.1. L⊕Π-norms

Informally, an L⊕Π-norm is obtained by sticking together Lukasiewicz’s t-norm and product t-norm on [0, 1], in the sense of the representation theorem for continuous t-norms, i.e., delimited with one idempotent element named h (standing for ‘half’). We require that h ≠ 0 and h ≠ 1; as stated below, it does not matter which element plays the role of h, as long as it is neither 0 nor 1.

3.1.1 Definition. A continuous t-norm *LΠ is an L⊕Π-norm iff there is an element h ∈ (0, 1) such that h *LΠ h = h, and bijective mappings f and g,

\[ f : [0, 1] \rightarrow [0, h], \quad g : [0, 1] \rightarrow [h, 1], \]

such that for a, b ∈ [0, 1],

\[ f(a *L b) = f(a) *LΠ f(b), \quad g(a *Π b) = g(a) *LΠ g(b) \]

3.1.2 Lemma. The mapping f from the previous definition is increasing and continuous. Moreover, if 0 ≤ x < y ≤ h, then also f(y ⇒ x) = f(y) ⇒ f(x). Similarly for g.

For proof see 4.1.4.

3.1.3 Theorem. Let *₁, *₂ be two L⊕Π-norms. Then

(i) *₁ and *₂ are isomorphic on [0, 1], i.e., there is a bijective mapping f such that ∀x, y ∈ [0, 1], f(x *₁ y) = f(x) *₂ f(y)

(ii) the two standard algebras [0, 1] *₁ and [0, 1] *₂ are isomorphic likewise (via f).

Proof. (i) is obvious; for proof of (ii), see 4.1.4. QED

3.2. Axioms of L⊕Π

In this section we spell out, as closely as possible, the particulars of the L⊕Π-norm by means of propositional formulas. These formulas should guarantee, in particular, the existence of a non-extremal idempotent, should specify that * on the “bottom” behaves like Lukasiewicz’s t-norm, and on the “top” like product t-norm. In the first attempt, we include all promising formulas, disregarding independence claims. Refinements will follow in section 3.5.
Let us introduce a new truth constant $\bar{h}$, and add an axiom

$$\bar{h} \& \bar{h} \equiv \bar{h}$$

(Id)

This states the existence of an idempotent element $h$. Adding another axiom $\neg \neg \bar{h}$ ascertains that $h \neq 0$. Whenever $h$ is idempotent (as secured by (Id) in this case), $e(\neg \neg \bar{h}) = 1$ iff $e(h) \neq 0$ for any t-norm $*$ and any evaluation $e$. There is however, as shown in 3.3.2, no formula expressing $h \neq 1$.

The rest of intended axioms will be “translated” axioms of Lukasiewicz and product logics. We introduce two functions $\flat$ and $\sharp$, operating on formulas, defined as follows:

\[
\begin{align*}
0^\flat &= 0 \\
1^\flat &= \bar{h} \\
p^\flat &= p \land \bar{h} \\
(\varphi \land \psi)^\flat &= \varphi^\flat \land \psi^\flat \\
(\varphi \rightarrow \psi)^\flat &= (\varphi^\flat \rightarrow \psi^\flat) \land \bar{h}
\end{align*}
\]

\[
\begin{align*}
0^\sharp &= h \\
1^\sharp &= 1 \\
p^\sharp &= p \lor h \\
(\varphi \land \psi)^\sharp &= \varphi^\sharp \land \psi^\sharp \\
(\varphi \rightarrow \psi)^\sharp &= \varphi^\sharp \rightarrow \psi^\sharp
\end{align*}
\]

3.2.1 Theorem. If $\flat$ and $\sharp$ are as above, $\varphi$ and $\psi$ are formulas, then:

(i) $\varphi \equiv \psi$ is a tautology of $[0, 1]_L$ iff $\varphi^\flat \equiv \psi^\flat$ is a tautology of $[0, 1]_{L\Pi}$

(ii) $\varphi \equiv \psi$ is a tautology of $[0, 1]_{L\Pi}$ iff $\varphi^\sharp \equiv \psi^\sharp$ is a tautology of $[0, 1]_L$

Proof. In $[0, 1]_{L\Pi}$, define new operations $1^h = h$ and $a \Rightarrow^h b = \min\{a \Rightarrow b, h\}$. Then $[0, h]$ is closed with respect to $*_{L\Pi}$ and $\Rightarrow^h$, and the structure

$$L^\flat = ([0, h], 0^\flat, 1^\flat, *_{L\Pi}, \Rightarrow^h)$$

is isomorphic to $[0, 1]_L$ using lemma 3.1.2.

Take a formula $\varphi(p_1, \ldots, p_n)$, and an evaluation $e$ in $[0, 1]_{L\Pi}$; suppose $e(p_i) = a_i$. By induction on $\varphi$ one proves that $\varphi^\flat(a_1, \ldots, a_n)$ evaluated in $[0, 1]_{L\Pi}$ yields the same element as $\varphi(\min(a_1, h), \ldots, \min(a_n, h))$ in $L^\flat$.

Therefore, for any $\varphi(p_1, \ldots, p_n)$ and $\psi(p_1, \ldots, p_n)$: $\varphi \equiv \psi$ is a tautology of $[0, 1]_L$ iff it is a tautology of $L^\flat$ if $\varphi^\flat \equiv \psi^\flat$ is a tautology of $[0, 1]_{L\Pi}$.

The case of $\sharp$ is analogous; define $0^h = h$, then the structure

$$L^\sharp = ([h, 1], 0^h, 1^h, *_{L\Pi}, \Rightarrow^h)$$

is isomorphic to $[0, 1]_L$. \hfill QED

In particular, if $\varphi$ is a tautology of $[0, 1]_L$, then $\varphi^\flat \equiv \bar{h}$ is a tautology of $[0, 1]_{L\Pi}$. By applying $\sharp$ to (A1)–(A7) one obtains a set of translated axioms of BL, which will be denoted BL$^\sharp$. As to $\flat$, let BL$^\flat$ be the set of formulas $\varphi^\flat \equiv \bar{h}$, where $\varphi$ is an axiom of BL.

\[1\] though $\bar{T}$ is definable in BL, I have included it for clarity’s sake.
Applying $\flat$ to $(L)$, one gets
\[ ([((\varphi \rightarrow \overline{h}) \land \overline{h}) \land \overline{h}) \rightarrow \varphi'] \land \overline{h} \]
which can be somewhat simplified using negation; we append $\equiv \overline{h}$ to this formula, obtaining
\[ ([\neg \neg \varphi' \land \overline{h}] \land \overline{h}) \rightarrow \varphi' \land \overline{h} \quad (L^\flat) \]
and $\sharp$ applied to (I1) and (I2) in turn yields
\[ ((\chi^\sharp \rightarrow \overline{h}) \rightarrow ((\varphi^\sharp \land \chi^\sharp \rightarrow \psi^\sharp \land \chi^\sharp) \rightarrow (\varphi^\sharp \rightarrow \psi^\sharp)) \]
\[ (\varphi^\sharp \land (\varphi^\sharp \rightarrow \overline{h})) \rightarrow \overline{h} \]
respectively; these formulas will be denoted $\Pi^1\sharp$ and $\Pi^2\sharp$ respectively.

3.2.2 Definition. The set of axioms of the logic $L^{\oplus} \Pi$ is as follows:
\[ L^{\oplus} \Pi = BL \cup \{h \land h \equiv h\} \cup \{\neg \neg h\} \cup BL^\flat \cup BL^\sharp \cup \{L^\flat\} \cup \{\Pi^1\sharp\} \cup \{\Pi^2\sharp\} \]

3.2.3 Lemma. The axioms of $L^{\oplus} \Pi$ are sound with respect to any $[0,1]_{L^{\oplus} \Pi}$.
Proof. The evaluation of $\overline{h}$ is fixed as a non-zero idempotent. By theorem 3.2.1, we need not prove soundness for $L^\flat$, $\Pi^1\sharp$, and $\Pi^2\sharp$. As to $(BL)^\sharp$ and $(BL)^\flat$, these formulas will be shown provable in $BL \cup \{\overline{h} \land \overline{h} \equiv \overline{h}\}$. $\square$

3.3. $L^{\oplus} \Pi$-algebras

3.3.1 Definition. An algebra $L = (L, 0, 1, h, \cup, \cap, *, \Rightarrow)$ is an $L^{\oplus} \Pi$-algebra if all axioms of $L^{\oplus} \Pi$ are $L$-tautologies.

$L^{\oplus} \Pi$-algebras form a variety (since $L^{\oplus} \Pi$ is an extension of $BL$, it is a subvariety of $BL$-algebras); the completeness theorem tells us that a formula which is valid in all $L^{\oplus} \Pi$-algebras is provable in $L^{\oplus} \Pi$. The aim of the following section will be to prove completeness with respect to $[0,1]_{L^{\oplus} \Pi}$.

Note that any MV-algebra is an $L^{\oplus} \Pi$-algebra with $h = 1$. Due to the axiom $\neg \neg \overline{h}$, the only $L^{\oplus} \Pi$-algebra satisfying $h = 0$ is the trivial one-element algebra. We cannot, however, prevent $h$ from being equal to 1:

3.3.2 Lemma. There is no formula which, added to $BL \cup \{Id\}$, guarantees that $h \neq 1$ in all models. The same is true for standard algebras; i.e., if the definition of an $L^{\oplus} \Pi$-norm did not contain the condition $h \neq 1$, it would be impossible to secure it by an axiom.

Proof. The proof will be carried out for standard algebras. (The claim is obvious for the variety of all $L^{\oplus} \Pi$-algebras, and the proof is analogous).
Add a constant \( h \) to the language of standard algebras and suppose there is a formula \( \varphi \) that holds in a standard algebra iff \( h \cdot h = h \) and \( h \neq 1 \); let \( \mathcal{A} \) be the class of all standard algebras where \( \varphi \) holds. Then any \([0, 1]_{\oplus \Pi}\) is in \( \mathcal{A} \); pick one standard \( \oplus \Pi \)-algebra \( A \). In \([0, 1]_{L} \), set \( h = 1 \). Define a mapping \( f : A \rightarrow [0, 1]_{L} \), sending \([0, h]\) in \( A \) isomorphically to \([0, 1]_{L} \) (by definition of \( \oplus \Pi \)-norms and 3.1.2, such isomorphism always exists), and all elements of \([h, 1]\) to \( h \). Then \( f \) is a homomorphism: if \( x \leq y \leq h \), we get \( f(x \cdot y) = f(\min(x, y)) = f(x) = f(x)_{*L} 1 = f(x)_{*L} f(y) \); as to \( \Rightarrow \), if \( x < h < y \), then \( f(y \Rightarrow x) = f(x) = 1 \Rightarrow_{L} f(x) = f(y) \Rightarrow_{L} f(x) ; \) if \( h \leq x \leq y \), then \( f(y \Rightarrow x) \geq f(x) = 1 \), and \( f(y) \Rightarrow_{L} f(x) = 1 \Rightarrow_{L} 1 = 1 \). Since homomorphisms preserve validity of formulas, \( \varphi \) must hold in \([0, 1]_{L} \). But in \([0, 1]_{L} \), \( h = 1 \); this is contradictory, and therefore, no such \( \varphi \) exists.

This means that all \( \oplus \Pi \)-tautologies not containing \( \overline{h} \) are also \( M \)-tautologies; moreover, all \( \oplus \Pi \)-tautologies without \( h \) are also \( H \)-tautologies (this is treated in general in section 4.4); this simple case is easily verified by observing that in any (linearly ordered \( \oplus \Pi \)-algebra, the subalgebra \([0] \cup (h, 1] \) is a linearly ordered \( \Pi \)-algebra). By the completeness theorem therefore, for any formula \( \varphi \) not containing \( \overline{h} \), if \( \oplus \Pi \vdash \varphi \) then \( L \vdash \varphi \) and \( \Pi \vdash \varphi \). The same is true for standard algebras: any tautology of \([0, 1]_{\oplus \Pi} \) is a tautology of \([0, 1]_{L} \) and \([0, 1]_{\Pi} \) (as will be discussed in general in 4.4).

3.4. Standard completeness for \( \oplus \Pi \)

We aim now at proving the set of axioms \( \oplus \Pi \) to be complete with respect to any \( \oplus \Pi \)-norm \( *_{\oplus \Pi} \). Our definition of an \( \oplus \Pi \)-norm does not allow \( h \) to be 0 or 1; this is necessary, since, if \( h = 1 \), we get \( \mbox{Lu\k{a}siewicz t-norm} \)—but \( \oplus \Pi \) is obviously not complete with respect to the standard algebra given by \mbox{Lu\k{a}siewicz t-norm} \mbox{ (} \neg \neg \varphi \rightarrow \varphi \mbox{) is not provable in \( \oplus \Pi \)} \mbox{).}

Thanks to the completeness theorem (for \( \oplus \Pi \)-algebras), the question whether \( \oplus \Pi \) is complete with \([0, 1]_{\oplus \Pi} \) can be reformulated thus: if a formula is a tautology for all \([0, 1]_{\oplus \Pi} \), is it also an \( L \)-tautology for any \( \oplus \Pi \)-algebra \( L \)? (If so, then it is provable in \( \oplus \Pi \).) To be able to answer this question in the affirmative, we follow in the footsteps of [6], namely, we use local embeddings. Note that, again using the completeness theorem, it suffices to take all linearly ordered \( \oplus \Pi \)-algebras into account.

3.4.1 Theorem (Standard completeness for \( \oplus \Pi \)). A formula \( \varphi \) is provable within \( \oplus \Pi \) iff it is a 1-tautology of any \([0, 1]_{\oplus \Pi} \).

Proof. We have observed already that all algebras given by \( \oplus \Pi \)-norms are isomorphic, and hence have the same sets of tautologies. It is therefore sufficient to prove the theorem for any \([0, 1]_{L_{\oplus \Pi}} \).

Suppose there is a formula \( \varphi \) (in the language of \( \oplus \Pi \)) and a linearly ordered \( \oplus \Pi \)-algebra \( L \) such that \( L \not\models \varphi \), i.e., there is an \( L \)-evaluation \( e \) such that \( e(\varphi) \neq 1 \) (hence \( \varphi \) is not provable in \( \oplus \Pi \)). Fix the algebra \( L = (L, 0, 1, \cup, \cap, *, \Rightarrow) \), the evaluation \( e \) in \( L \), and the formula \( \varphi \) (note that \( L \) is non-trivial). We shall exhibit an evaluation \( e_{s} \) in \([0, 1]_{L_{\oplus \Pi}} \) such that \( e_{s}(\varphi) \neq 1 \), and hence \([0, 1]_{L_{\Pi}} \not\models \varphi \) either. Therefore, no formula which is not provable in \( \oplus \Pi \) can be valid in \([0, 1]_{L_{\oplus \Pi}} \).
As mentioned already, formulas translate into terms, i.e., for each formula \( \varphi \) there is a term \( \sigma \) in the language of \( \mathbb{L} \oplus \Pi \)-algebras such that, for any \( \mathbb{L} \oplus \Pi \)-algebra \( \mathbf{L} \), \( \mathbf{L} \models \varphi \) iff \( \mathbf{L} \models (\sigma = 1) \).

Let \( \sigma = 1 \) be the equation corresponding to \( \varphi \), and let \( \{x_1, \ldots, x_n\} \) be all the variables in \( \sigma \). Since the evaluation \( e \) is fixed, let \( a_i = e(x_i) \), and \( \mathcal{A} = \{a_1, \ldots, a_n\} \). Let \( \mathcal{A} = \{a \in \mathcal{L} : a = e(\tau)\} \), \( \tau \) being any subterm of \( \sigma \) (i.e., \( \mathcal{A} \) is the set of evaluations of all subterms of \( \sigma \) under \( e \)).

Now we are going to determine an isomorphic embedding \( \alpha \) of \( \mathcal{A} \) into \( [0,1]_{\mathbb{L} \oplus \Pi} \) such that \( \alpha(x_1, \ldots, x_n) = \tau(\alpha(x_1), \ldots, \alpha(x_n)) \) for any subterm \( \tau \) of \( \sigma \). Once this is accomplished, we put \( e_\ast(x_i) = \alpha(a_i) \); this defines an evaluation in \( [0,1]_{\mathbb{L} \oplus \Pi} \) such that \( [0,1]_{\mathbb{L} \oplus \Pi} \models \sigma = 1 \) [8]. Hence, \( [0,1]_{\mathbb{L} \oplus \Pi} \models \varphi \).

It is shown in detail in [6] that any finite part of an \( \mathbb{M} \)-algebra (\( \Pi \)-algebra) can be isomorphically embedded in \( [0,1]_{\mathbb{L}} \). Since the \( \mathbb{L} \oplus \Pi \)-algebra is an ordered sum\(^2\) of an \( \mathbb{M} \)-algebra and a \( \Pi \)-algebra, the embedding may, with slight modifications, be performed separately for each. The modifications are as follows: define \( \mathcal{L}^1 = \{x \in \mathcal{L} : x \leq h\} \) and \( \mathcal{L}^2 = \{x \in \mathcal{L} : x \geq h\} \). Neither of these is a subalgebra of \( \mathcal{L} \); \( \mathcal{L}^1 \) lacks the constant 1, and \( \mathcal{L}^2 \) lacks the constant 0. Therefore, we define (as in 3.2.1) \( 1^h = h, \; 0^h = h \), and an operation \( x \Rightarrow y = \min(x \Rightarrow y, h) \) (and optionally also \( -x = \max(-x, h) \)). Note that for \( x, y \in \mathcal{L}^2, \; x \Rightarrow y = 1 \) iff \( x \Rightarrow y = h \). Define \( \mathcal{L}^0 = (\mathcal{L}^0, 0^h, 1^h, \cup, \cap, \ast, \Rightarrow) \) and \( \mathcal{L}^2 = (\mathcal{L}^2, 0^h, 1, \cup, \cap, \ast, \Rightarrow) \). Then \( \mathcal{L}^1 \) is an \( \mathbb{M} \)-algebra and \( \mathcal{L}^2 \) is a \( \Pi \)-algebra.

If \( h = 1 \) in \( \mathcal{L} \), the above construction gives \( \mathcal{L}^0 \) as a trivial one-element algebra. In that case, take an arbitrary linearly ordered \( \Pi \)-algebra \( \mathcal{M} \) and replace \( \mathcal{L}^0 \) with \( \mathcal{M} \) in the ordered sum \( (\mathcal{L} = \mathcal{L}^0 \oplus \mathcal{M}) \). This construction will not affect any element of \( \mathcal{M} \) (since those are all in \( \mathcal{L}^0 \)), but it will allow us to suppose that \( \mathcal{L}^0 \) has at least two elements, namely, that \( 0^h \neq 1 \).

Let \( \mathcal{A}_1 = \{a \in \mathcal{A} : a < h\} \) and \( \mathcal{A}_2 = \{a \in \mathcal{A} : a \geq h\} \). Then \( \mathcal{A}_1 \) is a finite subset of \( \mathcal{L}^1 \); by [6], an embedding \( f_1 : \mathcal{A}_1 \rightarrow [0,1]_{\mathbb{L}} \) exists so that for \( a, b, c \in \mathcal{A}_1, \; a \ast b = c \iff f_1(a) \ast_1 f_1(b) = f_1(c) \) and \( a \Rightarrow b = c \iff f_1(a) \Rightarrow_1 f_1(b) = f_1(c) \); similarly \( f_2 \) for \( \mathcal{A}_2 \). We have seen in 3.2.1 that there are isomorphic mappings \( g_1 \) and \( g_2 \) of the standard algebras \( [0,1]_{\mathbb{L}} \) and \( [0,1]_{\mathbb{L} \oplus \Pi} \) into \( [0,1]_{\mathbb{L} \oplus \Pi} \), preserving the operations \( a \ast b \) and \( a \Rightarrow b \) for \( a > b \). Fix (an arbitrary) \( [0,1]_{\mathbb{L} \oplus \Pi} \); set \( 1^h_{\mathbb{L} \oplus \Pi} = h_{\mathbb{L} \oplus \Pi}, \; 0^h_{\mathbb{L} \oplus \Pi} = h_{\mathbb{L} \oplus \Pi} \), and \( x \neq y_{\mathbb{L} \oplus \Pi} = \min(x \Rightarrow_{\mathbb{L} \oplus \Pi} y, h) \) in it. Define \( \alpha_1 = g_1 \circ f_1 \) and \( \alpha_2 = g_2 \circ f_2 \), and, finally, \( \alpha = \alpha_1 \cup \alpha_2 \). We shall verify the following:

(i) \( \forall x, y, z \in \mathcal{A}, \; x \ast y = z \iff \alpha(x) \ast_{\mathbb{L} \oplus \Pi} \alpha(y) = \alpha(z) \). Suppose \( x \ast y = z \). Let us spell out the case \( x, y \in \mathcal{A}_1 \) first (then also \( z \in \mathcal{A}_1 \)): \( \alpha(z) = \alpha(x \ast y) = g_1(f_1(x) \ast_1 f_1(y)) = \alpha(x) \ast_{\mathbb{L} \oplus \Pi} \alpha(y) \). Analogously for \( \mathcal{A}_2 \).

(ii) \( \forall x, y, z \in \mathcal{A}_1, \; x \Rightarrow y = z \iff \alpha(x) \Rightarrow_{\mathbb{L} \oplus \Pi} \alpha(y) = \alpha(z) \). Suppose \( x \Rightarrow y = z \). The

\(^2\) for the detailed definition of the term ‘ordered sum’ of \( \mathbb{B} \)-algebras, see either [4] or 4.3.1

\(^3\) its operations will be subscripted with \( \mathbb{L} \oplus \Pi \)
3.5. Reducing the axioms of L⊕Π

First of all, note that the definition of functions ⨯ and ⨯ only uses the one non-extremal idempotent h; it is independent of the exact “layout” of the t-norm. Accordingly, we extend BL with (Id) only; this extension will be denoted BL+. 

3.5.1 Definition. BL+ is the logic BL ∪ {♭ & ⨯ ≡ ⨯}. 

As shown below, it is possible to prove the whole of (BL)♭ and (BL)♮ in BL+. Hence, by the completeness theorem, any BL+-tautology is provable in BL+. We give detailed syntactic proofs; note, however, that we need not do so—it might be easier to verify that the formula is a tautology (a BL+-tautology in this case) and content with the fact that the proof exists, by the completeness theorem.

We are going to make use of these simple auxiliary statements:

3.5.2 Lemma. Let T be a schematic extension of BL. If T ⊢ ϕ and if ϕ is any formula (in the language of T), then T ⊢ ψ → ϕ. 

Proof. 

BL ⊢ ϕ → (ψ → ϕ) (see 2.2.2 (i)). Hence 

T ⊢ ψ → ϕ by modus ponens.  

3.5.3 Lemma. Let T be a schematic extension of BL. If T ⊢ ϕ and T ⊢ ψ, then T ⊢ ϕ ∧ ψ. 

Proof. Combine 2.2.2 (iii) and 2.2.2 (vi) using the transitivity of implication (A1). 

3.5.4 Lemma. BL ⊢ (α → β) → ((α ∧ γ) → (β ∧ γ)). 

Proof. By (A5), the desired statement is BL-equivalent to [(α → β) ∧ (α ∧ γ)] → (β ∧ γ). By 2.2.2 (vii), it is enough to show the two implications 

BL ⊢ ((α → β) ∧ (α ∧ γ)) → β and 

BL ⊢ ((α → β) ∧ (α ∧ γ)) → γ. 

QED
Take the first one first:
\[ \text{BL} \vdash ((\alpha \to \beta) \land (\alpha \land \gamma)) \to [\alpha \land (\alpha \to \beta)] \] by 2.2.2 (ix).

We get the desired formula by transitivity of the implication and 2.2.2 (iv).

The second case is equally easy:
\[ \text{BL} \vdash ((\alpha \to \beta) \land (\alpha \land \gamma)) \to (\alpha \land \gamma) \]
\[ \text{BL} \vdash ((\alpha \to \beta) \land (\alpha \land \gamma)) \to \gamma \] by transitivity and 2.2.2 (vi).

\[ \text{QED} \]

Now let us prove the translated axioms. We shall start with \( \sharp \), because it is easier. Note that \((A1)^\sharp-(A6)^\sharp\) are obtained from \((A1)-(A6)\) by substitution. It only remains to prove \((A7)\).

**3.5.5 Theorem.** \( \text{BL}^+ \vdash \overline{\alpha} \to \varphi^\sharp \) for any formula \( \varphi \).

**Proof.** By induction on the structure of \( \varphi \).

Let \( \varphi \) be atomic: then \( \varphi^\sharp = (\varphi \lor \overline{\alpha}) \), and
\[ \text{BL}^+ \vdash \overline{\alpha} \to (\varphi \lor \overline{\alpha}) \] by 2.2.2 (viii).

Suppose \( \text{BL}^+ \vdash \overline{\alpha} \to \varphi^\sharp \), \( \text{BL}^+ \vdash \overline{\alpha} \to \psi^\sharp \). Then by 2.2.2 (v),
\[ \text{BL}^+ \vdash \overline{\alpha} \land \overline{\alpha} \to \varphi^\sharp \land \psi^\sharp, \text{ and using (Id),} \]
\[ \text{BL}^+ \vdash \overline{\alpha} \to \varphi^\sharp \land \psi^\sharp. \]

As to the implication, by 2.2.2 (i),
\[ \text{BL}^+ \vdash (\overline{\alpha} \to \psi^\sharp) \to (\varphi^\sharp \to (\overline{\alpha} \to \psi^\sharp)); \text{ by 3.5.4,} \]
\[ \text{BL}^+ \vdash \varphi^\sharp \to (\overline{\alpha} \to \psi^\sharp) \]
and, using 2.2.2 (ii),
\[ \text{BL}^+ \vdash \overline{\alpha} \to (\varphi^\sharp \to \psi^\sharp) \]
\[ \text{QED} \]

Now we are going to prove the \( \beta \)-translations.

**3.5.6 Theorem.** \( \text{BL}^+ \) proves these formulas:

\[ (i) \quad \left[ (\varphi^\beta \to \psi^\beta) \land \overline{\alpha} \right] \to \left[ (\varphi^\beta \to \chi^\beta) \land \overline{\alpha} \right] \to \left[ (\varphi^\beta \to \chi^\beta) \land \overline{\alpha} \right] \equiv \overline{\alpha} \]
\[ (ii) \quad \left[ (\varphi^\beta \land \psi^\beta) \to \varphi^\beta \right] \equiv \overline{\alpha} \]
\[ (iii) \quad \left[ (\varphi^\beta \to \psi^\beta) \to (\psi^\beta \land \varphi^\beta) \land \overline{\alpha} \right] \equiv \overline{\alpha} \]
\[ (iv) \quad \left[ (\varphi^\beta \land \psi^\beta \land \chi^\beta) \to (\varphi^\beta \land \psi^\beta \land \chi^\beta) \right] \equiv \overline{\alpha} \]
\[ (va) \quad \left[ (\varphi^\beta \to (\psi^\beta \land \chi^\beta)) \land \overline{\alpha} \right] \to \left[ (\varphi^\beta \to (\psi^\beta \land \chi^\beta)) \land \overline{\alpha} \right] \equiv \overline{\alpha} \]
\[ (vb) \quad \left[ (\varphi^\beta \land \psi^\beta \to \chi^\beta) \land \overline{\alpha} \right] \to \left[ (\varphi^\beta \land \psi^\beta \to \chi^\beta) \land \overline{\alpha} \right] \equiv \overline{\alpha} \]
\[ (vi) \quad \left[ (\varphi^\beta \to (\psi^\beta \land \chi^\beta)) \land \overline{\alpha} \right] \to \left[ (\varphi^\beta \to (\psi^\beta \land \chi^\beta)) \land \overline{\alpha} \right] \equiv \overline{\alpha} \]
\[ (vii) \quad \left[ \overline{\varphi^\beta} \land \overline{\alpha} \right] \equiv \overline{\alpha} \]

**Proof.** All the formulas have the form \([\alpha \land \overline{\alpha}] \equiv \overline{\alpha}\). It is sufficient, of course, to prove \( \overline{\alpha} \to [\alpha \land \overline{\alpha}] \) (in \( \text{BL}^+ \)); then by 2.2.2 (viii), it is sufficient to prove \( \overline{\alpha} \to \alpha \) (for the given \( \alpha \)), or just to prove \( \alpha \) and then use 3.5.2. In all the proofs, we are going to leave the \( \beta \)'s out.
By 3.5.4, it is now sufficient to show that the same as Cases (iii), (iv) and (vii) are analogous. and 2.2.2 (v), it is enough to show that We will use (A6) now: it says (with (A5)) that By 2.2.2 (vii), it is sufficient to prove in $BL^+\to BL$. BL$^+$ also proves $\phi \to ((\phi \& \psi) \to \phi)$. Now $(\phi \& \psi) \to \phi$ is (A2), therefore provable. Cases (iii), (iv) and (vii) are analogous.

By 3.5.4, it is sufficient to prove $(\phi \to ((\psi \to \chi) \land \overline{\psi})) \to ((\phi \& \psi) \to \chi)$ in $BL^+$. We know that

$BL^+ \vdash [(\psi \to \chi) \land \overline{\psi}] \to (\psi \to \chi)$; therefore, using (A1) and 2.2.2 (i),

$BL^+ \vdash (\phi \to [((\psi \to \chi) \land \overline{\psi})]) \to (\phi \to (\psi \to \chi))$ and, using (A5a),

$BL^+ \vdash (\phi \to [((\psi \to \chi) \land \overline{\psi})]) \to ((\phi \& \psi) \to \chi)$.

We are going to adjust the antecedent of the desired implication first:

$BL^+ \vdash [((\phi \& \psi) \to \chi) \land \overline{\psi}] \to [((\phi \& \psi) \to (\psi \to \chi)) \land \overline{\psi}]$, using the fact that $\psi \to \phi$.

By 3.5.4, it is now sufficient to show that

$BL^+ \vdash [((\phi \& \psi) \to \chi) \land \overline{\psi}] \to (\phi \to [((\psi \to \chi) \land \overline{\psi})]);$ or, using (A5),

$BL^+ \vdash [((\phi \to (\psi \to \chi)) \land \overline{\psi})] \to (\phi \to [((\psi \to \chi) \land \overline{\psi})];$ or, using (A5a),

$BL^+ \vdash [((\phi \to (\psi \to \chi)) \land \overline{\psi})] \land \phi \to [((\psi \to \chi) \land \overline{\psi})].$

Using 2.2.2 (vii), we will get this statement from the two implications

$BL^+ \vdash [((\phi \to (\psi \to \chi)) \land \overline{\psi})] \land \phi \to \overline{\psi}$, which is quite clear, and

$BL^+ \vdash [((\phi \to (\psi \to \chi)) \land \overline{\psi})] \land \phi \to [\psi \to \chi]$, which follows from 2.2.2 (iv).

We are going to show that

$BL^+ \vdash [[((\phi \to (\psi \to \chi)) \land \overline{\psi}) \to (\psi \to \chi)] \land \overline{\psi}] \to [([[(\psi \to \phi) \land \overline{\psi}] \to (\chi \land \overline{\psi})] \to (\chi \land \overline{\psi})].$

By 2.2.2 (vii), it is sufficient to prove in $BL^+$ the two implications

$[((\phi \to (\psi \to \chi)) \land \overline{\psi}) \to (\psi \to \chi)] \land \overline{\psi}$, which is obvious, and

$[((\phi \to (\psi \to \chi)) \land \overline{\psi}) \to (\chi \land \overline{\psi})] \to [([[(\psi \to (\phi \land \overline{\psi}) \to (\chi \land \overline{\psi})] \to (\chi \land \overline{\psi})] \to (\chi \land \overline{\psi})] \to (\chi \land \overline{\psi})$, which is, by (A5), the same as

$[((\phi \to (\psi \to \chi)) \land \overline{\psi}) \to (\psi \to \chi)] \land \overline{\psi}] \to [([[(\psi \to (\phi \land \overline{\psi}) \to (\chi \land \overline{\psi})] \to (\chi \land \overline{\psi})] \to (\chi \land \overline{\psi})$.

We will use (A6) now: it says (with (A5)) that

$BL^+ \vdash [(\phi \to (\psi \to \chi)) \land (\psi \to \phi) \to (\chi \land \overline{\psi})]$; by transitivity of the implication and 2.2.2 (v), it is enough to show that

$BL^+ \vdash [[((\phi \to (\psi \to \chi)) \land (\psi \to \phi) \to (\chi \land \overline{\psi})] \to (\psi \to (\phi \to \chi))$ and

$BL^+ \vdash [(((\phi \to (\psi \to \chi)) \land (\psi \to (\phi \to \chi)) \to (\chi \land \overline{\psi})] \to (\psi \to (\phi \to \chi)).
Take the first case, rewrite it as:

\[
[(((\varphi \to \psi) \land \overline{\varphi}) \to \chi) \land \overline{\chi}] \& (\varphi \to \psi) \to \chi; \\
[([\alpha \land \overline{\alpha}] \to \chi) \land \overline{\beta}] \& \alpha \to \chi; \\
(\alpha \land \overline{\alpha} \& (\overline{\beta} \to ([\alpha \land \overline{\alpha}] \to \chi))) \to \chi; \\
(\alpha \land \overline{\alpha} \& (\overline{\beta} \to ([\alpha \land \overline{\alpha}] \to \chi))) \to \chi.
\]

By 2.2.2 (iv), it is now sufficient to show

\[
\text{BL}^+ \vdash (\alpha \land \overline{\alpha} \& (\overline{\beta} \to ([\alpha \land \overline{\alpha}] \to \chi))) \to \chi.
\]

If we replace the first \& by \& (which is acceptable by 2.2.2 (vi)), we get

\[
([\alpha \land \overline{\alpha}] \& ([\alpha \land \overline{\alpha}] \to \chi)) \to \chi,
\]

which is exactly 2.2.2 (iv), and hence provable in \text{BL}^+. The second case is analogous.

4. Axiomatizations of standard algebras

The research into the nature of continuous t-norms and standard algebras was primarily an attempt at classification, with regard to their respective sets of tautologies and the possibility of axiomatization. A question of particular interest was whether and how the tautologies of a standard algebra can be derived from the axioms of the three logics L, G, and Π, depending on the “layout” of the underlying t-norm. The theorem characterizing all continuous t-norms has come in extremely handy of course, since it allows one to calculate the behaviour of various (sometimes broad) classes of t-norms.

Before mentioning any formulas, we are going to have a close look into isomorphisms of standard algebras and their parts. Interesting questions arise from the existence of \(2^\omega\) pairwise non-isomorphic standard algebras (see Unsolved problems in the Appendix). Next we mention intersection logics; an interesting paper on these turned out lately (see [1]). Then we show that the sets of tautologies of individual standard algebras can be ordered into a hierarchy by inclusion, give some of its properties, and derive consequences of the inclusions. We also exhibit examples of standard algebras for which \text{BL} is complete.

4.1. Isomorphisms of standard algebras

In this section we investigate the cardinality of the set of pairwise non-isomorphic t-norms (and also the cardinalities of each cluster of isomorphic t-norms). Obviously there are \(2^\omega\) continuous t-norms in all. The upper bound is given by the cardinality of mappings of \(\omega\) into \(2^\omega\) (continuous functions are determined by their values on rational numbers). We shall exhibit \(2^\omega\) distinct continuous t-norms.

Later in this section we show that isomorphisms w.r.t. the operation \(\ast\) also preserve the operation \(\Rightarrow\). Thus the equivalence, defined by existence of isomorphisms between continuous t-norms, transfers to standard algebras; there are uncountably many pairwise non-isomorphic standard algebras.
4.1.1 Theorem. Let $f$ be a continuous, increasing, bijective mapping of $[0,1]$ onto itself (an increasing homeomorphism). Let $*$ be a continuous t-norm. For each $a,b \in [0,1]$, define $a*_1b = f(f^{-1}(a)*f^{-1}(b))$; then $*_1$ is a continuous t-norm (isomorphic to $*$).

Proof. From the definition of $*_1$ we get $f(a*b) = f(a)*_1f(b)$. Observe that $f^{-1}$ is also increasing and $f^{-1}(c*_1d) = f^{-1}(c)*f^{-1}(d)$. Indeed, put $a = f^{-1}(c), b = f^{-1}(d)$; then $f^{-1}(c*_1d) = f^{-1}(f(a)*_1f(b)) = f^{-1}(f(a*b)) = a*_1b = f^{-1}(c)*f^{-1}(d)$.

Now let us check that $*_1$ as defined above is a t-norm. Commutativity is clear; as for associativity,

$$a*(b*_1c) = f[f^{-1}(a)*f^{-1}(b*_1c)] = f[f^{-1}(a)*f^{-1}(f(f^{-1}(b)*f^{-1}(c)))] = f[f^{-1}(a)* (f^{-1}(b)*f^{-1}(c))] = f(f^{-1}(a)*f^{-1}(b))*f^{-1}(c) = f[f^{-1}(f^{-1}(a)*f^{-1}(b))]*f^{-1}(c)] = f[f^{-1}(a*_1b)]*1f(f^{-1}(c)) = (a*_1b)*_1c.$$  

Since $f$ and $f^{-1}$ are increasing, if $c \leq d$, then $c*_1a = f(f^{-1}(c)*f^{-1}(a)) \leq f(f^{-1}(d)*f^{-1}(a)) = a*_1d$; the rest follows from commutativity.

$1*_1x = f(1*f^{-1}(x)) = f(f^{-1}(x)) = x$, and $0*_1x = f(0*f^{-1}(x)) = f(0) = 0$.

Since $*_1$ is a composition of two continuous mappings on $[0,1]$, it is continuous. \(\Box\)

The relation of “being isomorphic” is an equivalence on the set of all continuous t-norms. One might well ask what the cardinalities of the equivalence classes are. The answer is given in the following lemma.

4.1.2 Lemma. Let $*$ be a continuous t-norm. If $a, b \in [0,1]$ exist so that $a*b < \min(a,b)$, then there are $2^\omega$ distinct continuous t-norms isomorphic to $*$; otherwise there are no $n$-tums isomorphic to $*$ (and distinct from $*$).

Proof. Take the two elements $a$ and $b$ for which $a*b < \min(a,b)$; let $a \leq b$ (a is nonzero). Define a homeomorphism $f$ on $[0,1]$ as follows: first choose $0 < f(a) \leq f(b) \leq 1$ (the last inequality depending on $b$); then choose $f(a*b)$ arbitrarily in $(0,f(a))$. There is a continuum of values to choose from. Then define the rest of $f$ (as an arbitrary continuous homeomorphism, e.g., connect $0, f(a*b), f(a), f(b), 1$).

This defines $2^\omega$ distinct homeomorphisms $f$, and these in turn define $2^\omega$ distinct continuous t-norms $*_i$ isomorphic to $*$ (the t-norms differ (at least) in the value of $f(a)**_i(f(b))$. If no such $a$ and $b$ exist, then $*$ is Gödel’s t-norm; the above construction fails, and there are indeed no $n$-tums isomorphic to $*$ (by checking). \(\Box\)

By the above lemma, the cardinality of each “cluster” (equivalence class) of isomorphic t-norms is $2^\omega$, but for Gödel’s t-norm, the equivalence class of which is a singleton.

How many equivalence classes are there? We give a recipe$^1$ for constructing $2^\omega$ pairwise non-isomorphic t-norms, using again the representation theorem.

4.1.3 Lemma. There are $2^\omega$ pairwise non-isomorphic t-norms.

Proof. Take an infinite, monotonous sequence of reals within $[0,1]$, converging, say, to 1; for example, $1 - 1/2^i, i \in N$. The elements of the sequence will be (the only)

$^1$ due to Petr Hájek
4.1.4 Theorem. Let \( *_a, *_b \) be two continuous t-norms, let \( 0 \leq a_1 < a_2 \leq 1 \), where \( a_1 \) and \( a_2 \) are idempotents of \( *_a \), let \( 0 \leq b_1 < b_2 \leq 1 \) and let \( f \) be a bijective mapping of \([a_1, a_2]\) onto \([b_1, b_2]\), such that \((\forall x, y \in [a_1, a_2]) f(x *_a y) = f(x) *_b f(y)\). Then

(i) \( f \) is increasing (and hence continuous) on \([a_1, a_2]\), and

(ii) \((\forall x, y \in [a_1, a_2]), x > y \implies f(x \Rightarrow_a y) = f(x) \Rightarrow_b f(y)\).

Proof.

(i) Suppose \( x < y, x, y \in [a_1, a_2] \). Then, by 2.1.3 (i), \( \exists z (z *_a y = x) \). In this case \( a_1 \leq z \leq a_2 \), because \( a_1 *_a y = a_1 \) and \( a_2 *_a y = y \). Thus \( f(z) *_b f(y) = f(x) \) and therefore \( f(x) \leq f(y) \). Because \( f(x) = f(y) \) is impossible, \( f(x) < f(y) \).

(ii) We want to prove \( f(x \Rightarrow_a y) = f(x) \Rightarrow_b f(y) \) for \( x > y, x, y \in [a_1, a_2] \).

By definition, \( f(x) \Rightarrow_b f(y) = \max z : z *_b f(x) \leq f(y) \). Because \( f \) is increasing, this is equal to \( \max u : u *_a x \leq y \), i.e., \( f(x \Rightarrow_a y) \).

Note, however, that an analogous statement cannot be formed for homomorphisms. For example, take an algebra \( A \) given by a t-norm with one non-extremal idempotent element \( \frac{1}{2} \). Define a mapping \( f : A \rightarrow [0, 1] \) such that \( f(x) = 0 \) for \( x \in [0, \frac{1}{2}] \), and \( f(x) = 2x - 1 \) otherwise. Then \( f \) is a homomorphism with respect to \(*\)'s, but not to \( \Rightarrow\)'s.

The theorem also takes subintervals of \([0, 1]\) into consideration, and states precisely in what sense they can be isomorphic. For the sake of brevity, we shall henceforward often say simply that two intervals are isomorphic, or that an interval is an isomorphic copy of a particular standard algebra; meaning that the isomorphism concerns the operations \( x * y \) for arbitrary arguments, and \( x \Rightarrow y \) for \( x > y \).

4.2. Intersection logics

We know from the standard completeness theorems that the three axiom systems (Łukasiewicz’s, Gödel’s, and product logic) are complete w.r.t. their standard algebras. In this section we take a look at their intersections. I have recently obtained a draft paper ([1]), containing, apart from the completeness result for BL, also some additional...
results for intersection logics. These are stronger than mine own, and are given at the close of this section.

The symbol $L$ here denotes Łukasiewicz’s logic; where the axiom $\neg\neg\varphi \to \varphi$ is referred to, we say ‘the axiom $L$’; similarly for $G$ and $\Pi$.

4.2.1 Theorem.

(i) The axiom $\Pi_1$ is a 1-tautology of $[0,1]_L$. Moreover, $\Pi_1$ holds in $[0,1]_*$ if $*$ is either $*_L$ or $*_\Pi$.

(ii) The axiom $\Pi_2$ is a 1-tautology of $[0,1]_G$. Moreover, $\Pi_2$ holds in $[0,1]_*$ if there is no $c \in [0,1]$ such that $*[0,c]$ is isomorphic to $*_L$.

(iii) $BL \subset (L \cap G \cap \Pi)$

Proof.

(i) $\Pi_1$ is provable from the axioms of $L$. Since $L$ proves $\neg\neg\chi \equiv \chi$, it is sufficient to prove the formula $\chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi))$ in $BL$.

$BL \vdash ((\varphi \& \chi) \& (\varphi \& \chi \rightarrow \psi \& \chi)) \rightarrow \psi$ (by application of 2.2.2 (iv))

$BL \vdash (\chi \& (\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi))$ by (A5b), and hence

$BL \vdash \chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi))$ by (A5b).

As to other t-norms, if $*$ is a continuous t-norm with a non-extremal idempotent $h$,
evaluate $e(\chi) = h$, $h < e(\varphi) < 1$, $h < e(\psi) < e(\varphi) < 1$. Then $e(\neg\neg\chi) = 1$, $e(\varphi \& \chi) = h$
e and $e(\psi \& \chi) = h$, hence $e(\varphi \& \chi \rightarrow \psi \& \chi) = 1$, but $e(\varphi \rightarrow \psi) = c < 1$, and we get

$1 \Rightarrow (1 \Rightarrow e)$, which is $c < 1$.

(ii) $BL$ proves $\varphi \& \neg\neg\varphi \equiv \overline{0}$; $G$ proves $\varphi \& \psi \equiv \varphi \& \psi$.

If $*$ does not start with Łukasiewicz, $e$ is an evaluation in $[0,1]_*$, and $e(\varphi) > 0$, then $e(\neg\varphi) = e(\varphi \rightarrow \overline{0}) = e(\varphi) \Rightarrow 0 = 0$, because either there is a non-extremal idempotent $h \leq e(\varphi)$, then use 2.1.3 (iv), or the $\Rightarrow$ is isomorphic to Gödel’s or product residuum, which both yield $0$. Therefore $\varphi \& \neg\varphi$ always evaluates to $0$ in $[0,1]_*$. 

(iii) The formula

$$[\neg\neg\varphi \to \varphi] \lor [\varphi \& \varphi \equiv \varphi] \lor [\neg\neg\chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi))]$$

is a 1-tautology of $[0,1]_L$, $[0,1]_G$, and $[0,1]_\Pi$ (obviously), but is not a t-tautology; it is not a 1-tautology of $[0,1]_{L \cap \Pi}$.

Evaluate $e(\chi) = h$, $h < e(\varphi) < 1$, $h < e(\psi) < e(\varphi) < 1$.

Then $e(\neg\neg\varphi) = 1$ and $e(\neg\neg\varphi \to \varphi) = e(\varphi) < 1$. Also, $e(\varphi \& \varphi) < e(\varphi)$ and $e(\varphi \rightarrow \varphi \& \varphi) < 1$. The $\Pi_1$ part does not evaluate to $1$ as shown above.

QED

This gives nice examples of formulas that are not t-tautologies, and fall in the intersection of $L$ and $\Pi$ (namely $\Pi_1$), or in the intersection of $G$ and $\Pi$ (namely $\Pi_2$). As to the intersection of $L$ and $G$, there is of course the cheap solution of $(\varphi \& \varphi \equiv \varphi) \lor (\neg\neg\varphi \to \varphi)$. Note that $L$ and $G$ are both characterized by a schema with only one (formula) variable $\varphi$, while $G$ has two axioms with three formulas. At present, we are unable to find an analogically simple axiom for $\Pi$.

The above theorem also shows that $(\neg\neg\varphi \to \varphi) \lor (\varphi \& \varphi \equiv \overline{0})$ is a t-tautology.
In [1], interesting completeness results have been obtained for intersection logics. The aim of the paper is primarily to show that BL is complete w.r.t. t-tautologies. This is done by proving an analogy of the representation theorem 2.1.2 for linearly ordered saturated BL-algebras; it is shown that each saturated linearly ordered BL-algebra is an ordered sum of (linearly ordered) MV-algebras, II-algebras, and G-algebras. This allows to transfer counterexamples using local embeddings (separately for each part of the sum), and therefore yields standard completeness.

Standard completeness is thereby established not only for BL, but also for certain schematic extensions; if $T$ is a schematic extension of BL such that there is at least one standard algebra satisfying $T$, then $T$ proves $\varphi$ iff $\varphi$ is a 1-tautology of all standard algebras satisfying $T$.

We give axioms of intersection logics now. First, the logic $L\Pi G$ is a schematic extension of BL, obtained by adding the formula

$$(\varphi \to \varphi \& \psi) \to [\neg \varphi \lor \psi \lor ((\varphi \to \varphi \& \varphi) \land (\psi \to \psi \& \psi))]$$

which holds in $[0,1]_L$ iff $*$ is $*_L$ or $*_G$ or $*_\Pi$; by the above considerations, $L\Pi G$ proves $\varphi$ iff $\varphi$ is a tautology of $[0,1]_L$ and $[0,1]_G$ and $[0,1]_\Pi$. (Thus $L\Pi G$ is the axiomatization of the intersection of tautologies of the three logics $L$, $G$, $\Pi$.)

The logic $L\Pi$ is obtained by adding to BL the axiom

$$(\varphi \to (\varphi \& \psi)) \to (\neg \varphi \lor \psi)$$

and is complete w.r.t. the set of tautologies of both $[0,1]_L$ and $[0,1]_\Pi$.

The logic $II G$ is obtained from $L\Pi G$ by adding the axiom

$$(\varphi \land \neg \varphi) \equiv \top$$

and is complete w.r.t. the intersection of tautologies of $[0,1]_G$ and $[0,1]_\Pi$.

Finally, $LG$ is obtained from $L\Pi G$ by adding the axiom

$$(\neg \neg \varphi \to \varphi) \lor (\varphi \to \varphi \& \varphi)$$

and is complete w.r.t. the intersection of tautologies of $[0,1]_G$ and $[0,1]_G$.

Observe that the axiom for $L\Pi I$ has the same properties as $II I$ (cf. 4.2.1 (i)), i.e., holds only in $[0,1]_L$ and $[0,1]_\Pi$.

4.3. Representation of t-norms

Standard algebras are ordered sums of isomorphic copies of $[0,1]_L$, $[0,1]_G$, and $[0,1]_\Pi$; we introduce a useful notation for ordered sums, and also a couple of notions which characterize each standard algebra up to an isomorphism. I borrow the definition of ordered sum from [4]; we shall only need its special case for standard algebras.
4.3.1 Definition. Let \( C \) be a linearly ordered set with least element 0 and largest element 1. For \( c \in C \), let \( c^+ \) be the upper neighbour of \( c \) in \( C \) if it exists, otherwise \( c^+ = c \). For each \( c \in C \), let \( A_c \) be an isomorphic copy of a linearly ordered BL-algebra with least element \( c \) and largest element \( c^+ \). Assume that the non-extremal elements of \( A_c; c \in C \) are not elements of any \( A_{c'}, c' \in C, c \neq c' \).

The ordered sum \( \bigoplus_{c \in C} A_c \) is defined as follows:

(i) the domain is \( \bigcup_{c \in C} A_c \)
(ii) for \( x \in A_{c_1}, y \in A_{c_2} \) define \( x \leq y \) iff \( c_1 = c_2 \) and \( x \leq c_1 \), or \( c_1 < c_2, c_1, c_2 \in C \)
(iii) \( x \ast y = x \ast c \) for \( x, y \in A_c, c \in C \), otherwise \( x \ast y = \min(x, y) \)
(iv) \( x \Rightarrow y = 1 \) iff \( x \leq y \)
(v) \( x \Rightarrow y = x \Rightarrow c \) for \( x, y \in A_c, x > y, c \in C \)
(vi) \( x \Rightarrow y = y \) for \( x \in A_{c_1}, y \in A_{c_2}, c_1, c_2 \in C \), and \( c_1 > c_2 \).

4.3.2 Lemma. In the above notation, \( A = \bigoplus_{c \in C} A_c \) is a linearly ordered BL-algebra.

For proof, see [2].

The representation theorem tells us that each standard algebra is an ordered sum of isomorphic copies of \([0, 1]_L, [0, 1]_G, \) and \([0, 1]_H, \) in the sense of the above definition. Indeed, if \([u, v], u < v, \) is an interval of idempotents, then it is isomorphic to \([0, 1]_G \) (as is easily verified). In the representation, we take each maximal interval of idempotents as an isomorphic copy of \([0, 1]_G \). Note that we have seen in 4.1.4 that each isomorphism w.r.t. \( \ast \) also preserves \( \Rightarrow \).

The (idempotent) elements delimiting the isomorphic copies of the three types of algebras will be called cutpoints.

4.3.3 Definition. Let \( x \) be an idempotent element of \([0, 1]_s \). We say that \( x \) is a cutpoint (of \( \ast \), or of \([0, 1]_s \)) iff there is no interval \([a, b], a < b, \) of idempotent elements such that \( x \in (a, b) \) (i.e., such that \( x \) is sharply within the interval).

\( x \) is a proper cutpoint if it is a non-extremal idempotent (i.e., distinct from 0 and 1).

Thus we dismiss idempotents lying sharply within an interval of idempotents. There are exactly three standard algebras without proper cutpoints, namely \([0, 1]_L, [0, 1]_G, [0, 1]_H \). For other continuous t-norms, the sets of their cutpoints are countable closed subsets of \([0, 1] \). (It is clearly countable: it is closed because, if \( \{c_i\}_{i=0}^\infty \) is a sequence of cutpoints, its limit is also a cutpoint: it is an idempotent, and it cannot lie sharply within an interval of idempotents).

The converse also holds: an arbitrary countable closed subset \( C \) of \([0, 1], \) such that \( 0 \in C, 1 \in C, \) is the set of cutpoints of a continuous t-norm (in fact, of uncountably many t-norms). To define a particular continuous t-norm \( \ast \), it is sufficient to define \( \ast \) on each \([c, c^+], c < c^+, c \in C \) via an isomorphism to Lukasiewicz’s, Gödel’s, or product t-norm on \([0, 1] \).

4.3.4 Definition. Let \( \mathcal{X}, \mathcal{Y} \) be countable closed subsets of \([0, 1] \). We say that \( \mathcal{X} \) and \( \mathcal{Y} \) are isomorphic (via \( f \)) iff \( f \) is a bijective mapping of \( \mathcal{X} \) onto \( \mathcal{Y} \), and \( \forall u, v \in \mathcal{X}, u \leq v \) iff \( f(u) \leq f(v) \).
Observe that if two t-norms are isomorphic, then their sets of cutpoints are isomorphic likewise.

We shall need one more term concerning t-norms, namely the term layout of a t-norm; informally, layout is the distribution of isomorphic copies of \([0,1]_L, [0,1]_G,\) and \([0,1]_Π\) between each two consecutive cutpoints. We are also going to introduce a convenient notation of layouts.

4.3.5 Definition. Let \(X\) be a countable closed subset of \([0,1]\), such that \(0 \in X, 1 \in X\). A layout for \(X\) is an arbitrary mapping \(λ\), assigning to each \([c,c^+],[c,c^+],[c,c^+]\) one of the symbols \(L, G, Π\), and the symbol \(U\) (for ‘undefined’) if \(x = x^+\).

4.3.6 Lemma. Let \(*\) be a continuous t-norm and \(C\) the set of its cutpoints. We say that \(λ\) is the layout of \(*\) if for each \(c \in C\), \(λ([c,c^+])\) is:

- \(L\) iff \(A_c\) is an isomorphic copy of \([0,1]_L]\,
- \(G\) iff \(A_c\) is an isomorphic copy of \([0,1]_G]\,
- \(Π\) iff \(A_c\) is an isomorphic copy of \([0,1]_Π]\,
- \(U\) iff \(A_c\) is trivial.

Proof.
(i) If \(*_1\) and \(*_2\) are two continuous t-norms with cutpoints \(C\) and layout \(λ\), then the isomorphism between them is an identity on \(C\), and on each \([c,c^+],[c,c^+],[c,c^+]\) it is a composition of the two isomorphisms (one for each t-norm) mapping these intervals onto one of the three standard algebras.

(ii) Clearly \(D\) and \(λ_1\) define a continuous t-norm (up to an isomorphism). If \([0,1]_a\) and \([0,1]_a\), are isomorphic via \(f\), then \(f\) maps isomorphically \(C\) onto \(D\); the rest is analogous to the proof of 4.1.3. Conversely, if we have an isomorphism between \(C\) and \(D\), and the layouts match on each of the intervals, the isomorphism of the standard algebras is a union of isomorphisms on each pair of intervals \([c,c^+],[f(c),f(c)^+]\), which is obtained by composing the isomorphisms mapping these intervals onto one of the standard algebras.

Each equivalence class of isomorphic standard algebras is thus determined by a particular layout, and an equivalence class of pairwise isomorphic sets of cutpoints.

For t-norms with finitely many cutpoints we may conveniently encompass both the exact number of cutpoints and the layout by an expression which is an “ordered sum” of symbols \(L, G, Π\). For example, \(L⊕Π\)-norms as defined in the preceding chapter have three cutpoints, one of them proper (called \(h\)). So \(C = \{0,h,1\}\), and \(λ([0,h]) = L,\)

\[\text{ technically, we prefer to have } λ\text{ total}\]
This notation may of course be introduced even for a few infinitely cut t-norms: for example, \( L \oplus L \oplus L \ldots \) is the layout of a t-norm with cutpoints \( \{0, c_1, c_2, c_3, \ldots \} \), where \( c_i < c_j \) for \( i < j \), and \( \lim_{i \to \infty} c_i = 1 \). In most cases however, there is little chance of using the infix notation with infinitely cut t-norms.

We shall sometimes use variables for the ordered sums of symbols, denoted with uppercase letters (like X, Y, ...); we shall at times need both variables for single symbols (one of the symbols L, G, or \( \Pi \)), and variables for sums of symbols, indexed by a closed subset of \([0, 1]\). Let us establish the convention that X’s match single symbols and Y’s match any non-empty (including one-element) sums of symbols. For example,

4.3.7 Definition. Let \( [0, 1]_{Y_1} \) and \( [0, 1]_{Y_2} \) be two standard algebras. Then \( [0, 1]_{Y_1 \oplus Y_2} \) denotes a standard algebra (up to an isomorphism) with an idempotent element \( h \) such that \( [0, h] \) is isomorphic to \( [0, 1]_{Y_1} \), and \( [h, 1] \) is isomorphic to \( [0, 1]_{Y_2} \).

Note that the evaluation of variables (X’s or Y’s respectively) is fixed at the time when we use them (as in many general statements: proving a general statement, we fix an arbitrary evaluation). Thus \( [0, 1]_Y \) always denotes a single standard algebra, up to an isomorphism.

We shall also at times use a wildcard B in infix notations. The letter B stands for ‘blank’ and behaves syntactically as a Y-variable, i.e., matches ordered sums of symbols L, G, and \( \Pi \) indexed by a countable subset of \([0, 1]\), but its evaluation is not fixed. Expressions containing B denote a class of continuous t-norms; more precisely, any continuous t-norm may be isomorphically embedded into the “blank” interval(s). For example, \( L \oplus B \) denotes all continuous t-norms \( * \) for which \( c > 0 \) exists such that \( *([0, c]) \) is isomorphic to \( *_1 \) (the rest of the layout may be arbitrary).

Wildcards will be used in statements where only a certain part of the layout is relevant, e.g., the initial segment.

4.4. Shrinking intervals

Now let us introduce an operation called shrinking on continuous t-norms. Shrinking takes two arguments: a continuous t-norm \( * \), and an interval \([u, v]\), \( u, v \in [0, 1] \), \( u \) and \( v \) being idempotents of \( * \), of which at least one is non-extremal. The output of shrinking is a continuous t-norm; formally, it is the t-norm which results from \( * \) by shrinking the interval \([u, v]\) into one point \( z \). We shall, later on, show some of the beautiful properties of shrinking. First, we give the formal definition.

Suppose \( * \) is a continuous t-norm with non-extremal idempotents; let \( C \) be the cutpoints and \( \lambda \) the layout of \( * \). Pick any two idempotents \( u, v \), such that \( u < v \) and at least one is non-extremal. We say that \( *_1 \) results from \( * \) by shrinking of \([u, v]\) (into \( z \)), and write \( *_1 = \text{shrink}(*, [u, v], z) \), if:

(i) the set \( D \) of cutpoints of \( *_1 \) satisfies the following conditions:

1. there are bijective mappings \( f_1 \) and \( f_2 \) such that \( C \cap [0, u] \) is isomorphic to \( D \cap [0, z] \) via \( f_1 \) and \( C \cap [v, 1] \) is isomorphic to \( D \cap [z, 1] \) via \( f_2 \). Note that if \( u = 0 \), then also \( f_1(u) = z = 0 \).
4.4.1 Lemma. Let \( \ast \) be a continuous t-norm, \( 0 \leq c_1 \leq c_2 \leq 1 \) two idempotents. Then:

(i) If \( x, y \in [c_1, c_2] \), then \( x \ast y \in [c_1, c_2] \), and it evaluates to \( c_2 \) iff \( x = y = c_2 \).

(ii) If \( x, y \in [c_1, c_2] \), \( x > y \), then \( x \Rightarrow y \in [y, c_2] \).

Proof. Let \( \lambda_1 = \ast \). We have seen in the preceding chapter that Taut([0,1]) is a subset of both \( \mathbf{L} \) and \( \mathbf{II} \); the following statements give a generalization of this observation. We are going to show, through shrinking, that (and how) the sets of tautologies of individual standard algebras are ordered by inclusion.

4.4.2 Lemma. Let \( \ast \) be a continuous t-norm, \( 0 < c_1 < c_2 \leq 1 \) its idempotents. Then:

(i) If \( x, y \in [0, c_1] \) or \( x, y \in [c_2, 1] \), \( x \ast y \leq c_2 \); if \( y < c_2 \), then \( x \ast y < c_2 \).

(ii) As to \( \Rightarrow \), the layout \( \lambda_1 \) of \( \ast \) satisfies, for each \( d \in \mathcal{D} \), \( \lambda_1([d, d^+]) = \lambda([f^{-1}(d), (f^{-1}(d))^+]) \); if \( \lambda(v) \) is not defined (because \( v \) is not a cutpoint of \( \ast \)) and \( z \in \mathcal{D} \), set \( \lambda_1([z, z^+]) = G \). This defines \( \ast_1 \) up to an isomorphism (by \( \mathcal{D} \) and \( \lambda_1 \)).

We have seen in the preceding chapter that Taut([0,1]) is a subset of both \( \mathbf{L} \) and \( \mathbf{II} \); the following statements give a generalization of this observation. We are going to show, through shrinking, that (and how) the sets of tautologies of individual standard algebras are ordered by inclusion.

4.4.3 Theorem. Let \( \ast \) be a continuous t-norm, \( C \) its cutpoints, \( \lambda \) its layout. Let \( 0 < u < v \leq 1 \) be two idempotents of \( \ast \). Let \( \ast_1 = \text{shrink}(\ast, [u, v], \mathbb{Z}) \). Then the structures \([0,1], \ast_1\) and \(([0, u] \cup [v, 1], 0, 1, \ast, \Rightarrow) \) are isomorphic.
Then \( \alpha \) is the desired isomorphism, since it is obviously an isomorphism between \([0, u]\) and \([0, z]\), and also between \([v, 1]\) and \([z, 1]\) (regardless of whether \( z \in \mathcal{D} \)). If \( x \in [0, u] \) and \( y \in [v, 1] \), then \( \alpha(x \ast y) = \alpha(x) = \min(\alpha(x) \ast_1 \alpha(y)) = \alpha(x) \ast_1 \alpha(y) \) and \( \alpha(y \Rightarrow x) = \alpha(y) \Rightarrow_1 \alpha(x) \) (because \( x < u \) and \( \alpha(x) < z \); see \( 2.3.2 \) (ii)). \[QED\]

4.4.4 Theorem. Let \( * \) be a continuous t-norm, \( C \) its cutpoints, \( \lambda \) its layout. Let \( 0 = u < v < 1 \) be two idempotents of \( * \). Let \( \ast_1 = \text{shrink}(*, [u, v], z) \). Then if there is no \( w \) such that \( \lambda([v, w]) = L \), the structures \([0, 1]_{\ast_1} \) and \( \{0\} \cup (v, 1], 0, 1, *, \Rightarrow) \) are isomorphic.

Proof. This proof will be a slight modification of the previous one.\(^2\) Take the mapping \( \alpha \) from the previous proof; set \( \beta = \alpha \setminus \{(v, 0) \} \cup \{(0, 0)\} \). Obviously \( \beta \) is an isomorphism w.r.t. each \( [f^{-1}(d), (f^{-1}(d))^+] \) and \( [d, d^+], d \in \mathcal{D}, d \neq 0 \). Moreover, if \( v^+ \neq v \) in \( C \), then \( \lambda_1([v, v^+]) \neq L \); therefore if \( a, b \in (v, v^+] \), then \( \beta(a \ast b) = \beta(a) \ast_1 \beta(b) \) by the definition of \( \beta \) (observe \( \beta(a) \ast_1 \beta(b) > 0 \)). Similarly for \( \Rightarrow \). If \( y \in (v, 1] \), then \( \beta(y \Rightarrow 0) = \beta(0) = \beta(y) \Rightarrow_1 0 = 0 \) (using again the condition that \( [0, 0^+] \) is not isomorphic to \( [0, 1]_{\beta} \)). \[QED\]

4.4.5 Corollary. Let \( * \) be a continuous t-norm, \( C \) its cutpoints, \( \lambda \) its layout, and \( 0 \leq u < v \leq 1 \) two idempotents of \( * \), of which at least one is non-extremal. Suppose that if \( u = 0 \), then there is no \( c \in C \) such that \( \lambda([v, c]) = L \). Let \( \ast_1 = \text{shrink}(*, [u, v], z) \). Then, for any formula \( \varphi \), if \([0, 1]_{\ast_1} \models \varphi \), then \([0, 1]_{\ast_1} \models \varphi \).

Proof. If \([0, 1]_{\ast_1} \not\models \varphi \) then \([0, 1]_{\ast_1} \not\models \varphi \) either, because any counterexample in \([0, 1]_{\ast_1} \) can be isomorphically embedded in \([0, 1]_{\ast_1} \). \[QED\]

4.5. The hierarchy

The last statement tells us that (and how) the sets of tautologies of individual standard algebras are ordered by inclusion into a hierarchy of sets of formulas. We are now going to list the inclusions; in the following statement, let the symbol \( X \) be fixed and stand for either \( L \) or \( G \) or \( \Pi \), whereas \( Y_1 \) and \( Y_2 \) are arbitrary (fixed) nonempty sums of symbols (see also the discussion before 4.3.7).

4.5.1 Theorem. For fixed \( X \) and fixed \( Y_1 \) and \( Y_2 \), the following inclusions hold:

(i) \( \text{Taut}(\{0, 1\}_{Y_1 \oplus X}) \subseteq \text{Taut}(\{0, 1\}_{Y_1 \oplus Y_2}) \)

(ii) \( \text{Taut}(\{0, 1\}_{Y_1 \oplus X}) \subseteq \text{Taut}(\{0, 1\}_{Y_1}) \); if \( Y_2 \) does not start with \( L \), then also

(iii) \( \text{Taut}(\{0, 1\}_{X \oplus Y_2}) \subseteq \text{Taut}(\{0, 1\}_{X \oplus Y_2}) \).

As a consequence of this theorem, if a continuous t-norm \( * \) has \( G \) (or \( \Pi \)) in its layout, then the tautologies of \([0, 1]_* \) are a subset of the tautologies of \([0, 1]_G \) (or \([0, 1]_\Pi \)). The result for \( L \) had to be stated somewhat more carefully, since, e.g., \( \Pi \) holds in \([0, 1]_\Pi \) but not in \([0, 1]_L \). Thus there are no standard algebras whose 1-tautologies would lie outside the three systems \( L \), \( G \), and \( \Pi \).

The above theorem has various consequences, and some questions arise of it. We start with a simple but nice one:

\(^2\) We have observed in the definition of shrinking that if \( u = 0 \), then also \( z = 0 \).
4.5.2 Theorem. There is no formula expressing that a continuous t-norm has a proper cutpoint; i.e., if a formula is valid in a standard algebra \([0,1]_X\), then it is also valid in \([0,1]_X\).

Proof. By 4.5.1, all tautologies of \([0,1]_X\) are also tautologies of \([0,1]_X\). QED

On the other hand, there is a formula which states that the t-norm has no proper cutpoints. We are now going to show that there is a non-collapsing chain within our hierarchy, i.e., a chain with infinitely many sharp inclusions, by exhibiting a formula which says that a t-norm (unfortunately only of a certain type) has at most \(k\) cutpoints.

4.5.3 Theorem. Let \(Id(\varphi)\) stand for \(\varphi \rightarrow \varphi \& \varphi\); let \(\Pi(\chi, \varphi, \psi)\) stand for

\[\neg \neg \chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi))\]

and finally, let \(\Pi(\alpha, \beta, \gamma)\) stand for

\[\Pi(\alpha, \beta, \gamma) \land \Pi(\alpha, \gamma, \beta) \land \Pi(\beta, \alpha, \gamma) \land \Pi(\beta, \gamma, \alpha) \land \Pi(\gamma, \alpha, \beta) \land \Pi(\gamma, \beta, \alpha)\]

Let \(k\) be fixed; take \(k + 2\) propositional variables \(\{p_i\}_{i=1}^{k+2}\) and a formula

\[\left( \bigwedge_{i=1}^{k+1} \Pi(\alpha, \beta, \gamma) \right) \lor \left( \bigvee_{j=i+1}^{k+2} \bigvee_{m=j+1}^{k+1} \Pi(p_i, p_j, p_m) \right) \quad (\Phi_k)\]

Suppose \(*\) is a continuous t-norm with cutpoints \(\mathcal{C}\) and a layout \(\lambda\), such that for \(c \in \mathcal{C}\), \(c \neq c^+\), \(\lambda(p, c^+) = \Pi\); then \(\Phi_k\) holds in \([0,1]_*\) if and only if the cardinality of \(\mathcal{C}\) is at most \(k + 1\), i.e., if \(*\) has at most \(k - 1\) proper cutpoints.

Proof. Suppose \(*\) is as above. First observe that if \(c \neq c^+\), \(c \in \mathcal{C}\) and \(e(p_i), e(p_j), e(p_m) \in (c, c^+)\), then \(e(\Pi(p_i, p_j, p_m)) = 1\) (evaluation in \((c, c^+)\) is as in \([0,1]_\Pi\)).

Also, if \(c \neq c^+\), \(c \in \mathcal{C}\), and \(e(p_i), e(p_j) \in (c, c^+)\), \(e(p_m) \in (d, d^+\) for some \(d > c, d \in \mathcal{C}\), then \(e(\Pi(p_i, p_j, p_m)) = 1\). Suppose \(\pi\) is the term corresponding to \(\Pi\), and \(e(p_i) = i, e(p_j) = j, e(p_m) = m\), and, e.g., \(i \leq j\). Observe that \((= x) = 1\) for any \(x > 0\). We check that all six cases evaluate to \(1\):

(i) \(\pi(1, m, j)\) evaluates to \(1\) because the last term \(j \Rightarrow m\) evaluates to \(1\); similarly for \(\pi(1, i, m)\) and \(\pi(1, m, i)\).

(ii) \(\pi(1, m, j)\): here \(m * i = i, ((m * i) \Rightarrow (j * i)) = (i \Rightarrow (j * i)) = j, and m \Rightarrow j = j\), hence we get \(1 \Rightarrow (j \Rightarrow j) = 1\).

(iii) \(\pi(1, m, i)\): here \(((j * m) \Rightarrow (i * m)) \Rightarrow (j \Rightarrow i) = (j \Rightarrow i) \Rightarrow (i \Rightarrow i) = 1\).

(iv) \(\pi(1, j, m)\): here \(((m * j) \Rightarrow (i * j)) = (j \Rightarrow i) \Rightarrow (i \Rightarrow j) = 1\).

Next, if \(e(p_i) < c\) and \(e(p_j) > c\), then \(e(\Pi(p_i, p_j, p_m)) < 1\) (the second implication evaluates to \(1\), but the third is sharply less than \(1\); see also the proof of 4.2.1) and hence also \(e(\Pi(p_i, p_j, p_m)) < 1\).

\(^2\) see the axiom \(\text{LIIG}\) for intersection logics
If $*$ has at most $k - 1$ proper cutpoints, $e$ is any evaluation in $[0, 1]_*$, then either

at least one $p_i$ is evaluated by an idempotent (then the left hand side of $\Phi_k$ evaluates to 1), or there is a $c \in C$ such that (at least) three $p_i$’s are evaluated in $(c, c^+]$, or there exist $c, d \in C$, $c < d$, such that $e(p_i), e(p_j) \in (c, c^+]$ and $e(p_m) \in (d, d^+]$ (which is the case investigated above); thus at least one member in the right hand side disjunction evaluates to 1. Therefore $\Phi_k$ holds under any evaluation in $[0, 1]_*$.

Suppose $*$ has $k$ proper cutpoints $\{c_1 < c_2 < \ldots < c_k\}$; we define an evaluation $e$ such that $e(\Phi_k) < 1$. Put $e(p_i) \in (0, c_1)$, $e(p_2) \in (c_1, c_2)$, ..., $e(p_{k+1}), e(p_{k+2}) \in (c_k, 1)$. Then none of the $e(p_i)$’s is idempotent and each member of the right hand side disjunction in $\Phi_k$ contains a subformula $\Pi_i(p_{i_1}, p_{i_2}, p_{i_3})$ such that for some $c \in C$, $e(p_{i_1}) < c$, $e(p_{i_2}, p_{i_3}) > c$. Therefore no member evaluates to 1, and the whole disjunction evaluates to a certain $n < 1$. \[QED\]

Thus $\text{Taut}([0, 1]_H) \supset \text{Taut}([0, 1]_{H\oplus H}) \supset \text{Taut}([0, 1]_{H\oplus H\oplus H}) \supset \ldots$ etc.

This theorem may be generalized also to t-norms with both $\Pi$’s and $G$’s in layouts, since in a layout with $k$ $\Pi$’s can accommodate at most $k + 1$ $G$’s ($G\oplus G$ makes no sense). The formula $\Phi_k$ holds for standard algebras without $L$’s iff the number of proper cutpoints is at most $2k$ (the layout would be then $G\oplus\Pi\oplus G\oplus\ldots\oplus G$, with $k$ occurrences of $\Pi$).

The construction of the formula $\Phi_k$ used in the previous statement is very like the one used in [3].

Other consequences of the above corollary will be given in separate sections.

### 4.6. Completeness results for BL

The title of this section announces completeness results; we use the fact that BL is complete w.r.t. t-automorphisms (as shown in [1]). Two things will be done, based upon that result. First, we exhibit examples of standard algebras for which the set of t-automorphisms is exactly Thm(BL). There are uncountably many pairwise non-isomorphic such algebras. Second, we show that BL is complete with respect to standard algebras of type $[0, 1]_{L\oplus B}$.

We outline some easy prerequisites.

#### 4.6.1 Lemma. If $\varphi$ is not a t-automorphism, then there is a continuous $t$-norm $*$ with finitely many cutpoints such that $\varphi$ is not a 1-automorphism in $[0, 1]_*$.

**Proof.** Though the shrinking method could be applied (with better estimate of the number of cutpoints), it would require special care. There is a simpler way. Let $\varphi$ be non-tautological. Take a standard algebra $[0, 1]_{a_1}$ and an evaluation $e_1$ in $[0, 1]_{a_1}$ such that $e_1(\varphi(p_1, \ldots, p_n)) < 1$. Let $a_i = e_1(p_i)$. Suppose $a_{i_1} \leq a_{i_2} \leq \ldots \leq a_{i_m}$. Take those cutpoints of $*_1$ for which $a_{i_1} \in [c_j, c^+_j]$. By 4.4.1, $\{\emptyset\} \cup [c_1, c^+_1] \cup \ldots \cup [c_n, c^+_n] \cup \{1\}$ is a subalgebra of $[0, 1]_{a_1}$. Define $*_2$: its cutpoints will be $\{0, c_1, c^+_1, c_2, \ldots, c^+_n, 1\}$. The layout is the same as in $*_1$ on each $[c_i, c^+_i]$, and arbitrary elsewhere. Let $e_2$ be an evaluation in $[0, 1]_{a_2}$, such that $e_2(p_i) = a_i$; then $e_2(\varphi) < 1$. $*_2$ has finitely many cutpoints.

We are now looking for standard algebras into which any finite counterexample can be embedded. In particular, each finite sum $X_1 \oplus X_2 \oplus \ldots \oplus X_n$ must be embeddable.
into the layout of such t-norms. It is obviously sufficient if the t-norm $\ast$, with cutpoints $C$ and layout $\lambda$, satisfies the following: there is an increasing (or decreasing) sequence $\{c_i\}_{i=0}^\infty$, $c_i \in C$, such that for each $i$, if $\lambda[c_i, c_i^+] = L$, then there are $c_j$, $j > i$ and $c_k$, $k > i$, such that $\lambda[c_j, c_j^+] = G$ and $\lambda[c_k, c_k^+] = \Pi$, and similarly for $G$ and $\Pi$. We will call such t-norms $[L, G, \Pi]$-periodical. Of course a t-norm is $[L, G, \Pi]$-periodical iff it is $[\Pi, G, L]$-periodical, etc. A similar definition could be produced for $[L, \Pi]$-periodical t-norms, and for other combinations; needless to say, if a t-norm is $[L, G, \Pi]$-periodical then it is also $[L, \Pi]$-periodical.

4.6.2 Theorem. Let $\ast$ be a continuous t-norm, $C$ its cutpoints and $\lambda$ its layout. If $\lambda[0, 0^+] = L$ and $\ast$ is $[L, G, \Pi]$-periodical, then $\text{Taut}([0, 1]_\ast) = \text{Thm}(\text{BL})$.

Proof. First, let us remark that $\ast$ must have the layout $L \oplus B$, for otherwise, the formula $\Pi 2$ would be valid in it.

If $\varphi$ is not a t-tautology, it has a finite counterexample (a t-norm $\ast_1$ with finitely many cutpoints $D = \{0 = d_0 < d_1 < \ldots < d_n = 1\}$ and layout $\lambda_1$). We shall show that $\ast_1$ is obtained from $\ast$ by finitely many shrink operations, which satisfy the conditions of 4.4.5, and consequently, if a formula is not valid in $[0, 1]_{\ast_1}$, it is not valid in $[0, 1]_\ast$.

To each $d_i \in D$ assign $c_i$ in $C$ so that:

(i) the set $C = \{c_0, c_1 < c_2 < \ldots < c_n = 1\}$ is isomorphic to $D$
(ii) if $\lambda_1([d_0, d_0^+]) = L$, then $c_0 = 0$
(iii) for each $i = 0\ldots n$, $\lambda([c_i, c_i^+]) = \lambda_1([d_i, d_i^+])$.

This is obviously possible due to the definition of $\ast$. Now we shrinking each of the intervals $[0, c_0]$, and $[c_{i-1}, c_i]$, $i = 1, \ldots, n$ (if of nonzero length); this amounts to finitely many shrinkings. The resulting t-norm is $\ast_1$. We always shrink an interval between two idempotents; if $[0, c_0]$ is nonzero, then $\lambda([c_0, c_0^+]) \neq L$ (otherwise the interval $[d_0, d_0^+]$ maps to the initial $L$ in $\ast$ and $0 = c_0$).

For example, $L \oplus G \oplus I \oplus L \oplus G \ldots$ is an example of layout for an $[L, G, I]$-periodical t-norm.

Moreover, a slight improvement may be done to the lemma about finite counterexamples; we may even request that the continuous t-norm have finitely many idempotents. The resulting theorem is proved in exactly the same way as 4.6.2.

4.6.3 Lemma. If $\varphi$ is not a t-tautology, then there is a continuous t-norm with finitely many idempotents such that $\varphi$ is not a $1$-tautology in $[0, 1]_\ast$.

Proof. As in 3.5.1; from the original set of idempotents, we retain each cutpoint $c_i$ and $c_i^+$ such that there is an $a_i$ within $[c_i, c_i^+]$, and $\lambda[c_i, c_i^+]$ is $L$ or $\Pi$, and each idempotent $a_i$. We preserve the original layouts on each $[c_i, c_i^+]$, and fill in arbitrarily $L$’s and $\Pi$’s inbetween.

4.6.4 Theorem. Let $\ast$ be a continuous t-norm, $C$ its cutpoints and $\lambda$ its layout. If $\lambda[0, 0^+] = L$ and $\ast$ is $[L, \Pi]$-periodical, then $\text{Taut}([0, 1]_\ast) = \text{Thm}(\text{BL})$.

Now we shift to the second promised result.

4.6.5 Theorem. BL is complete with respect to $[0, 1]_{\ast \oplus B}$, i.e., a formula $\varphi$ is provable in BL iff it is valid in all standard algebras starting with an interval isomorphic to $[0, 1]_L$. 
Proof. Suppose \( \varphi \) is not a theorem of BL (i.e., not a t-tdautology). Take an arbitrary counterexample \( \varphi \) in a standard algebra \([0, 1]_{Y_0}\) (the number of cutpoints may be arbitrary this time). We prove that there is an \( Y \) such that in the standard algebra \([0, 1]_{L \oplus Y} \), \( \varphi \) is not a 1-tautology.

If \( Y_0 = L \oplus Y_1 \) for some \( Y_1 \) then we have finished. Suppose not; take the standard algebra \([0, 1]_{L \oplus Y_0}\) (a sum of \([0, 1]_L\) and the above counterexample), and its cutpoint \( c \) such that \([0, c]\) is isomorphic to \([0, 1]_L\). Then since \([0, 1]_{Y_0}\) results by shrinking of the interval \([0, c]\) from \([0, 1]_{L \oplus Y_0}\) (satisfying the conditions of 4.4.5, since the initial segment of \( Y_0 \) is not \( L \)), if \( \varphi \) is not a 1-tautology of \([0, 1]_{Y_0}\), it is not a 1-tautology of \([0, 1]_{L \oplus Y_0}\) either.

QED

4.7. Other completeness results

The results of this section are rather fragmentary. We just briefly list what has been achieved; see Unsolved problems for the rest.

First, let \( LG(\varphi) \) stand for

\[
(\neg \neg \varphi \rightarrow \varphi) \lor (\varphi \rightarrow \varphi \land \varphi)
\]

We are going to use this formula in several ways.

4.7.1 Lemma. BL is not complete w.r.t. \([0, 1]_{L \oplus \Pi}\).

Proof. Let \( \Pi(\chi, \varphi, \psi) \) stand for

\[
\neg \neg \chi \rightarrow ((\varphi \land \chi \rightarrow \psi \land \chi) \rightarrow (\varphi \rightarrow \psi)).
\]

Then

\[
LG(\varphi) \lor LG(\psi) \lor LG(\chi) \lor \Pi(\chi, \varphi, \psi)
\]

is a 1-tautology of \([0, 1]_{L \oplus \Pi}\), but not, e.g., of \([0, 1]_{L \oplus L}\). Indeed, if in \([0, 1]_{L \oplus L}\) none of the \( LG\)'s evaluates to 1, then the values for \( \varphi, \psi \) and \( \chi \) must be from the interval \((h, 1)\); but then the \( \Pi\)-part evaluates to 1, because the evaluation copies evaluations in \([0, 1]_{L \oplus L}\).

Now take \([0, 1]_{L \oplus L}\), suppose \( h \) is the proper cutpoint in it. Observe that we may assign to \( \varphi, \psi \) and \( \chi \) values from \((h, 1)\) in such a way that \( e(\varphi \land \chi) = e(\psi \land \chi) = h \) and \( e(\varphi) > e(\psi) \). Then the evaluation yields \( 1 \Rightarrow (1 \Rightarrow (e(\varphi) \Rightarrow e(\psi))) \), which is certainly not 1.

QED

Note that the tautologies of \([0, 1]_{L \oplus \Pi}\) are sharply within the intersection of tautologies of \([0, 1]_L\) and \([0, 1]_G\), e.g., \( \Pi_1 \) does not hold in \([0, 1]_{L \oplus \Pi}\). We now show that, on the contrary, the tautologies of \([0, 1]_{L \oplus G}\) are exactly the intersection of tautologies of \([0, 1]_L\) and \([0, 1]_G\).

4.7.2 Lemma. The formula

\[
(\neg \neg \varphi \rightarrow \varphi) \lor (\varphi \rightarrow \varphi \land \varphi)
\]

is a 1-tautology of \([0, 1]_L\) if \( * \) is \( *_L \) or \( *_G \) or \( *_{L \oplus G} \).

Proof. Obviously, in \([0, 1]_L, [0, 1]_G, \) and \([0, 1]_{L \oplus G}\) (LG) holds. Conversely, if the layout contains a \( \Pi \) or a non-initial \( L \), then (LG) does not hold. Therefore it can only contain a \( G \) and/or an initial \( L \) (note that \( L \oplus G \oplus G \) makes no sense).

QED
4.7.3 Corollary. The logic $\text{BL} \cup LG(\varphi)$ is complete w.r.t. $[0,1]_{L \oplus G}$.

Proof. These axioms define the smallest variety containing all $\text{MV}$-algebras, $G$-algebras, and algebras which are an ordered sum of an $\text{MV}$-algebra and a $G$-algebra. (As above). This variety contains standard algebras; standard completeness is a consequence of the results of [1]^3.

4.7.4 Theorem. Fix $X$ as either $L$ or $\Pi$. For $i = 1, 2$ let $\ast_i$ be a continuous $t$-norm with an infinite set of cutpoints $C_i$ and a layout $\lambda_i$ such that for each $c \in C_i$, $c \neq c^+$, $\lambda_i[c,c^+] = X$. Then $\text{Taut}([0,1]_{\ast_1}) = \text{Taut}([0,1]_{\ast_2})$.

Proof. Suppose $\varphi$ is not a tautology of $[0,1]_{\ast_1}$. Then there is a standard algebra $A$ with a finite number of cutpoints and only $X$’s in its layout, such that $\varphi$ is not a tautology of $A$ (see the proof of 4.6.1). Since the tautologies of $[0,1]_{\ast_2}$ are a subset of the tautologies of $A$, $\varphi$ does not hold in $[0,1]_{\ast_2}$ either.

\[ \square \]

^3 indicated also in 4.2
Appendix

Unsolved Problems

(i) Find a single, simple axiom schema, preferably with one propositional variable, that could be used as an axiom of product logic instead of III and II2.

(ii) Find a complete axiomatics for $[0, 1]_{\leq 2}$.

(iii) Characterize the class of standard algebras $[0, 1]_*$ for which

$$\text{Taut}([0, 1]_*) = \text{Thm}(BL)$$

(iv) Find an analogy of 4.5.3 for sums of L’s, or prove that the inclusions are not sharp (that the chain collapses).

(v) Characterize standard algebras for which a recursive axiomatics exists.

These problems are of course not independent. For example, (v) is based on the fact that there can be only countably many recursive axiomatics; but, if the “majority” of standard algebras has Thm(BL) for tautologies, then (v) is solved.

References


I include references to drafts since either they are as yet unpublished, or I am not familiar with the contents of the published version. Drafts can be obtained by asking.