Varieties generated by standard BL-algebras

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Abstract

If $\mathbb{V}$ is a subvariety of $\mathbb{BL}$ generated by a class of standard BL-algebras, then $\mathbb{V}$ is generated by a *finite* class of standard BL-algebras.

1 Introduction

This paper is a contribution to the theory of BL-algebras, structures that form the equivalent algebraic semantics of Hájek's Basic Logic (BL), introduced in [12]. BL belongs to the family of substructural logics, being the semilinear, divisible axiomatic extension of the logic $\mathrm{FL_{ew}}$—Full Lambek Calculus with exchange and weakening (see [10], [16] for an overview). BL was designed as an example of a formal deductive system of fuzzy logic, forming a common fragment of, e.g., Łukasiewicz logic and Gödel logic.

Presently, the amount of knowledge collected about both BL and the variety $\mathbb{BL}$ of BL-algebras may fairly be labelled ample ([5] offers a comprehensive collection of the state of the art). Some of these results, particularly those concerning the ordinal-sum structure of BL-chains, are of key importance in this paper.

BL was conceived, and is often presented, as the logic of continuous t-norms. Those BL-algebras that are given by continuous t-norms (i.e., the domain of the algebra is the real unit interval $[0, 1]$, its monoidal operation is continuous w.r.t. the order topology, and its lattice order is just the usual order of reals) are called *standard*. While [12] was confident that indeed the variety $\mathbb{BL}$ of BL-algebras was generated by its standard members, the actual knowledge was ascertained later in [6], using the ideas of [11]. These papers succeeded in extending an existing knowledge of the ordinal-sum structure of standard BL-algebras ([15]) to the class of all linearly ordered BL-algebras (called BL-*chains*). It follows from this result that the class of

BL-chains is partially embeddable into the class of standard BL-algebras; since the variety $\mathbb{BL}$ is generated by its chains, it is generated by its standard algebras.

BL-algebras can be viewed as ordinal sums of Wajsberg hoops. This fact is explored in the papers [2] and [1]. In particular, the former work gives conditions that are necessary and sufficient for any BL-chain to generate the variety $\mathbb{BL}$; it is apparent already from this work that a large family or pairwise non-isomorphic standard BL-algebras generate the same variety. The work also investigates the lattice of subvarieties of $\mathbb{BL}$, showing that it has the cardinality of the continuum.

On the other hand, the cardinality of the family of subvarieties of $\mathbb{BL}$ that are generated by a standard BL-algebra is countably infinite. This is not obvious, taking into account the fact that there is a continuum of classes of isomorphism of standard BL-algebras. It is shown in [13] that there is a one-one correspondence between subvarieties of $\mathbb{BL}$ generated by a standard BL-algebra and finite words in a certain finite alphabet; this fact is employed to obtain coNP-completeness of the equational theories of these subvarieties. Importantly, [7] shows that each of these subvarieties is finitely based. [9] implies that this result can be extended to subvarieties of $\mathbb{BL}$ generated by finite classes of standard BL-algebras.

In this paper we address the question which subvarieties of $\mathbb{BL}$ are generated by *arbitrary* classes of standard BL-algebras. We argue that passing from finite to arbitrary classes of standard BL-algebras yields no new subvarieties of $\mathbb{BL}$. As stated in the abstract, we show that any variety generated by an arbitrary class of standard BL-algebras is already generated by some finite class of standard BL-algebras (where the latter need not be a subclass of the former). Combined with knowledge obtained before, this result offers answers to some interesting questions, such as:

(i) What is the cardinality of the class of subvarieties of $\mathbb{BL}$ generated by arbitrary classes of its standard algebras?
(ii) Are any of these subvarieties finitely based?
(iii) What is the computational complexity of the equational theories of these subvarieties?

The result stated in the abstract of this paper is obtained by investigating the (quasi)order imposed on the class standard BL-algebras by the inclusion relation on the subvarieties of $\mathbb{BL}$ these algebras generate.

This paper is organized as follows. Section 2 collects relevant well-known facts about standard BL-algebras and gives references. Section 3 gives necessary and sufficient conditions for a class of standard BL-algebras to generate the whole variety $\mathbb{BL}$ or its subvariety $\mathbb{SBL}$. Section 4, presenting the core of our proof, addresses the "remaining cases", i.e., classes of standard BL-algebras that generate neither $\mathbb{BL}$ nor $\mathbb{SBL}$. Section 5 offers some reflections on the obtained knowledge and Section 6 lists acknowledgements.

2 Background

This section gives some definitions and results that are used in this paper but introduced elsewhere. It is intended as a brief overview, with references to works where the topics mentioned just in passing here are discussed in depth and with proofs. It targets mainly the ordinal-sum structure of standard BL-algebras.

We fix notation. Algebras are denoted $\mathbf{A}$, $\mathbf{B}$, $\dots$, while the domain of an algebra $\mathbf{A}$ is denoted A, and for each function symbol f in the language, its $\mathbf{A}$-interpretation is denoted $f^{\mathbf{A}}$. The only predicate symbol in the language is the equality symbol $\approx$; we use $=$ for metamathematical identity. Classes of algebras are denoted $\mathbb{K}$, $\mathbb{L}$, $\dots$. For $\mathbb{K}$ a non-empty class of algebras in the same language, the variety given by $\mathbb{K}$ is denoted $\mathbf{Var}(\mathbb{K})$; we write $\mathbf{Var}(\mathbf{A})$ for $\mathbf{Var}(\{\mathbf{A}\})$.

For $\mathbf{A}, \mathbf{B}$ two algebras in the same language, $\mathbf{A}$ is *partially embeddable* into $\mathbf{B}$ iff each finite partial subalgebra of $\mathbf{A}$ is embeddable into $\mathbf{B}$, that is, for each finite set $A_0 \subseteq A$ there is a one-one mapping $f : A_0 \longrightarrow B$ such that for each n-ary function symbol g in the language, if for $a_1, \dots, a_n \in A_0$ we have $g^{\mathbf{A}}(a_1, \dots, a_n) \in A_0$, then $f(g^{\mathbf{A}}(a_1, \dots, a_n)) = g^{\mathbf{B}}(f(a_1), \dots, f(a_n))$. For $\mathbb{K}$, $\mathbb{L}$ two classes of algebras in the same language, $\mathbb{K}$ is partially embeddable into $\mathbb{L}$ iff each finite partial subalgebra of a member of $\mathbb{K}$ is embeddable into a member of $\mathbb{L}$.

We use ω for the set of natural numbers (i.e., ordinal numbers of finite cardinality).

The language of BL consists of four binary function symbols $\cdot, \rightarrow, \wedge, \vee$ and two constant symbols 0 and 1; $\approx$ is the only predicate symbol. We define $\neg x$ to stand for $x \rightarrow 0$.

Definition 1. (BL-algebra) *A* BL-*algebra* $\mathbf{A} = \langle A, \cdot^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *has four binary operations and two constants and satisfies:*

- *(i)* $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *is a bounded lattice with the least element* $0^{\mathbf{A}}$ *and the greatest element* $1^{\mathbf{A}}$
- *(ii)* $\langle A, \cdot^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *is a commutative monoid*

(iii) $\cdot^{\mathbf{A}}$ *and* $\rightarrow^{\mathbf{A}}$ *form an adjoint pair, i.e., for all* $x, y, z \in A$, $z \leq^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} y)$ *iff* $x \cdot^{\mathbf{A}} z \leq^{\mathbf{A}} y$

(iv) *for all* $x, y \in A$, $x \wedge^{\mathbf{A}} y = x \cdot^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} y)$

(v) *for all* $x, y \in A$, $(x \rightarrow^{\mathbf{A}} y) \vee^{\mathbf{A}} (y \rightarrow^{\mathbf{A}} x) = 1^{\mathbf{A}}$

BL-algebras form a variety, denoted $\mathbb{BL}$. Linearly ordered BL-algebras are called BL-chains; as each BL-algebra is a subdirect product of BL-chains, the variety $\mathbb{BL}$ is generated by BL-chains. If the domain of a BL-algebra is the real unit interval $[0, 1]$, and its lattice order is the usual order of the reals, we speak of a *standard* BL-algebra. As explained in [6], in any standard BL-algebra $\mathbf{A}$ the operation $\cdot^{\mathbf{A}}$ turns out to be a continuous t-norm: a binary operation $*$ on $[0, 1]$ that is commutative, associative, non-decreasing in both arguments, satisfies $1^{\mathbf{A}} * x = x$ and $0^{\mathbf{A}} * x = 0^{\mathbf{A}}$ for each $x \in A$, and is continuous on $[0, 1]^2$. Since $*$ is continuous, there is a unique operation $\rightarrow_*$ satisfying adjointness, namely, $x \rightarrow_* y = \max\{z \mid x * z \leq y\}$; this operation is the *residuum* of $*$. Therefore, each standard BL-algebra is uniquely determined by its continuous t-norm $*$, and the notation $[0, 1]_*$ is often used.

Standard BL-algebras originally motivated the study of the logic BL and the corresponding variety $\mathbb{BL}$, as explained in [12]; a bit later, it was shown that the propositional logic BL was indeed the logic of standard BL-algebras, and consequently, the variety $\mathbb{BL}$ was generated by standard BL-algebras. These results were obtained by the combined efforts of [11] and [6].

Three examples of continuous t-norms stand out; these, with the respective residua, are summed up in Table 1. The table only gives the value of the residuum for the case when the first argument is strictly greater than the second, as it is easy to see that in any BL-algebra $\mathbf{A}$ and for any $x, y \in A$, we have $x \rightarrow^{\mathbf{A}} y = 1^{\mathbf{A}}$ iff $x \leq^{\mathbf{A}} y$.

t-norm	$x * y$	$x \rightarrow y$
Łukasiewicz	$\max(0, x + y - 1)$	$1 - x + y$
Gödel	$\min(x, y)$	y
product	$x \cdot y$	y/x

Table 1: Three examples of continuous t-norms

The algebra $[0, 1]_{\text{Ł}}$, given by the Łukasiewicz t-norm $*_{\text{Ł}}$, is the standard MV-algebra, which generates the variety of MV-algebras (as each MV-chain is partially embeddable into $[0, 1]_{\text{Ł}}$). MV-algebras can be rendered as BL-algebras satisfying the identity $\neg\neg x \approx x$, and they form the equivalent algebraic semantics of Łukasiewicz logic. Analogously, Gödel algebras, the equivalent algebraic semantics of Gödel logic, satisfy the identity $x \cdot x \approx x$, and $[0,1]_{\mathrm{G}}$, the algebra given by Gödel t-norm, generates the variety of Gödel algebras as each Gödel chain is partially embeddable into $[0,1]_{\mathrm{G}}$. Product algebras satisfy the identity $(\varphi \to \chi) \vee ((\varphi \to (\varphi \cdot \psi)) \to \psi) \approx 1$, forming the equivalent algebraic semantics of product logic, and the standard product algebra $[0,1]_{\Pi}$, given by the product t-norm, generates the variety of product algebras as each product chain is partially embeddable into $[0,1]_{\Pi}$.

Consider a standard BL-algebra $[0,1]_*$. We say that an element $x \in [0,1]$ is *idempotent* (w.r.t. $*$) iff $x * x = x$. For $[0,1]_*$, the set of its idempotents is a closed subset of $[0,1]$. The complement is a union of countably many pairwise disjoint open intervals; on the closure of each of these intervals, the operation $*$ is isomorphic either to the Łukasiewicz t-norm $*_{\mathrm{L}}$ on $[0,1]$, or to the product t-norm $*_{\Pi}$ on $[0,1]$ (and the isomorphism preserves also the operation $\to$ for each x, y such that $x > y$). This is a result due to Mostert and Shields [15].

It follows that for each continuous t-norm $*$, one can decompose the domain of $[0,1]_*$ into closed subintervals where $*$ is isomorphic either to $*_{\mathrm{L}}$ or to $*_{\Pi}$ or to $*_{\mathrm{G}}$ (in the last case, it is reasonable to limit oneself to those intervals of idempotent elements that are maximal w.r.t. inclusion) and points idempotent w.r.t. $*$ that do not belong to any of the three types of intervals (for example, if the idempotent elements of $*$ are $\{\{0\} \cup \{1/2^n\}_{n\in\omega}\}$, and on each $[1/2^{n+1}, 1/2^n]$, $*$ is isomorphic to $*_{\mathrm{L}}$, then 0 is an example of such a point). The import of the decomposition statement given above is captured by the notion of an *ordinal sum*; it can be found in [8] concerning hoops. Here, we relate it to BL-chains.[1]

Definition 2. (Ordinal sum of BL-chains) *Let I be a linearly ordered set with the smallest element i_0 and let $\{\mathbf{A}_i\}_{i\in I}$ be a family of* BL*-chains such that for each $i \neq j \in I$, we have $A_i \cap A_j = \{1^{\mathbf{A}_i}\} = \{1^{\mathbf{A}_j}\}$. The* ordinal sum *$\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$ of $\{\mathbf{A}_i\}_{i\in I}$ is an algebra in the language of* BL*, defined as follows:*

(i) the domain is $A = \bigcup_{i\in I} A_i$

(ii) $0^{\mathbf{A}} = 0^{\mathbf{A}_{i_0}}$ and $1^{\mathbf{A}} = 1^{\mathbf{A}_{i_0}}$

[1] We depart slightly from the usual definition of ordinal sum of BL-chains as given in [6]: here, we consider the greatest element of each of the algebras to be the same in all the summands, and we make it the greatest element of the sum; this is usual when considering ordinal sums of hoops.

(iii) $x \leq^{\mathbf{A}} y$ *iff* $\begin{cases} x, y \in A_i \textit{ and } x \leq^{\mathbf{A}_i} y \\ x \in A_i \setminus \{1^{\mathbf{A}_i}\} \textit{ and } y \in A_j \textit{ and } i < j \end{cases}$

(iv) $x \cdot^{\mathbf{A}} y = \begin{cases} x \cdot^{\mathbf{A}_i} y & \textit{if } x, y \in A_i \\ \min^{\mathbf{A}}(x, y) & \textit{otherwise} \end{cases}$

(v) $x \to^{\mathbf{A}} y = \begin{cases} 1^{\mathbf{A}} & \textit{if } x \leq^{\mathbf{A}} y \\ x \to^{\mathbf{A}_i} y & \textit{if } x, y \in A_i \\ y & \textit{otherwise} \end{cases}$

The elements of the sum are called *components*. For I as above, $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$, and $i, j \in I$, we write $\mathbf{A}_i < \mathbf{A}_j$ whenever $i < j$: the total order on I imposes an order on the components of $\mathbf{A}$. One can think of ordinal sums as being positioned "horizontally", in an increasing manner from left to right, so that the least component is the leftmost one; this is our approach in this paper, as we eventually want to think of ordinal sums as expressions written on a line. We speak of the *first* or *initial* component in the sense of the leftmost (i.e., the least) component. Another possibility is to think of a vertical ordering, where the least component is at the bottom.

An ordinal sum of a non-empty family of BL-chains is a BL-chain (possibly taking isomorphic copies to meet the condition that domains are pairwise disjoint except for the greatest element). A converse result holds in a stronger way: each saturated BL-chain can be decomposed as an ordinal sum of MV-chains, Gödel chains, product chains, and copies of the two-element Boolean algebra; this is shown in [11], [6]. The result, generalizing a result of Mostert and Shields, captures the structure of BL-chains and makes it possible to show that standard BL-algebras generate the variety $\mathbb{BL}$. We do not need the general result, as our paper focuses on standard BL-algebras.

Theorem 3. ([15, 12]) *Each standard* BL*-algebra is an ordinal sum of a family of* BL*-algebras, each of whom is an isomorphic copy of either* $[0,1]_{\text{Ł}}$ *or* $[0,1]_{\text{G}}$ *or* $[0,1]_{\Pi}$ *or* 2 *(the two-element Boolean algebra).*

In a standard BL-algebra, the *type* of a component is one of Ł (isomorphic to $[0,1]_{\text{Ł}}$), G (isomorphic to $[0,1]_{\text{G}}$), Π (isomorphic to $[0,1]_{\Pi}$), or 2 (isomorphic to the two-element Boolean algebra).

Moreover, with the proviso that only maximal Gödel components are considered, the decomposition is unique: for each standard BL-algebra $\mathbf{A}$, one can write $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ where the linearly ordered set I, as well as the type of each of the $\mathbf{A}_i$'s, are uniquely determined by $\mathbf{A}$.

Lemma 4. *Two standard* BL-*algebras* $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ *and* $\mathbf{B} = \bigoplus_{j \in J} \mathbf{B}_j$ *are isomorphic if and only if* I *and* J *are order-isomorphic via some* f*, and for each* $i \in I$ *we have that* $\mathbf{A}_i$ *is isomorphic to* $\mathbf{B}_{f(i)}$ *(that is,* $\mathbf{A}_i$ *and* $\mathbf{B}_{f(i)}$ *are of the same type).*

Since we are mainly interested in the equational theory given by a class of standard BL-algebras, we need not distinguish between isomorphic copies of a single algebra; rather, we represent classes of isomorphism of standard BL-algebras with ordinal sums of the four symbols Ł, G, Π, and 2. We shall refer to such sums as BL-*expressions*.

For example, the expression Ł ⊕ G ⊕ Ł represents the isomorphism class of a standard BL-algebra with an Ł-component followed by a G-component followed by another Ł-component. The expression ωŁ represents an infinite sum ordered by the ordinal ω where all components are Ł-components (analogously for $\omega\Pi$). Needless to say, not every standard BL-algebra can be rendered by a nice finite string, as there is a continuum of pairwise non-isomorphic standard BL-algebras. We also remark here that, while there can be at most countably many Ł-, G-, and Π-components, there can in general be uncountably many 2-components; for example, the Cantor set can be taken as the set of idempotents of a continuous t-norm. However, as far as the varieties generated by standard BL-algebras are concerned, we shall be able to restrict ourselves to algebras with tangible finite-string representations (see canonical BL-algebras below).

Obviously, each G-component can be further decomposed as an ordinal sum of continuum many 2-components; on the other hand, Ł- and Π-components are sum-indecomposable as BL-algebras. However, Π-components can be decomposed as a sum of two Wajsberg hoops, as explained below.

Definition 5. (Hoop) *A hoop is an algebra* $\mathbf{A} = \langle A, \cdot^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$*, where* $\langle A, \cdot^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *is a commutative monoid and where the following identities hold:*

(i) $x \rightarrow x \approx 1$
(ii) $x \cdot (x \rightarrow y) \approx y \cdot (y \rightarrow x)$
(iii) $x \rightarrow (y \rightarrow z) \approx (x \cdot y) \rightarrow z$

$\mathbf{A}$ *is a* Wajsberg *hoop iff the identity* $(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$ *holds in it.*

Ordinal sums of (linearly ordered) hoops are defined analogously to ordinal sums of BL-chains (though they need not have a first component to form a hoop), and any BL-chain can be decomposed as an ordinal sum of Wajsberg hoops; see [2]. In particular, each standard BL-algebra can be decomposed as an ordinal sum of isomorphic copies of three particular Wajsberg hoops: $[0,1]_{\mathrm{Ł}}$, $(0,1]_{\mathrm{C}}$, and 2, where $[0,1]_{\mathrm{Ł}}$ and 2 are as before and $(0,1]_{\mathrm{C}}$ is the standard cancellative hoop, i.e., the positive part of $[0,1]_{\Pi}$ (thus $[0,1]_{\Pi} = 2 \oplus (0,1]_{\mathrm{C}}$). We speak of Ł-, C-, and 2-components. The Wajsberg-hoop decomposition of a standard BL-algebra is obtained by replacing each component Π with the ordinal sum $2 \oplus \mathrm{C}$ and each component G with the sum of continuum many 2's ordered in the natural order of reals on $[0,1]$.

Not only does the class of standard BL-algebras generate the variety $\mathbb{BL}$, but the same variety is generated by particular examples of standard BL-algebras. This is easily observed using the fact (shown in [12]) that $(0,1]_{\mathrm{C}}$ is partially embeddable into $[0,1]_{\mathrm{Ł}}$ (hence $[0,1]_{\Pi}$ is partially embeddable into $[0,1]_{\mathrm{Ł}} \oplus [0,1]_{\mathrm{Ł}}$) and the trivial fact that 2 is a subalgebra of, and hence embeddable into, $[0,1]_{\mathrm{Ł}}$. However, $[0,1]_{\mathrm{Ł}}$ is not partially embeddable to $[0,1]_{\Pi}$. Due to partial embeddability, any standard BL-algebra with infinitely many Ł-components, one of whom is the first component, generates $\mathbb{BL}$: indeed, given an identity not valid in a BL-chain, one can pass, via partial embeddings, first to a standard BL-algebra and then to a standard BL-algebra with infinitely many L-components, where the identity is not valid either. The paper [2] shows that the converse implication also holds, so it characterizes those standard BL-algebras that generate $\mathbb{BL}$.[2]

Theorem 6. ([2]) *A standard* BL*-algebra* $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$*, where* I *is linearly ordered with minimum* i_0*, generates the variety* $\mathbb{BL}$ *iff* $\mathbf{A}_{i_0}$ *is an* Ł*-component and for infinitely many* $i \in I$*,* $\mathbf{A}_i$ *is an* Ł*-component.*

The variety $\mathbb{SBL}$ is a subvariety of $\mathbb{BL}$ given by the identity $x \wedge \neg x \approx 0$. Easily, a standard BL-algebra is an SBL-algebra iff the first component in its ordinal sum is *not* an Ł-component (hence, in the Wajsberg hoop decomposition, it is a 2-component). An analogy of the above theorem for those standard BL-algebras that generate $\mathbb{SBL}$ reads as follows.

Theorem 7. *A standard* SBL*-algebra* $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$*, where* I *is linearly ordered with minimum* i_0*, generates the variety* $\mathbb{SBL}$ *iff* $\mathbf{A}_{i_0}$ *is* not *an* Ł*-component and for infinitely many* $i \in I$*,* $i \neq i_0$*,* $\mathbf{A}_i$ *is an* Ł*-component.*

In the rest of this section, we reproduce some material from [7], a paper discussing in detail some crucial consequences of the ordinal sum representation of standard BL-algebras.

[2] The result in [2] is much more general, giving the characterization for any BL-chain viewed as an ordinal sum of Wajsberg hoops.

Definition 8. *Let* $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$*, where* I *is linearly ordered with minimum* i_0*, be a standard* BL*-algebra. We denote* $\mathbf{Fin}(\mathbf{A})$ *the class of finite ordinal sums* $\bigoplus_{0\le j\le n} \mathbf{W}_j$ *of Wajsberg hoops among isomorphic copies of* $[0,1]_{\text{Ł}}$, $(0,1]_{\text{C}}$*, and* 2 *(i.e.,* Ł*-,* C*-, and* 2*-components), such that* $\mathbf{W}_0$ *is either a* 2*- or an* Ł*-component and there are components* $\mathbf{A}_{i_0} < \mathbf{A}_{i_1} \cdots < \mathbf{A}_{i_n}$ *of* $\mathbf{A}$*, such that for each* $j = 0, \ldots, n$*,*

(i) *if* $\mathbf{A}_{i_j}$ *is an* Ł*-component, then* $\mathbf{W}_j$ *is an* Ł*-, a* C*-, or a* 2*-component;*
(ii) *if* $\mathbf{A}_{i_j}$ *is a* C*-component, then* $\mathbf{W}_j$ *is a* C*-component;*
(iii) *if* $\mathbf{A}_{i_j}$ *is a* 2*-component, then* $\mathbf{W}_j$ *is a* 2*-component.*

For a class $\mathbb{K}$ *of standard* BL*-algebras, we define* $\mathbf{Fin}(\mathbb{K}) = \bigcup_{\mathbf{A}\in\mathbb{K}} \mathbf{Fin}(\mathbf{A})$.

The next theorem establishes the **Fin** operator as a tool of investigating partial embeddability and hence, inclusion of varieties.

Theorem 9. [7] *Let* $\mathbb{K}$, $\mathbb{L}$ *be two classes of standard* BL*-algebras. Then the following are equivalent:*

(i) $\mathbf{Var}(\mathbb{K}) \subseteq \mathbf{Var}(\mathbb{L})$*;*
(ii) $\mathbb{K}$ *is partially embeddable into* $\mathbb{L}$*;*
(iii) $\mathbf{Fin}(\mathbb{K}) \subseteq \mathbf{Fin}(\mathbb{L})$.

Note that Theorems 6 and 7 follow from Theorem 9, using the fact that each standard BL-algebra (SBL-algebra) is partially embeddable into the standard algebra of type ωŁ ($\Pi \oplus \omega$Ł respectively).

Definition 10. (Canonical BL-algebra) *A standard* BL*-algebra is* canonical *iff its ordinal-sum decomposition is either* ωŁ*, or* $\Pi \oplus \omega$Ł*, or a finite* $\oplus$*-sum of expressions from among* Ł*,* G*,* Π *and* $\omega\Pi$*, where no* G *is preceded or followed by another* G *and no* $\omega\Pi$ *is preceded or followed by a* G*, a* Π *or another* $\omega\Pi$.

As all canonical BL-algebras are standard, we may use the short 'canonical BL-algebra' for 'canonical standard BL-algebra'.

Using the above-given conditions sufficient for a standard BL-algebra to generate either of the varieties $\mathbb{BL}$ or $\mathbb{SBL}$, and observing that any standard BL-algebra without Ł-components but with infinitely many Π-components is partially embeddable into any other standard BL-algebra with the same property, one may conclude:

Lemma 11. *For each standard* BL*-algebra, there is a canonical* BL*-algebra generating the same variety.*

It follows that there are only countably many subvarieties of $\mathbb{BL}$ that are generated by a single standard BL-algebra. Moreover, as a consequence of Theorem 9, non-isomorphic canonical BL-algebras generate distinct subvarieties of $\mathbb{BL}$ (since, for two non-isomorphic canonical BL-algebras $\mathbf{A}$ and $\mathbf{B}$, clearly $\mathbf{Fin}(\mathbf{A}) \neq \mathbf{Fin}(\mathbf{B})$). Hence, there is a 1-1 correspondence between those subvarieties of $\mathbb{BL}$ that are given by a single standard BL-algebra, and isomorphism classes of canonical BL-algebras, given by the expressions from Definition 10.

In [7], it has been shown that each subvariety of $\mathbb{BL}$ generated by a standard BL-algebra is finitely based. This result extends to finite classes of standard BL-algebras, using the result of [9].

3 A case analysis

In view of ideas and results presented above, we start this section by introducing some terminology and conventions that will facilitate our proof by revealing its combinatorial nature. Then we give two lemmata which give necessary and sufficient conditions for a class of standard BL-algebras to generate the varieties $\mathbb{BL}$ and $\mathbb{SBL}$, respectively.

First of all, a simple yet essential lemma on replacing generators of varieties of algebras.

Lemma 12. *Let* $\mathbb{K} = \bigcup_{i\in I} \mathbb{K}_i$, $\mathbb{L} = \bigcup_{i\in I} \mathbb{L}_i$ *be classes of algebras in the same language. Assume* $\mathbf{Var}(\mathbb{K}_i) = \mathbf{Var}(\mathbb{L}_i)$ *for each* $i \in I$*. Then* $\mathbf{Var}(\mathbb{K}) = \mathbf{Var}(\mathbb{L})$*.*

Proof. $\mathbf{Var}(\mathbb{K}) = \mathbf{Var}(\bigcup_{i\in I} \mathbb{K}_i) = \mathbf{Var}(\bigcup_{i\in I} \mathbf{Var}(\mathbb{K}_i)) =$
$= \mathbf{Var}(\bigcup_{i\in I} \mathbf{Var}(\mathbb{L}_i)) = \mathbf{Var}(\bigcup_{i\in I} \mathbb{L}_i) = \mathbf{Var}(\mathbb{L})$. □

The following statement can then be obtained as a natural generalization of Lemma 11 for classes of standard BL-algebras.

Corollary 13. *Let* $\mathbb{C}$ *be a class of standard* BL*-algebras and let* $\mathbb{C}'$ *result from* $\mathbb{C}$ *by replacing each standard* BL*-algebra* $\mathbf{A} \in \mathbb{C}$ *with a canonical standard* BL*-algebra* $\mathbf{A}'$ *generating the same variety (i.e., one such that* $\mathbf{Var}(\mathbf{A}) = \mathbf{Var}(\mathbf{A}')$*). Then* $\mathbf{Var}(\mathbb{C}) = \mathbf{Var}(\mathbb{C}')$*.*

Given an arbitrary class $\mathbb{C}$ of standard BL-algebras, when interested in the variety it generates, we are able to make the following two simplifications without loss of generality. Firstly, we may work with isomorphism classes in $\mathbb{C}$, that is, we may prefer thinking about the BL-expressions representing the algebras in $\mathbb{C}$ rather than the algebras themselves; we rely on Lemma 4 and the discussion following it. Secondly, we may assume that all elements of $\mathbb{C}$ are canonical BL-algebras; in view of Definition 10, the expressions representing the classes of isomorphism in $\mathbb{C}$ are then among ωŁ, $\Pi \oplus \omega$Ł, and finite sum of expressions from among Ł, G, Π and $\omega\Pi$. (Hence the class of BL-expressions representing $\mathbb{C}$ is countable.)

Definition 14. (Canonical BL-expression) *The class of canonical* BL-*expressions consists of* ωŁ, $\Pi \oplus \omega$Ł, *and finite* $\oplus$*-sums of expressions* Ł, G, Π *and* $\omega\Pi$, *where no* G *is preceded or followed by another* G, *and no* $\omega\Pi$ *is preceded or followed by a* G, *a* Π *or another* $\omega\Pi$. *The class of canonical* BL-*expressions is denoted* $\mathbb{L}$.

In other words, canonical BL-expressions are just the BL-expressions representing canonical BL-algebras (cf. Definition 10). The symbol $\oplus$, functioning as a concatenation symbol, may be omitted.

Our technical statements will be presented using canonical BL-expressions alongside the corresponding (isomorphism classes of) canonical BL-algebras. We take the liberty of passing freely between the algebras and the expressions. We use the same notation; thus working with a class $\mathbb{C}$ of canonical BL-algebras, the corresponding class of canonical BL-expressions is also denoted $\mathbb{C}$. Moreover, we use some of the concepts defined for classes of algebras also for the expressions; in particular, if $\mathbb{C}$ is a class of expressions, then also $\mathbf{Fin}(\mathbb{C})$ is viewed as the finite class of expressions given in Definition 8.

From what has been presented so far, it is clear that, as regards the variety generated, a key characteristic of a canonical BL-expression is the number of its Ł-components. Another key characteristic is whether or not one of its Ł-components is the initial component. We partition the class $\mathbb{L}$ of canonical BL-expressions according to these characteristics.

Definition 15. *Let* $i \in \omega \cup \{\omega\}$. *We denote*

(i) $\mathbb{L}^i_{\text{Ł}}$ *the class of canonical* BL-*expressions whose initial component is an* Ł*-component and with exactly* i Ł*-components;*
(ii) $\mathbb{L}^i_{\overline{\text{Ł}}}$ *the class of canonical* BL-*expressions whose initial component is not an* Ł*-component and with exactly* i Ł*-components;*
(iii) $\mathbb{L}^i = \mathbb{L}^i_{\text{Ł}} \cup \mathbb{L}^i_{\overline{\text{Ł}}}$.

Moreover, let

$$\mathbb{L}_{\text{Ł}} = \bigcup_{i \in \omega \cup \{\omega\}} \mathbb{L}^i_{\text{Ł}} \quad \textit{and} \quad \mathbb{L}_{\overline{\text{Ł}}} = \bigcup_{i \in \omega \cup \{\omega\}} \mathbb{L}^i_{\overline{\text{Ł}}}$$

Note that expressions in the class $\mathbb{L}^0$ have no Ł-components (and $\mathbb{L}^0_{\text{Ł}}$ is empty); for each $1 \leq i < \omega$, the class $\mathbb{L}^i$ collects all canonical BL-expressions with exactly i Ł-components; and $\mathbb{L}^\omega$ consists of two expressions ωŁ and $\Pi \oplus \omega$Ł.

For an arbitrary class $\mathbb{C}$ of canonical BL-expressions, one can introduce a partition on $\mathbb{C}$ along the lines of the above given partition of $\mathbb{L}$. That is, let $\mathbb{C}^i_{\text{Ł}} = \mathbb{C} \cap \mathbb{L}^i_{\text{Ł}}$ and $\mathbb{C}^i_{\overline{\text{Ł}}} = \mathbb{C} \cap \mathbb{L}^i_{\overline{\text{Ł}}}$ for each $i \in \omega \cup \{\omega\}$. Moreover, let $\mathbb{C}_{\text{Ł}} = \mathbb{C} \cap \mathbb{L}_{\text{Ł}}$ and $\mathbb{C}_{\overline{\text{Ł}}} = \mathbb{C} \cap \mathbb{L}_{\overline{\text{Ł}}}$.

Relying on Lemma 12, we shall address the classes $\mathbb{C}_{\text{Ł}}$ and $\mathbb{C}_{\overline{\text{Ł}}}$ separately, trying to replace each one with a suitable finite counterpart that generates the same variety. Clearly, $\mathbb{C}_{\text{Ł}}$ generates a subvariety of $\mathbb{BL}$ and $\mathbb{C}_{\overline{\text{Ł}}}$ generates a subvariety of $\mathbb{SBL}$ (in either case, the subvariety is not necessarily proper).

The following two lemmata are consequences of Theorem 9 (and generalizations of Theorems 6 and 7 to classes of canonical BL-expressions). For both, we consider $\mathbb{C}$ a given class of canonical BL-expressions, with $\mathbb{C}^i_{\text{Ł}}$ and $\mathbb{C}^i_{\overline{\text{Ł}}}$, $i \in \omega \cup \{\omega\}$, as above.

Lemma 16. *If the set* $\{k \in \omega \,|\, \mathbb{C}^k_{\text{Ł}}$ *is nonempty*$\}$ *is infinite, or if* $\mathbb{C}^\omega_{\text{Ł}}$ *is nonempty, then* $\mathbf{Var}(\mathbb{C}_{\text{Ł}}) = \mathbb{BL}$.

If the class $\mathbb{C}_{\text{Ł}}$ generates the variety $\mathbb{BL}$, then $\mathbf{Var}(\mathbb{C}_{\text{Ł}}) = \mathbf{Var}(\omega\text{Ł})$; in other words, the variety $\mathbb{BL}$ generated by $\mathbb{C}_{\text{Ł}}$ is also generated by the single canonical BL-expression ωŁ.

On the other hand, if $\mathbb{C}_{\text{Ł}}$ does not satisfy any of the conditions of Lemma 16, then there is a $k_0 \in \omega$ such that each expression $\mathbf{A} \in \mathbb{C}_{\text{Ł}}$ has at most k_0 Ł-components. In such a case, $\mathbb{C}_{\text{Ł}}$ generates a proper subvariety of $\mathbb{BL}$ (by Theorem 9).

Lemma 17. *If the set* $\{k \in \omega \,|\, \mathbb{C}^k_{\overline{\text{Ł}}}$ *is nonempty*$\}$ *is infinite, or if* $\mathbb{C}^\omega_{\overline{\text{Ł}}}$ *is nonempty, then* $\mathbf{Var}(\mathbb{C}_{\overline{\text{Ł}}}) = \mathbb{SBL}$.

If the class $\mathbb{C}_{\overline{\text{Ł}}}$ generates the variety $\mathbb{SBL}$, then $\mathbf{Var}(\mathbb{C}_{\overline{\text{Ł}}}) = \mathbf{Var}(\Pi \oplus \omega\text{Ł})$, hence the variety $\mathbb{SBL}$ generated by $\mathbb{C}_{\overline{\text{Ł}}}$ is generated by the single canonical BL-expression $\Pi \oplus \omega$Ł.

If $\mathbb{C}_{\overline{\text{Ł}}}$ does not satisfy any of the conditions of Lemma 17, then there is a $k_1 \in \omega$ such that each $\mathbf{A} \in \mathbb{C}_{\overline{\text{Ł}}}$ has at most k_1 Ł-components. $\mathbb{C}_{\overline{\text{Ł}}}$ then generates a proper subvariety of $\mathbb{SBL}$.

We may conclude now that, for either of the classes $\mathbb{C}_{\text{Ł}}$ and $\mathbb{C}_{\overline{\text{Ł}}}$, the situation when the number of Ł-components in its members is unbounded has been addressed by the above lemmata, and it remains to address the case when there is an upper bound on the number of Ł-components in any of its members.

In the next section, we shall rely upon the assumption of such an upper bound. We stress that the two classes $\mathbb{C}_{\text{Ł}}$ and $\mathbb{C}_{\overline{\text{Ł}}}$ are handled separately, and so are the corresponding upper bound assumptions (that is, we do not assume that there is an upper bound on the number of Ł-components in all members of $\mathbb{C}$). Indeed, if $\mathbf{Var}(\mathbb{C}_{\text{Ł}}) = \mathbb{BL}$ (i.e., the number of Ł-components in $\mathbb{C}_{\text{Ł}}$ is unbounded), then also $\mathbf{Var}(\mathbb{C}) = \mathbb{BL}$ no matter what $\mathbb{C}_{\overline{\text{Ł}}}$ may be. However, if $\mathbf{Var}(\mathbb{C}_{\overline{\text{Ł}}}) = \mathbb{SBL}$ (i.e., the number of Ł-components in $\mathbb{C}_{\overline{\text{Ł}}}$ is unbounded), then $\mathbb{SBL} \subseteq \mathbf{Var}(\mathbb{C}) \subseteq \mathbb{BL}$, where the first inclusion is proper iff $\mathbb{C}_{\text{Ł}}$ is nonempty; this interesting case deserves investigation. Note also that it follows from Theorem 9 that the second inclusion in the above formula is proper unless $\mathbf{Var}(\mathbb{C}_{\text{Ł}}) = \mathbb{BL}$.

4 Finitely many Ł-components

In this section we present the core of our proof. We assume a class of canonical BL-expressions $\mathbb{C}$ is given. We maintain the partition of $\mathbb{C}$ into $\mathbb{C}_{\text{Ł}}$ and $\mathbb{C}_{\overline{\text{Ł}}}$, depending on the nature of the first component, and we handle the two cases separately (though using the same method).

In either case, we rely on the assumption, developed at the close of the previous section, of an existing finite upper bound on the number of Ł-components, in all members of the class $\mathbb{C}_{\text{Ł}}$ (the class $\mathbb{C}_{\overline{\text{Ł}}}$). In either case, we show how to find a *finite* class of canonical BL-expressions generating the same variety.

We also maintain the refined partition of the two classes above into $\mathbb{C}^i_{\text{Ł}}$ ($\mathbb{C}^i_{\overline{\text{Ł}}}$ resp.), where i is a natural number giving the number of Ł-components in all members of the class. Importantly, by our assumption, i only runs up to a certain k_0 (k_1 resp.), so in either case the partition is finite. Relying on Lemma 12, for each $0 \leq i \leq k_0$ and the class $\mathbb{C}^i_{\text{Ł}}$, we seek to find a finite class $(\mathbb{C}^i_{\text{Ł}})'$ of canonical BL-expressions s. t. $\mathbf{Var}(\mathbb{C}^i_{\text{Ł}}) = \mathbf{Var}((\mathbb{C}^i_{\text{Ł}})')$. (If the class $\mathbb{C}^i_{\text{Ł}}$ is finite, or even empty, we need not do anything.) Analogously for $\mathbb{C}^i_{\overline{\text{Ł}}}$.

We are now prepared to lay out our strategy for the proof and to give some key statements. The following relation $\preceq$ on canonical BL-expressions will play a key part.

Definition 18. *For canonical* BL-*expressions* $\mathbf{A}$ *and* $\mathbf{B}$*, let* $\mathbf{A} \preceq \mathbf{B}$ *iff* $\mathbf{Var}(\mathbf{A}) \subseteq \mathbf{Var}(\mathbf{B})$*.*

According to Theorem 9, for any two canonical BL-expressions $\mathbf{A}$ and $\mathbf{B}$ we also have $\mathbf{A} \preceq \mathbf{B}$ iff $\mathbf{Fin}(\mathbf{A}) \subseteq \mathbf{Fin}(\mathbf{B})$ iff $\mathbf{A}$ is partially embeddable to $\mathbf{B}$.

The relation $\preceq$ is clearly reflexive and transitive (on any class of algebras in a fixed language); it follows from Theorem 9 that it is antisymmetric on canonical BL-expressions, hence it is a partial order on these.

Our proof is based on the following simple fact: for any two canonical BL-expressions $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A} \preceq \mathbf{B}$ iff $\mathbf{Var}(\mathbf{A}) \subseteq \mathbf{Var}(\mathbf{B})$ iff $\mathbf{Var}(\{\mathbf{A},\mathbf{B}\}) = \mathbf{Var}(\mathbf{B})$. In particular, if $\mathbf{A} \preceq \mathbf{B}$, then the variety generated by the class $\{\mathbf{A},\mathbf{B}\}$ depends solely on $\mathbf{B}$. Therefore, given a class $\mathbb{C}$ of canonical BL-expressions, when investigating the variety $\mathbf{Var}(\mathbb{C})$, we may forget about those elements of $\mathbb{C}$ that are $\preceq$-bounded from above by another element of $\mathbb{C}$.

This observation does not really solve our problem, but it does indicate a path toward solving it, through turning our attention to the behaviour of $\preceq$. It turns out that the relation $\preceq$ is rather well-behaved on each $\mathbb{L}^i_{\text{Ł}}$ (and each $\mathbb{L}^i_{\overline{\text{Ł}}}$), and using its properties, it is not difficult to find, for each $\mathbb{C}^i_{\text{Ł}} \subseteq \mathbb{L}^i_{\text{Ł}}$ (where $1 \leq i \leq k_0$) and each $\mathbb{C}^i_{\overline{\text{Ł}}} \subseteq \mathbb{L}^i_{\overline{\text{Ł}}}$ (where $i \leq k_1$), a finite counterpart generating the same variety. The rest of this section will be devoted to constructing such counterparts.

If successful, the construction will indeed solve the problem: in view of Lemma 12, if $\mathbf{Var}(\mathbb{C}^i_{\text{Ł}}) = \mathbf{Var}((\mathbb{C}^i_{\text{Ł}})')$ for each $1 \leq i \leq k_0$, then $\mathbf{Var}(\mathbb{C}_{\text{Ł}}) = \mathbf{Var}(\mathbb{C}'_{\text{Ł}})$; if each $(\mathbb{C}^i_{\text{Ł}})'$ is finite, then also $\mathbb{C}'_{\text{Ł}} = \bigcup_{1\leq i\leq k_0}(\mathbb{C}^i_{\text{Ł}})'$ is finite. Analogously for $\mathbb{C}_{\overline{\text{Ł}}}$, k_1, and $\mathbb{C}'_{\overline{\text{Ł}}}$.

We define $\emptyset \prec \mathbf{A}$ for any canonical BL-expression $\mathbf{A}$. The symbol $\emptyset$ can be regarded as an empty expression, corresponding to a trivial algebra $\{1\}$. In particular, $\emptyset \oplus \mathbf{A} = \mathbf{A} \oplus \emptyset = \mathbf{A}$ for any canonical BL-expression $\mathbf{A}$, and $\emptyset \oplus \emptyset = \emptyset$.

Fix an arbitrary $k \in \omega$. For a canonical BL-expression $\mathbf{A} \in \mathbb{L}^k$, we may write

$$\mathbf{A} = \mathbf{A}_0 \oplus \text{Ł} \oplus \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{k-1} \oplus \text{Ł} \oplus \mathbf{A}_k$$

where each $\mathbf{A}_j$, $j \leq k$ is without Ł-components, i.e., it is either the empty sum $\emptyset$, or a finite ordinal sum of G's and Π's, or $\omega\Pi$. Taking each index $j \leq k$ as an argument (and ignoring the Ł-components), one may think of the expressions in $\mathbb{L}^k$ as *functions* on a finite domain $[k] = \{0, \ldots, k\}$, where the function value on an argument $j \leq k$ is the expression $\mathbf{A}_j$. Needless to say, for $\mathbf{A} \in \mathbb{L}^k$ we have $\mathbf{A} \in \mathbb{L}^k_{\text{Ł}}$ iff $\mathbf{A}_0 = \emptyset$.

Theorem 19. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{L}^k_{\overline{\text{Ł}}}$, where $\mathbf{A} = \mathbf{A}_0 \oplus \text{Ł} \oplus \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{k-1} \oplus \text{Ł} \oplus \mathbf{A}_k$ and $\mathbf{B} = \mathbf{B}_0 \oplus \text{Ł} \oplus \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_{k-1} \oplus \text{Ł} \oplus \mathbf{B}_k$. Then $\mathbf{A} \preceq \mathbf{B}$ iff, for each $j \leq k$, $\mathbf{A}_j \preceq \mathbf{B}_j$. In other words, $\preceq$ on $\mathbb{L}^k_{\overline{\text{Ł}}}$ is the product order of $\preceq$ on $\mathbb{L}^0_{\overline{\text{Ł}}}$ with $k+1$ factors (the first factor considered without $\emptyset$). Analogously, $\preceq$ on $\mathbb{L}^k_{\text{Ł}}$ is the product order of $\preceq$ on $\mathbb{L}^0_{\text{Ł}}$ with k factors.*

Proof. We perform the proof for $\mathbb{L}^k_{\overline{\text{Ł}}}$; for $\mathbb{L}^k_{\text{Ł}}$ it is analogous, the only difference being that each expression $\mathbf{A} \in \mathbb{L}^k_{\text{Ł}}$ starts with an Ł, so $\mathbf{A}_0$ is $\emptyset$, hence the number of factors decreases by one, while for $\mathbb{L}^k_{\overline{\text{Ł}}}$, the value in the first factor is always distinct from $\emptyset$.

Observe that one cannot state the result for $\mathbb{L}^k$ as a whole. Indeed, consider the sums $\mathbf{A} = \text{Ł} \oplus \Pi$ and $\mathbf{B} = \Pi \oplus \text{Ł} \oplus \Pi$, both in $\mathbb{L}^1$. Certainly one has $\mathbf{A}_i \preceq \mathbf{B}_i$ for $i = 0, 1$, but it is not true that $\mathbf{A} \preceq \mathbf{B}$, because $\mathbf{B}$ is an SBL-algebra, while $\mathbf{A}$ is not.

First let $\mathbf{A} \preceq \mathbf{B}$, i.e., using Theorem 9, $\mathbf{Fin}(\mathbf{A}) \subseteq \mathbf{Fin}(\mathbf{B})$. For any $j \leq k$, we want to show $\mathbf{A}_j \preceq \mathbf{B}_j$. Fix j. If $\mathbf{A}_j$ is the empty sum $\emptyset$, the statement holds by definition (in such a case, our assumptions imply $j > 0$ for the case of $\mathbb{L}^k_{\overline{\text{Ł}}}$). If $\mathbf{A}_j$ is a nonempty sum in $\mathbb{L}^0$, we show $\mathbf{Fin}(\mathbf{A}_j) \subseteq \mathbf{Fin}(\mathbf{B}_j)$, which yields the desired statement. Let $\mathbf{C} \in \mathbf{Fin}(\mathbf{A}_j)$. Then for some $\mathbf{C}_0, \ldots, \mathbf{C}_{j-1}, \mathbf{C}_{j+1}, \ldots, \mathbf{C}_k$, where $\mathbf{C}_i = \emptyset$ whenever $\mathbf{A}_i = \emptyset$, otherwise $\mathbf{C}_i \in \mathbf{Fin}(\mathbf{A}_i)$, $i = 0, \ldots, j-1, j+1, \ldots, k$, we have that $\mathbf{C}_0 \oplus \text{Ł} \oplus \ldots \mathbf{C}_{j-1} \oplus \text{Ł} \oplus \mathbf{C} \oplus \text{Ł} \oplus \mathbf{C}_{j+1} \oplus \cdots \oplus \text{Ł} \oplus \mathbf{C}_k$ is in $\mathbf{Fin}(\mathbf{A})$, and hence also in $\mathbf{Fin}(\mathbf{B})$. Then certainly $\mathbf{C} \in \mathbf{Fin}(\mathbf{B}_j)$.

For the converse implication, assume $\mathbf{A}_j \preceq \mathbf{B}_j$ for each $j \leq k$, so $\mathbf{A}_j$ is partially embeddable into $\mathbf{B}_j$ for each $j \leq k$. Then certainly $\mathbf{A}$ is partially embeddable into $\mathbf{B}$ (component-wise), and hence, using Theorem 9, $\mathbf{A} \preceq \mathbf{B}$. □

At this point, it will be illuminating to look at the properties of $\preceq$ on the class $\mathbb{L}^0$ (i.e., canonical BL-expressions without Ł-components).

4.1 A case study: no Ł-components

The elements of $\mathbb{L}^0$ are canonical BL-expressions among the empty sum $\emptyset$, finite ordinal sums of G- and Π-components, and the expression $\omega\Pi$.

Lemma 20. *(i) For any $\mathbf{A} \in \mathbb{L}^0$, we have $\emptyset \preceq \mathbf{A} \preceq \omega\Pi$, so $\omega\Pi$ is the top element of $\mathbb{L}^0$ w.r.t. $\preceq$, while $\emptyset$ is the bottom element.*

(ii) If $\mathbf{A}, \mathbf{B} \in \mathbb{L}^0$ are finite sums of G- *and* Π-*components, then $\mathbf{A} \preceq \mathbf{B}$ iff $\mathbf{A}$ is a subsum of $\mathbf{B}$.*

(iii) Each $\preceq$-chain in $\mathbb{L}^0$ is ordered by the ordinal $\omega \cup \{\omega\}$ or one of its elements.

Proof. (i) $\emptyset$ is the bottom element by definition. As for $\omega\Pi$ being top, the fact is easily observed using a partial embedding argument, or in another guise, the easy fact that if $\mathbf{A}$ is any finite sum of G- and Π-components (i.e., of 2- and C-components), then $\mathbf{Fin}(\mathbf{A}) \subseteq \mathbf{Fin}(\omega\Pi)$.

(ii) In the given equivalence, the right-to-left implication is clear, as the property of being a subsum yields a componentwise partial embedding, and one can use Theorem 9.

Conversely, assume $\mathbf{A} \preceq \mathbf{B}$. If $\mathbf{A}$ is the empty sum, the statement holds by definition. Let us now assume $\mathbf{A} = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{m_{\mathbf{A}}}$, $\mathbf{B} = \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_{m_{\mathbf{B}}}$ are non-empty sums, where $m_{\mathbf{A}}$, $m_{\mathbf{B}}$ denote the respective numbers of their components ('components' here refers to G- and Π-components). From the assumption we have $\mathbf{Fin}(\mathbf{A}) \subseteq \mathbf{Fin}(\mathbf{B})$ by Theorem 9. In order to show that $\mathbf{A}$ is a subsum of $\mathbf{B}$, let us construct a 1-1 order-preserving mapping f from the G- and Π-components of $\mathbf{A}$ into the G- and Π-components of $\mathbf{B}$ (i.e., to each index $1 \leq i \leq m_{\mathbf{A}}$, we assign an index $1 \leq f(i) \leq m_{\mathbf{B}}$ in such a way that $1 \leq i_1 < i_2 \leq m_{\mathbf{A}}$ implies $1 \leq f(i_1) < f(i_2) \leq m_{\mathbf{B}}$), in such a way that $\mathbf{A}_i$ is a G-component iff so is $\mathbf{B}_{f(i)}$.

Consider a (Wajsberg hoop) subsum $\mathbf{S}$ of $\mathbf{A}$ such that it contains all Π-components of $\mathbf{A}$ (each Π-component appearing as $2 \oplus \mathrm{C}$ in $\mathbf{S}$), and instead of each G-component of $\mathbf{A}$, $\mathbf{S}$ contains a connected finite subsum $2 \oplus 2 \oplus \cdots \oplus 2$, where the number of 2's is some $m_0 > m_{\mathbf{B}}$. Thus we can write $\mathbf{S} = \mathbf{S}_1 \oplus \cdots \oplus \mathbf{S}_{m_{\mathbf{A}}}$, where for each $1 \leq i \leq m_{\mathbf{A}}$, $\mathbf{S}_i$ is a $2 \oplus \mathrm{C}$ whenever $\mathbf{A}_i$ is a Π, while $\mathbf{S}_i$ is a $\bigoplus_{k=1}^{m_0} 2$ whenever $\mathbf{A}_i$ is a G. It is elementary to observe that $\mathbf{S} \in \mathbf{Fin}(\mathbf{A})$.

Since we assume $\mathbf{S} \in \mathbf{Fin}(\mathbf{B})$, by definition there is a 1-1 order-preserving mapping from the hoop components in $\mathbf{S}$ to the hoop components in $\mathbf{B}$, thus in particular, from the C-components in $\mathbf{S}$ to the C-components in $\mathbf{B}$ (recall that $\mathbf{B}$ has no Ł-components), and hence, from the Π-components in $\mathbf{S}$ (and in $\mathbf{A}$) to the Π-components in $\mathbf{B}$; this yields the value $f(i)$ whenever $\mathbf{A}_i$ is a Π-component of $\mathbf{A}$. It remains to show that for each i such that $\mathbf{A}_i$ is a G-component, there is a suitable $f(i)$ such that $\mathbf{B}_{f(i)}$ is a G-component. If $\mathbf{A}_i$ is a G-component, then $\mathbf{S}_i$ is sum of m_0 copies of 2; each of these corresponds to some 2 in $\mathbf{B}$. Now some of the copies of 2 in $\mathbf{B}$ must be a part of a G-component, because $\mathbf{B}$ has at most $m_{\mathbf{B}}$ Π-components and $m_0 > m_{\mathbf{B}}$. Choosing any such G-component in $\mathbf{B}$ yields the value $f(i)$ for each $\mathbf{A}_i$ that is a G-component.

(iii) A finite chain satisfies the statement. Any infinite chain has a first element by (ii) and thus, any infinite chain include a subchain of type ω; but then, using (i) and (ii), the only expression in $\mathbb{L}^0$ that can sit atop an infinite $\preceq$-chain is $\omega\Pi$; hence, an infinite chain in $\mathbb{L}^0$ is ordered by either ω or $\omega \cup \{\omega\}$. □

We employ Higman's theorem in order to show that $\preceq$ on $\mathbb{L}^0$ has no infinite antichains. Recall that a quasiorder (i.e., a binary relation that is reflexive and transitive) is a well quasiorder (w.q.o.) whenever it has no infinite descending chains and no infinite antichains. A well partial order (w.p.o.) is a w.q.o. that is antisymmetric. ([3] provides an exposition of basic properties of well-quasiordered sets).

If Σ is an alphabet, Σ^+ denotes finite words over Σ. Assume $\leq$ is a quasi-order on Σ, and define, for $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_m)$ in Σ^+, the quasi-order $\leq^+$ by $(u_1, \ldots, u_m) \leq^+ (v_1 \ldots, v_n)$ iff there are $1 \leq l_1 < \cdots < l_m \leq n$ such that for each $1 \leq i \leq m$, we have $u_i \leq v_{l_i}$.

Theorem 21. (Higman) *If $(\Sigma, \leq)$ is a w.q.o., then so is $(\Sigma^+, \leq^+)$.*

We apply this theorem to $\Sigma = \{\mathrm{G}, \Pi\}$ with the ordering $\preceq$ (thus we have two incomparable elements). Thus $\preceq^+$ is just the ordering given by the property of being a subsequence, and on finite sums, it coincides with $\preceq$.

Theorem 22. *$\preceq$ on $\mathbb{L}^0$ is a well partial order.*

Proof. We know $\preceq$ is a partial order on $\mathbb{L}^0$. By Lemma 20, for finite sums of G- and Π-components $\mathbf{A}, \mathbf{B} \in \mathbb{L}^0$, we have $\mathbf{A} \preceq \mathbf{B}$ iff $\mathbf{A}$ is a subsum of $\mathbf{B}$. By Theorem 21, $\preceq$ is a well partial order on the finite sums in $\mathbb{L}^0$, and hence also on $\mathbb{L}^0$ as such: the presence of $\emptyset$ as the bottom and $\omega\Pi$ as the top element makes no difference. □

Hence there are no infinite $\preceq$-antichains in $\mathbb{L}^0$. Note that for each $n \in \omega$, $\mathbb{L}^0$ contains a $\preceq$-antichain of cardinality greater than n. Indeed, for arbitrary $n \in \omega$, consider the canonical BL-expressions with exactly n Π-components and one G-component, of the following types:

$\mathbf{A}_0 = \mathrm{G} \oplus n\Pi$
$\mathbf{A}_1 = \Pi \oplus \mathrm{G} \oplus (n-1)\Pi$
$\mathbf{A}_2 = \Pi \oplus \Pi \oplus \mathrm{G} \oplus (n-2)\Pi$
$\ldots$
$\mathbf{A}_n = n\Pi \oplus \mathrm{G}$

This defines $n+1$ canonical BL-expressions in $\mathbb{L}^0$. For $i \neq j \leq n$ we have $\mathbf{A}_i \not\preceq \mathbf{A}_j$, because (as is easily observed) $\mathbf{Fin}(\mathbf{A}_i) \not\subseteq \mathbf{Fin}(\mathbf{A}_j)$.

Now we examine the role of $\preceq$-chains. In view of the previous discussion, we shall be particularly interested in $\preceq$-chains without a top element.

Lemma 23. *Let $\{\mathbf{A}_i\}_{i\in I}$ be a nonempty $\preceq$-chain in $\mathbb{L}^0$. Then*

(i) $\{\mathbf{A}_i\}_{i\in I}$ *has a $\preceq$-supremum in* $\mathbb{L}^0$
(ii) $\mathbf{Var}(\{\mathbf{A}_i\}_{i\in I}) = \mathbf{Var}(\sup\{\mathbf{A}_i\}_{i\in I})$

Proof. (i) Follows from Lemma 20.
(ii) If $\{\mathbf{A}_i\}_{i\in I}$ has a top element, then the statement follows from the definition of $\preceq$. If $\{\mathbf{A}_i\}_{i\in I}$ has no top element, then $\sup\{\mathbf{A}_i\}_{i\in I} = \omega\Pi$. We rely on Theorem 9: on the one hand, it is not difficult to see that $\mathbf{Fin}(\mathbf{A}_i) \subseteq \mathbf{Fin}(\omega\Pi)$ for each $i \in I$, because $\omega\Pi$ has an infinite alternating sequence of 2's and C's. We prove the converse, i.e., $\mathbf{Fin}(\omega\Pi) \subseteq \mathbf{Fin}(\{\mathbf{A}_i\}_{i\in I}) = \bigcup_{i\in I}\mathbf{Fin}(\mathbf{A}_i)$. If $\mathbf{C} \in \mathbf{Fin}(\omega\Pi)$, then $\mathbf{C}$ is a finite sum of hoops of type C, 2 of cardinality n_0; since $\{\mathbf{A}_i\}_{i\in I}$ is infinite, for each n there is an $i \in I$ s. t. $\mathbf{A}_i$ has more than n Π-components; and so if $n \geq n_0$, we get $\mathbf{C} \in \mathbf{Fin}(\mathbf{A}_i)$. □

4.2 The general setting: k Ł-components

We come back to investigating the properties of $\preceq$ on $\mathbb{L}^k$ for an arbitrary but fixed $k \in \omega$. By previous results in this paper, $\preceq$ on $\mathbb{L}^0$ is a w.p.o. We now employ a well-known statement to obtain the same for $\preceq$ on $\mathbb{L}^k$ (see [3] for an exposition).

Theorem 24. *If $(L_1, \leq_1)$, $(L_2, \leq_2)$ are w.p.o.'s, so is their product $(L_1, \leq_1) \times (L_2, \leq_2)$.*

Corollary 25. *$\mathbb{L}^k_{\text{Ł}}$ and $\mathbb{L}^k_{\overline{\text{Ł}}}$ have no infinite $\preceq$-antichains.*

Proof. Theorem 19 states that $(\mathbb{L}^k_{\text{Ł}}, \preceq)$ is the k-th power of $(\mathbb{L}^0, \preceq)$ (there are k factors as $\mathbf{A}_0$ is always the empty expression $\emptyset$), while $(\mathbb{L}^k_{\overline{\text{Ł}}}, \preceq)$ is $(\mathbb{L}^0 \setminus \{\emptyset\}, \preceq) \times ((\mathbb{L}^0, \preceq)^k)$ ($\mathbf{A}_0$ is always non-empty). Then Theorem 24 implies that both the above products are w.p.o.'s. The statement follows. □

Lemma 26. *Let $\{\mathbf{A}_i\}_{i\in I}$ be a $\preceq$-chain in $\mathbb{L}^k_{\overline{\text{Ł}}}$. Then $\sup\{\mathbf{A}_i\}_{i\in I}$ is defined in $\mathbb{L}^k_{\text{Ł}}$, and $\mathbf{Var}(\{\mathbf{A}_i\}_{i\in I}) = \mathbf{Var}(\sup\{\mathbf{A}_i\}_{i\in I})$. Analogously for $\mathbb{L}^k_{\overline{\text{Ł}}}$.*

Proof. The existence of a supremum of the chain $\{\mathbf{A}_i\}_{i\in I}$ in $\mathbb{L}^k_{\text{Ł}}$ follows from Theorem 19: we have

$$\sup\{\mathbf{A}_i\}_{i\in I} = \sup\{(\mathbf{A}_i)_0\}_{i\in I} \oplus \text{Ł} \oplus \cdots \oplus \text{Ł} \oplus \sup\{(\mathbf{A}_i)_k\}_{i\in I}.$$

For the second part, clearly $\mathbf{Var}(\{\mathbf{A}_i\}_{i\in I}) \subseteq \mathbf{Var}(\sup\{\mathbf{A}_i\}_{i\in I})$ since for each $i \in I$, $\mathbf{Var}(\mathbf{A}_i) \subseteq \mathbf{Var}(\sup(\{\mathbf{A}_i\}_{i\in I}))$ by definition of $\preceq$. On the other hand, recall that by Theorem 9, $\mathbf{Var}(\sup\{\mathbf{A}_i\}_{i\in I}) \subseteq \mathbf{Var}(\{\mathbf{A}_i\}_{i\in I})$ iff $\sup\{\mathbf{A}_i\}_{i\in I}$ is partially embeddable into the class $\{\mathbf{A}_i\}_{i\in I}$. Using Lemma 23, for each $j \in [k]$, $\sup\{(\mathbf{A}_i)_j\}_{i\in I}$ is partially embeddable into the class $\{(\mathbf{A}_i)_j\}_{i\in I}$. If $A^\star$ is a finite partial subalgebra of $\sup\{\mathbf{A}_i\}_{i\in I}$, then for each j such that the intersection of $A^\star$ with the domain of $\sup\{(\mathbf{A}_i)_j\}_{i\in I}$ is nonempty (some $(A^\star)^j$), there is an i_j such that $(A^\star)^j$ is embeddable into the algebra $(\mathbf{A}_{i_j})_j$. Hence, setting $i_{\max} = \max\{i_j \mid (A^\star)^j \text{ is nonempty}\}$, the partial algebra $A^\star$ is embeddable into $\mathbf{A}_{i_{\max}}$ componentwise. This concludes the proof.

The proof for $\mathbb{L}^k_{\text{Ł}}$ is analogous, considering $1 \leq j \leq k$. $\square$

Let $\mathbb{K} \subseteq \mathbb{L}^k_{\text{Ł}}$ (or $\mathbb{K} \subseteq \mathbb{L}^k_{\overline{\text{Ł}}}$) be a nonempty class of canonical BL-expressions. Let $\{\mathbf{A}_i\}_{i\in I}$ be a $\preceq$-chain in $\mathbb{K}$. We say that $\{\mathbf{A}_i\}_{i\in I}$ is *maximal* in $\mathbb{K}$ iff no element of $\mathbb{K}$ can be added on top of it, i.e., there is no $\mathbf{B} \in \mathbb{K}$ such that $\mathbf{A}_i \prec \mathbf{B}$ for each $i \in I$. Obviously, each $\mathbf{A} \in \mathbb{K}$ belongs to some maximal chain.

Lemma 27. *Let $\mathbb{K} \subseteq \mathbb{L}^k_{\text{Ł}}$ (or $\mathbb{K} \subseteq \mathbb{L}^k_{\overline{\text{Ł}}}$) be a nonempty class of canonical* BL*-expressions. Let $\{\mathbf{A}_i\}_{i\in I}$, $\{\mathbf{B}_{i'}\}_{i'\in I'}$ be two maximal $\preceq$-chains in $\mathbb{K}$. If $\{\mathbf{B}_{i'}\}_{i'\in I'}$ has a top element in $\mathbb{K}$, then* $\sup(\{\mathbf{A}_i\}_{i\in I}) \not\prec \sup(\{\mathbf{B}_{i'}\}_{i'\in I'})$.

Proof. If $\mathbf{B} \in \mathbb{K}$ is the top element of $\{\mathbf{B}_{i'}\}_{i'\in I'}$ (so $\mathbf{B} = \sup(\{\mathbf{B}_{i'}\}_{i'\in I'})$), then assuming $\sup(\{\mathbf{A}_i\}_{i\in I}) \prec \mathbf{B}$ contradicts the assumption that $\{\mathbf{A}_i\}_{i\in I}$ is a maximal chain in $\mathbb{K}$, as $\mathbf{B}$ can be added. $\square$

Lemma 28. *Let $\mathbb{K} \subseteq \mathbb{L}^k_{\overline{\text{Ł}}}$ be a nonempty class of canonical* BL*-expressions. Let $\{\mathbf{B}_i\}_{i\in I}$ be a $\preceq$-chain without a top element in $\mathbb{K}$. Then there is a $j \in [k]$ such that for each $i \in I$, $(\mathbf{B}_i)_j$ is either empty or a finite sum of* G*- and* Π*-components, whereas* $(\sup(\{\mathbf{B}_i\}_{i\in I}))_j = \omega\Pi$. *Analogously for $\mathbb{K} \subseteq \mathbb{L}^k_{\text{Ł}}$.*

Proof. If a chain has no top element in $\mathbb{K}$, then it is infinite. By pigeonhole principle, there is a $j \in [k]$ such that $\{(\mathbf{B}_i)_j\}_{i\in I}$ is infinite; in particular, it is an infinite $\preceq$-chain in $\mathbb{L}^0$. If this $\preceq$-chain has $\preceq$-greatest element, then this element is an $\omega\Pi$ and there is an $i_j \in I$ s.t. $(\mathbf{B}_{i_j})_j = \omega\Pi$. Since $\{\mathbf{B}_i\}_{i\in I}$ has no top element, there are infinitely many elements $\preceq$-greater than $\mathbf{B}_{i_j}$; one may iterate the above for the remaining $[k] \setminus \{j\}$. Since k is finite, after finitely many steps one arrives at a j s.t. $\{(\mathbf{B}_i)_j\}_{i\in I}$ is infinite without a $\preceq$-greatest element, i.e., each $(\mathbf{B}_i)_j$ is either empty or a finite sum of G- and Π-components. This completes the proof. $\square$

Now we are ready to prove the main statement. Assume a given class $\mathbb{C} \subseteq \mathbb{L}^k_{\overline{\text{Ł}}}$ of canonical BL-expressions. (If, on the other hand, $\mathbb{C} \subseteq \mathbb{L}^k_{\text{Ł}}$, then $\mathbf{A}_0$ is empty for all $\mathbf{A} \in \mathbb{C}^k_{\text{Ł}}$. The proof will be analogous). We need to find a finite $\mathbb{C}' \subseteq \mathbb{L}^k_{\overline{\text{Ł}}}$ such that $\mathbf{Var}(\mathbb{C}) = \mathbf{Var}(\mathbb{C}')$. Let us denote $\mathbb{C}_0 = \mathbb{C}$.

Definition 29. *Let $\mathbb{C}_0 \subseteq \mathbb{L}^k_{\overline{\text{Ł}}}$. For $n \in \omega$, define*

$$\mathbb{C}_{n+1} = \{\mathbf{A} \mid \mathbf{A} = \sup\{\mathbf{A}_i\}_{i\in I} \text{ for some maximal chain } \{\mathbf{A}_i\}_{i\in I} \text{ in } \mathbb{C}_n\}$$

Theorem 30. *If the classes $\mathbb{C}_n$, $n \in \omega$, are defined as above (in particular, $\mathbb{C}_0 \subseteq \mathbb{L}^k_{\overline{\text{Ł}}}$), then*

(i) $\mathbf{Var}(\mathbb{C}_n) = \mathbf{Var}(\mathbb{C}_{n+1})$ *for each $n \in \omega$*

(ii) There is an $n \leq k + 2$ such that

- *(a)* $\mathbb{C}_n = \mathbb{C}_{n+1}$
- *(b)* $\mathbb{C}_n$ *is finite*

Proof. (i) Fix $n \in \omega$. Let $Q_n \subseteq \omega$ be an enumeration of maximal chains in $\mathbb{C}_n$, so $q \in Q_n$ iff $\{A_i\}_{i\in I_q}$ is a maximal chain in $\mathbb{C}_n$. In particular,

$$\mathbb{C}_n = \bigcup_{q\in Q_n} \{\mathbf{A}_i\}_{i\in I_q}$$

For each $q \in Q_n$, let $\mathbf{A}_q = \sup\{\mathbf{A}_i\}_{i\in I_q}$. (We remark that, while $\mathbf{A}_q \in \mathbb{L}^k_{\overline{\text{Ł}}}$, it need not be the case that $\mathbf{A}_q \in \mathbb{C}_n$.) By Lemma 26, $\mathbf{Var}(\{\mathbf{A}_i\}_{i\in I_q}) = \mathbf{Var}(\sup\{\mathbf{A}_i\}_{i\in I_q})$. Therefore, by Lemma 12, we have $\mathbf{Var}(\mathbb{C}_n) = \mathbf{Var}(\{\mathbf{A}_q \mid \mathbf{A}_q = \sup\{\mathbf{A}_i\}_{i\in I_q}\}_{q\in Q_n}) = \mathbf{Var}(\mathbb{C}_{n+1})$.

(ii) First observe that, for any $n \in \omega$, if $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n+1}$, then $\mathbf{A} \prec \mathbf{B}$ implies $\mathbf{B} \notin \mathbb{C}_n$. This follows directly from Lemma 27. For each $n \in \omega$, the expressions in $\mathbb{C}_{n+1} \setminus \mathbb{C}_n$ are referred to as the 'new expressions in $\mathbb{C}_{n+1}$', and by definition they are exactly the suprema of maximal chains without a top element in $\mathbb{C}_n$.

We prove the following statement: if, for $n \in \omega$, we have $\mathbf{A} \in \mathbb{C}_{n+1} \setminus \mathbb{C}_n$, then, for at least $n+1$ distinct elements $j \in [k]$, we have $(\mathbf{A})_j = \omega\Pi$. The statement will be proved by induction on n.

For $n = 0$ the statement holds: the new expressions in $\mathbb{C}_1$ are the suprema of infinite chains $\{\mathbf{A}_i\}_{i\in I}$ without top elements in $\mathbb{C}_0$; because each of these chains is infinite, there is a $j \in [k]$ (not necessarily unique) s.t. $(\sup\{\mathbf{A}_i\}_{i\in I})_j = \omega\Pi$. Now let us assume the statement for some $n \in \omega$ and prove it for $n+1$ on this assumption. If $\mathbb{C}_{n+2} \setminus \mathbb{C}_{n+1}$ is non-empty, then $\mathbb{C}_{n+1}$ contains a chain without a top element (in particular, an infinite chain). In any chain in $\mathbb{C}_{n+1}$ with at least two elements, all of its elements except the bottom one are expressions in $\mathbb{C}_{n+1} \setminus \mathbb{C}_n$, because for any two algebras $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{n+1}$, we have $\mathbf{A} = \sup(\{\mathbf{A}_i\}_{i \in I})$, $\mathbf{B} = \sup(\{\mathbf{B}_{i'}\}_{i' \in I'})$, where $\mathbf{A}_i, \mathbf{B}_{i'} \in \mathbb{C}_n$, and then the assumption $\mathbf{A} \prec \mathbf{B}$ implies $\{\mathbf{B}_{i'}\}_{i' \in I'}$ is a chain in $\mathbb{C}_n$ without a top element by Lemma 27. Therefore, for a maximal chain $\{\mathbf{B}_i\}_{i \in I}$ without a top element in $\mathbb{C}_{n+1}$, the induction assumption applies: for at least $n+1$ distinct elements j of $[k]$, for each of the elements $\mathbf{B}_i$ of the chain except the bottom, we have $(\mathbf{B}_i)_j = \omega\Pi$. Using Lemma 28, on the other hand there is a $j' \in [k]$ s.t. each $(\mathbf{B}_i)_{j'}$ is $\emptyset$ or a finite sum of G's and Π's, while $(\mathbf{B})_{j'} = \omega\Pi$. In other words, the supremum of this (maximal, infinite, topless) chain adds a new $\omega\Pi$ in the j'-th factor. This closes the induction proof.

Thus for any $n \geq 1$, any nontrivial chain in $\mathbb{C}_n$ has the property that all of its elements distinct from the bottom feature the value $\omega\Pi$ in at least n different indices $j \in [k]$. $\mathbb{C}_{k+1}$ contains no chains without a top element (each chain in $\mathbb{C}_{k+1}$ is finite, of length at most 2). It follows that $\mathbb{C}_{k+2}$ is an antichain, and hence, it is finite by Corollary 25. □

We set $\mathbb{C}' = \mathbb{C}_{k+2}$. This concludes our efforts, since $\mathbb{C}'$ is the finite class of canonical BL-expressions in $\mathbb{L}^k_{\text{Ł}}$ generating the same variety as $\mathbb{C}$.

5 Concluding remarks

The result proferred in this paper extends our knowledge on standard BL-algebras, in showing that the equational theory of any class of standard BL-algebras is contained among the equational theories of finite classes of standard BL-algebras. As a consequence, we may apply results available for the latter to the former. We mention some results due to such applications.

There are countably many subvarieties of $\mathbb{BL}$ that are generated by classes of standard BL-algebras. This is in contrast to the fact that the whole variety $\mathbb{BL}$ has a continuum of distinct subvarieties.

The equational theory of each class of standard BL-algebras is finitely based. This fact is obtained by combining the result in this paper with earlier results of [7] (for subvarieties of $\mathbb{BL}$ given by a standard BL-algebra) and of [9] (which applies to joins of finitely many subvarieties of $\mathbb{BL}$ given by a standard BL-algebra).

The equational theory of each class of standard BL-algebras is coNP-complete. This builds on earlier results of [4] concerning the computational complexity of the equational theory of $\mathbb{BL}$ (and hence, of many single standard BL-algebras) and [13], which extends the coNP-completeness to the equational theory of any standard BL-algebra. The equational theory of each finite class of standard BL-algebras is coNP-complete: the class coNP is closed under finite intersections, while for hardness, the results of [14] apply.

One may wonder about quasivarieties given by standard BL-algebras. As a matter of fact, each quasivariety generated by a class of standard BL-algebras is generated as a variety: i.e., for each class $\mathbb{K}$ of standard BL-algebras, we have $\mathbf{Var}(\mathbb{K}) = \mathbf{QVar}(\mathbb{K})$. This fact follows from [7]. Thus the presented result concerns also quasivarieties: quasivarieties generated by classes of standard BL-algebras are generated by finite classes of standard BL-algebras.

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