Expanding Basic Fuzzy Logic with truth constants for component delimiters

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Abstract

In this paper we investigate the propositional logic of standard algebras for Hájek's Basic Fuzzy Logic BL in a language expanded with propositional constants for the idempotent elements delimiting the Ł-, G-, and Π-components of the algebra. We start from a fixed standard BL-algebra; introduce new propositional constants; present a suitable set of axioms; investigate completeness results; and give some complexity results.

1 Introduction

Expansions of Basic Fuzzy Logic, or indeed of many other systems of many-valued logic, with propositional truth constants have been studied from various points of view. Pavelka [19] defined a deductive system allowing for reasoning with *degrees of truth* via including constants for all truth values into the propositional language, and his completeness result (the so-called Pavelka-style completeness) stated, for each theory and each formula, the equality of the provability degree and the validity degree of the formula with respect to the theory. In [11], Hájek presents a similar completeness result for a simpler system over the standard MV-algebra with constants for $\mathbb{Q}$, with *bookkeeping axioms* added to propositional Łukasiewicz logic ($\overline{r}$ interpreted as $r \in \mathbb{Q}$):

$$\overline{r} \,\&\, \overline{s} \equiv \overline{r * s}$$

$$\overline{r} \to \overline{s} \equiv \overline{r \Rightarrow s}$$

These axioms yield both Pavelka-style completeness and a finite strong completeness with respect to the canonical algebra $[0,1]_{\text{Ł}}$.

The elegant simplicity of bookkeeping axioms motivated a subsequent study of their effect in combination with logics other than Łukasiewicz; cf. [8], [15], [20], [7]. The last reference is particularly relevant for our paper: the authors start from a standard BL-algebra $[0,1]_*$ and introduce propositional constants for a countable subalgebra $C \subset [0,1]_*$; they restrict their attention to cases where the continuous t-norm $*$ is a finite ordinal sum, there is at least one constant inside each non-singleton component of the sum, and constants in Ł-components behave as rationals. They take the logic L for $[0,1]_*$ in the BL-language (which is finitely axiomatizable, cf. [9]), add bookkeeping axioms for C, and study the properties of the resulting deductive system $L(C)$, mainly from a completeness point of view. Interestingly, it turns out that the logic $L(C)$ thus obtained is typically not complete with respect to its canonical algebra, the only unconditional exceptions being Ł(C), G(C), $\Pi(C)$. (Meanwhile, only Ł(C) enjoys the finite strong canonical completeness.)

In this paper, also starting with a fixed standard BL-algebra $[0,1]_*$, we introduce propositional constants for idempotent elements delimiting its non-singleton Ł-, G-, and Π-components. We study the properties of the set of tautologies of the given algebra in the expanded language, focusing on axiomatizations. By way of comparison, the approach of [7] is more general than ours in its initial choice of the set of truth values for the new constants. Interestingly, as to axioms of the desired logic, [7] use the axioms of $[0,1]_*$ in the BL-language and bookkeeping. In this paper, we start with the axioms of BL, bookkeeping, and an additional set of axioms obtained by translating the axioms of the axiomatic extensions of BL into parts of the ordinal sum (this additional set of axioms is thus infinite for infinite ordinal sums). A notable difference is that out axioms yield canonical completeness (though not finite strong canonical completeness). While [7] is a comprehensive study of the outcome of adding bookkeeping axioms to logics of continuous t-norm, our paper is targeted towards canonical completeness.

It is worth stressing here that our approach relies strongly on decomposition of BL-chains (particularly the standard ones) into well-known components. For that reason, it applies best to Basic Logic, where suitable decomposition theorems are available.

A particular case of the general setting of this paper was studied in [14], where only one particular standard BL-algebra $[0,1]_{\text{Ł}\oplus\Pi}$ was considered; the present paper can be viewed as a continuation of [14]. Let us now sketch a broader context for the methods and results. Hájek's paper on *hedges* [12] presents a logic which makes it possible (like it is possible in our system) to express the isomorphism type of the n-th component with a formula, for each $n \in \mathbb{N}$; this provides a simple method of constructing algebras whose tautologies in the expanded language are of arbitrarily high degree of undecidability. A more general setting for these operators was presented in [17]. In [21], the author uses hedges in BL to construct a Gentzen proof system for BL; a recent comprehensive study along these lines is to be found in [4]. In [16], the authors present a way of defininig each continuous t-norm that is a finite ordinal sum inside the logic $\text{Ł}\Pi 1/2$.

This paper is organized as follows. A review of some basic definitions and results is presented in Section 2. In Section 3, the language is expanded with new propositional constants, and bookkeeping axioms, as well as algebraic semantics and general and standard completeness results, are introduced. The axioms and algebraization in this section are a rephrasing of [7] for the case of C being component delimiters, except for the fact that we consider also infinite ordinal sums; we prove the partial embeddability property for this more general case in Lemma 3.6. Section 4 presents new axioms that reflect the isomorphism type of the components, and gives canonical completeness results. Section 5 gives complexity results for the logics. Section 6 points out possible directions for continuation of the topic. Section 7 covers some auxiliary statements.

2 Background

Well-known facts are presented in this section that form a necessary background against which the content of this paper is shaped. The presentation is concise; throughout, references are given to comprehensive works on the topics. Our approach is rather in the style of [11], taking BL as the base logic, relying on Hilbert-style calculi, and understanding logics as sets of formulas.

A propositional language is a set of propositional connectives (propositional constants being taken as nullary connectives). Formulas in a language $\mathcal{L}$ are well-formed strings in an alphabet consisting of a countably infinite set of propositional variables, propositional connectives in $\mathcal{L}$, and parentheses. If $\mathcal{L}$ is a propositional language, a *logic* in the language $\mathcal{L}$ is a set of formulas in $\mathcal{L}$ closed under substitution and the rule of modus ponens. In particular, if $\mathbf{A}$ is an $\mathcal{L}$-algebra, then the *logic of* $\mathbf{A}$ is the set of tautologies of $\mathbf{A}$.

A logic L is *axiomatized* by a set $Ax \subseteq L$ iff each φ in L is provable from the schemata Ax using modus ponens. A theory in a language $\mathcal{L}$ is a set of formulas in $\mathcal{L}$. If T is a theory, φ a formula, and L a logic, all in a language $\mathcal{L}$, we write $T \vdash_L \varphi$ and say T proves φ in L iff φ is provable from the axioms $T \cup L$ using modus ponens.

If L_1 is a logic in a language $\mathcal{L}_1$ and L_2 is a logic in the language $\mathcal{L}_2$, we say L_2 is an axiomatic[1] expansion of L_1 iff $L_1 \subseteq L_2$ (entailing $\mathcal{L}_1 \subseteq \mathcal{L}_2$); if $\mathcal{L}_1 = \mathcal{L}_2$ we speak about extensions rather than expansions. In particular, if $Ax_1 \subseteq Ax_2$ are two sets of formulas, then the logic axiomatized by Ax_2 is an axiomatic expansion of the logic axiomatized by Ax_1. If a logic L_2 in a language $\mathcal{L}_2$ expands a logic L_1 in a language $\mathcal{L}_1$, we say the expansion is *conservative* iff $L_2 \upharpoonright \mathcal{L}_1 = L_1$ (i.e., L_1 is the $\mathcal{L}_1$-fragment of L_2).

Now we define the propositional Basic Fuzzy Logic, denoted BL, which has been introduced and discussed in detail in [11]; see also [10], [1]. The basic language of BL is $\{\overline{0}, \&, \rightarrow\}$, and one defines:

$$
\begin{array}{rll}
\neg\varphi & \text{as} & \varphi \rightarrow \overline{0} \\
\varphi \wedge \psi & \text{as} & \varphi \,\&\, (\varphi \rightarrow \psi) \\
\varphi \vee \psi & \text{as} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\
\varphi \equiv \psi & \text{as} & (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi) \\
\overline{1} & \text{as} & \overline{0} \rightarrow \overline{0}
\end{array}
$$

Definition 2.1. (The Basic Fuzzy Logic BL**)** *The propositional logic* BL *is axiomatized by the following schemata of axioms.*[2]
(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi \,\&\, \psi) \rightarrow \varphi$
(A3) $(\varphi \,\&\, \psi) \rightarrow (\psi \,\&\, \varphi)$
(A4) $(\varphi \,\&\, (\varphi \rightarrow \psi)) \rightarrow (\psi \,\&\, (\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \,\&\, \psi) \rightarrow \chi)$
(A5b) $((\varphi \,\&\, \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $\overline{0} \rightarrow \varphi$
The deduction rule of BL *is modus ponens.*

Łukasiewicz logic Ł is axiomatized by BL plus the axiom schema $\neg\neg\varphi \rightarrow \varphi$. Gödel logic G is axiomatized by BL plus the axiom schema $\varphi \rightarrow \varphi \,\&\, \varphi$. Product logic Π is axiomatized by BL plus the axiom schema $(\varphi \rightarrow \chi) \vee ((\varphi \rightarrow (\varphi \,\&\, \psi)) \rightarrow \psi)$.

Definition 2.2. (BL**-algebra)**
A BL*-algebra is an algebra* $\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *such that:*

[1] Since all extensions/expansions used in this paper are axiomatic (i.e., no new rules are added), the term 'axiomatic' is sometimes omitted.

[2] See [3] for a revised set of axioms.

(i) $\langle A, \wedge, \vee, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *is a bounded lattice*
(ii) $\langle A, *, 1^{\mathbf{A}} \rangle$ *is a commutative monoid*
(iii) for all $x, y, z \in A$, $z \leq (x \Rightarrow y)$ *iff* $x * z \leq y$
(iv) for all $x, y \in A$, $x \wedge y = x * (x \Rightarrow y)$
(v) for all $x, y \in A$, $(x \Rightarrow y) \vee (y \Rightarrow x) = 1^{\mathbf{A}}$
Linearly ordered BL-*algebras are called* BL-chains.

The variety $\mathbb{BL}$ of BL-algebras forms the *equivalent algebraic semantics* of BL (cf. [2]), where $*$ interprets the strong conjunction $\&$, the residuum $\Rightarrow$ interprets the implication $\rightarrow$, $0^{\mathbf{A}}$ interprets the constant $\overline{0}$. Evaluations are defined in the usual way, and we say a formula φ is a *tautology* of an algebra $\mathbf{A}$ (or that it is valid in $\mathbf{A}$), and write $\models_{\mathbf{A}} \varphi$, iff φ yields the value $1^{\mathbf{A}}$ under all evaluations in $\mathbf{A}$. Analogously, for a theory T and a formula φ, we write $T \models_{\mathbf{A}} \varphi$ iff φ yields the value $1^{\mathbf{A}}$ under all $\mathbf{A}$-evaluations satisfying T.

If L is an axiomatic extension of BL, an L-algebra is a BL-algebra validating all formulas in L. Each axiomatic extension of BL corresponds to a subvariety of $\mathbb{BL}$. In particular, the variety of MV-algebras (Gödel algebras, product algebras) corresponds to Łukasiewicz logic (Gödel logic, product logic respectively).

Theorem 2.3. (Strong completeness theorem) [11] *Let* L *be an axiomatic extension of* BL. *Let* $T \cup \{\varphi\}$ *be a set of formulas of* BL. *Then* $T \vdash_L \varphi$ *iff* $T \models_{\mathbf{A}} \varphi$ *for each (linearly ordered)* L*-algebra* $\mathbf{A}$.

BL-algebras whose domain is the real unit interval $[0,1]$ are of special interest as they constitute the *intended semantics* of BL. If $*$ is a continuous t-norm on $[0,1]$ (i.e., a binary operation that is associative, commutative, nondecreasing, satisfies boundary conditions $x * 0 = 0$, $x * 1 = x$ and is continuous), then $[0,1]_* = \langle [0,1], \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$, where $\wedge$, $\vee$ are order-determined and $x \Rightarrow y = \max\{z \,|\, x * z \leq y\}$ (the *residuum* of $*$), is a BL-algebra (fully determined by $*$). It turns out that in any BL-algebra on $[0,1]$, the monoidal operation $*$ is always continuous, and that BL-algebras on $[0,1]$ coincide with algebras given by continuous t-norms on $[0,1]$. They are commonly referred to as *standard* BL-algebras. The standard BL-algebra given by a continuous t-norm $*$ is denoted $[0,1]_*$.

The continuous t-norms corresponding to the three abovementioned extensions of BL are as follows:

	$x * y$	$x \Rightarrow y$ for $x > y$
Łukasiewicz	$\max(x + y - 1, 0)$	$1 - x + y$
Gödel	$\min(x, y)$	y
product	$x.y$	y/x

The next theorem, due to Mostert and Shields (cf. [18]) and often referred to as the *representation theorem for continuous t-norms*, explains the importance of the three examples above. For a continuous t-norm $*$, the set of its idempotents is a closed subset of $[0,1]$, its complement is a union of countably many pairwise disjoint open intervals; denote this set of intervals $\mathcal{I}_{\text{o}}$. Let $\mathcal{I}$ be the set of closures of the elements of $\mathcal{I}_{\text{o}}$.

Theorem 2.4. [18] *Let $*$ be a continuous t-norm.*
(i) *For each $[a,b] \in \mathcal{I}$, $* \upharpoonright [a,b]$ is isomorphic either to the product t-norm on $[0,1]$ or to the Łukasiewicz t-norm on $[0,1]$.*
(ii) *For $x, y \in [0,1]$, if $x, y \notin [a,b] \in \mathcal{I}$, then $x * y = \min(x,y)$.*

For each continuous t-norm, the maximal, closed intervals which are isomorphic copies of the Łukasiewicz, Gödel, or product t-norm are called the non-singleton *components* of the t-norm; we use the terms Ł-, G-, or Π-components and the three letters to denote the *isomorphism type* of a component. Not every element of $[0,1]$ belongs to a non-singleton component; those that do not are called the singleton components. We use $\oplus$ as the addition symbol for components.

Apart from the algebraic completeness given above, the logics BL, Ł, G, and Π enjoy *standard* completeness, which rests on partial embeddability results and (in the case of BL) on a variant of the above representation theorem for saturated BL-chains (defined below). For $\mathbf{A}$ and $\mathbf{B}$ two algebras with the same signature, we say $\mathbf{A}$ is partially embeddable into $\mathbf{B}$ iff every finite partial subalgebra of $\mathbf{A}$ is embeddable into $\mathbf{B}$, that is, for each finite set $A_0 \subseteq A$ there is a one-to-one mapping $f : A_0 \longrightarrow B$ such that for each n-ary function symbol g in the language, if for $a_1, \ldots, a_n \in A_0$ we have $g(a_1, \ldots, a_n) \in A_0$, then $f(g^{\mathbf{A}}(a_1, \ldots, a_n)) = g^{\mathbf{B}}(f(a_1), \ldots, f(a_n))$. For $\mathbb{K}$, $\mathbb{L}$ two classes of algebras with the same signature, we say $\mathbb{K}$ is partially embeddable into $\mathbb{L}$ iff each finite partial subalgebra of a member of $\mathbb{K}$ is embeddable into a member of $\mathbb{L}$.

The following theorem is elementary (but cf. Proposition 11 of [7], conditioning the converse implication).

Theorem 2.5. *Let L be an axiomatic extension (or an algebraizable axiomatic expansion) of* BL. *If the class of all L-chains is partially embeddable into the class of all standard L-algebras, then L enjoys the* finite strong standard completeness (FSSC)*: for a finite theory T and a formula φ in the language of L, $T \vdash_L \varphi$ iff $T \models_{\mathbf{A}} \varphi$ for each standard L-algebra $\mathbf{A}$.*

All MV-chains are partially embeddable into $[0,1]_{\text{Ł}}$ (the standard MV-algebra, given by the Łukasiewicz t-norm), a result due to Chang; it follows that Łukasiewicz logic Ł is complete w.r.t. tautologies of $[0,1]_{\text{Ł}}$, and the completeness extends to finite theories over Ł. Likewise, Gödel chains are partially embeddable into $[0,1]_{\text{G}}$, hence Gödel logic G enjoys FSSC (in fact, G is the only logic given by a continuous t-norm that has strong standard completeness also for infinite theories); product chains are partially embeddable into $[0,1]_\Pi$, and product logic Π enjoys FSSC; see [11] for details.

On the basis of that, one can show partial embeddability of BL-chains into standard BL-algebras, using the following results from [10] and [5]. A BL-chain $\mathbf{A}$ is *saturated* iff for each cut X, Y in $\mathbf{A}$, there is an idempotent d s. t. $x \in X$ and $y \in Y$ implies $x \le d \le y$, where a pair X, Y forms a *cut* in $\mathbf{A}$ iff $X \cup Y = A$, $x \in X$ and $y \in Y$ implies $x \le y$, Y is closed under $*$, and $x \in X$, $y \in Y$ implies $x * y = x$. Each BL-chain can be embedded in a saturated BL-chain in such a way that the image is dense.

Let $\mathbf{A}$ be a BL-chain, $a \in A$. For any $x, y \in A$ we denote $x \Rightarrow^a y = \min(x \Rightarrow y, a)$. If c, d are idempotent elements of $\mathbf{A}$, then $\mathbf{A}|[c,d]$ denotes the BL-algebra $\langle A, \wedge^{[c,d]}, \vee^{[c,d]}, *^{[c,d]}, (\Rightarrow^d)^{[c,d]} c, d\rangle$, where $f^{[c,d]}$ denotes the restriction of the operation f on A to $[c,d]$.

Lemma 2.6. [5] *Let $\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a saturated* BL*-chain and $E \subseteq A$ be the set of idempotent elements of A.*
(i) For each $a < b \in E$, $\mathbf{A}|[a,b]$ is a BL*-chain*
(ii) If $a \in E$, then there is a greatest closed interval $[c,d] \subseteq E$ such that $a \in [c,d]$ (where $c = d$ is an option); $\mathbf{A}|[c,d]$ is a Gödel algebra
(iii) If $a < b \in E$, $(a,b) \cap E = \emptyset$, then $\mathbf{A}|[a,b]$ is an MV-algebra or a product algebra.

One can decompose saturated BL-chains into components in a way analogous to standard BL-algebras, taking maximal (nontrivial) Gödel chains as single components.

Theorem 2.7. *Let $\mathbf{A}$ be a saturated* BL*-chain. Then $\mathbf{A} = \bigoplus_{j \in J} \mathbf{A}_j$, where each $\mathbf{A}_j$, $j \in J$ is an MV-chain, a product chain, a maximal Gödel chain, or a singleton.*

From this one immediately gets (componentwise) partial embeddability of BL-chains into standard BL-algebras, and hence FSSC for BL. See Corollary 7.4 in the Appendix for a general FSSC result for extensions of BL given by a single continuous t-norm.

Needless to say, expansions of BL correspond to expansions of BL-algebras with the corresponding operations. If $\mathcal{L}_1 \subseteq \mathcal{L}_2$ are two (algebraic) languages and the algebra $\mathbf{A}$ is an $\mathcal{L}_2$-structure, we denote $\mathbf{A} \upharpoonright \mathcal{L}_1$ its $\mathcal{L}_1$-reduct.

3 Logic of constants

According to Theorem 2.4, each continuous t-norm determines a set of components on the real unit interval $[0, 1]$. Each component is delimited by two *endpoints.*

Definition 3.1. (Endpoints of $*$) *For a given continuous t-norm $*$, let $\mathrm{EP}_0(*)$ denote the set of elements of $[0, 1]$ that are idempotent w.r.t. $*$ and delimit its non-singleton Ł-, G-, and Π-components. Further, let $\mathrm{EP}(*) = \mathrm{EP}_0(*) \cup \{0, 1\}$. The elements of $\mathrm{EP}(*)$ are called endpoints of $*$.*

For each $*$, the set $\mathrm{EP}(*)$ is countable (while this need not be the case for all idempotents of $*$, even if $*$ has no non-singleton Gödel components). The union of all non-singleton components of $*$ forms a dense set in $[0, 1]$. If, for two standard BL-algebras $[0, 1]_{*_1}$ and $[0, 1]_{*_2}$, their sets of endpoints $\mathrm{EP}(*_1)$ and $\mathrm{EP}(*_2)$ are order-isomorphic via an f and for $x, y \in \mathrm{EP}(*_1)$ we have $[x, y]$ is an Ł-component, (G-component, Π-component) in $[0, 1]_{*_1}$ iff $[f(x), f(y)]$ is an Ł-component (G-component, Π-component respectively) in $[0, 1]_{*_2}$, then $[0, 1]_{*_1}$ and $[0, 1]_{*_2}$ are isomorphic.

It is convenient to enumerate the endpoints with rational numbers. Indeed, from a class of pairwise isomorphic continuous t-norms, one can choose a representative $*$ with $\mathrm{EP}(*) \subseteq \mathbb{Q}$ (cf. Lemma 7.1 in the Appendix). For any continuous t-norm $*$ under consideration, we shall assume all elements of $\mathrm{EP}(*)$ to be rational, unless explicitly stated otherwise. Note that $\mathrm{EP}(*)$ is closed under the operations of the BL-algebra $[0, 1]_*$: the elements of $\mathrm{EP}(*)$ are idempotent, hence the operations are determined by the total ordering of $[0, 1]_*$.

Definition 3.2. (Successor function for EP) *Fix $*$ with endpoints $\mathrm{EP}(*)$. Let $+ : \mathrm{EP}(*) \longrightarrow \mathrm{EP}(*)$ be the function assigning to each $r \in \mathrm{EP}(*)$ an $r' \in \mathrm{EP}(*)$ s. t. $r' = \min\{x : x \in \mathrm{EP}(*) \,\mathrm{and}\, r < x\}$; set $+(r) = r$ if no such r' exists. We write r^+ for $+(r)$.*

Given a continuous t-norm $*$ with rational endpoints $\mathrm{EP}(*)$, introduce a set of new propositional truth constants $\mathcal{C}_* = \{c_r : r \in \mathrm{EP}(*)\}$ into the propositional language of BL. (As for the new constants c_0 and c_1, we shall stipulate $c_0 \equiv \overline{0}$ and $c_1 \equiv \overline{1}$ in the set of axioms.) The *canonical semantics* in $[0, 1]_*$ is $e(c_r) = r$ for any evaluation e. We can define the $+$ function on $\mathcal{C}_*$, by putting $c_r^+ = c_{(r^+)}$.

For each continuous t-norm $*$ we define a propositional logic $\mathrm{BL}_{\mathrm{EP}(*)}$ in the language of BL expanded with $\mathcal{C}_*$. Our new axioms will capture the idempotence and the ordering of the endpoints, but not yet the type of components. As a matter of fact, the axioms for constants introduced are (equivalent to) bookkeeping axioms for the set EP($*$), as given in [7]. Therefore the results of that paper (obtained for a much more general case) hold for the logic $\mathrm{BL}_{\mathrm{EP}(*)}$.

Definition 3.3. *Let $*$ be a continuous t-norm with endpoints* EP($*$). *The logic* $\mathrm{BL}_{\mathrm{EP}(*)}$ *expands the language of* BL *with a set of new propositional constants $\mathcal{C}_*$. The axioms of the logic* $\mathrm{BL}_{\mathrm{EP}(*)}$ *are the axioms of* BL *plus the following formulas:*

(EP_1^r)	$c_r \mathbin{\&} c_r \equiv c_r$	for each $r \in \mathrm{EP}(*)$
$(\mathrm{EP}_2^{r,s})$	$c_r \to c_s$	for each $r, s \in \mathrm{EP}(*)$ s.t. $r \leq s$
$(\mathrm{EP}_3^{r,s})$	$(c_s \to c_r) \to c_r$	for each $r, s \in \mathrm{EP}(*)$ s.t. $r < s$
(EP_4)	$c_0 \equiv \overline{0}, c_1 \equiv \overline{1}$	

The deduction rule is modus ponens.

Definition 3.4. ($\mathrm{BL}_{\mathrm{EP}(*)}$-algebra) *Let $*$ be a continuous t-norm with endpoints* EP($*$). *A* $\mathrm{BL}_{\mathrm{EP}(*)}$*-algebra is a structure*

$$\mathbf{A} = \langle A, \wedge, \vee, *, \Rightarrow, 0^{\mathbf{A}}, 1^{\mathbf{A}}, \{r^{\mathbf{A}}\}^{r \in \mathrm{EP}(*)} \rangle$$

such that $\langle A, \wedge, \vee, *, \Rightarrow, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ *is a* BL*-algebra and all axioms of* $\mathrm{BL}_{\mathrm{EP}(*)}$ *hold under* $e(c_r) = r^{\mathbf{A}}$ *for all* $r \in \mathrm{EP}(*)$ *and all evaluations* e.

A *standard* $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra is any $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra on $[0, 1]$. A *canonical* $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra is a standard algebra where $e(c_r) = r$ for all $r \in \mathrm{EP}(*)$ and all evaluations e.

In any $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra $\mathbf{A}$, some of its constants may coincide with $1^{\mathbf{A}}$. In particular, any BL-algebra $\mathbf{A}$ can be viewed as a $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra for any $*$, where $r^{\mathbf{A}} = 1$ for all $r \in \mathrm{EP}(*), r \neq 0$. Moreover, the following is easily observed:

Lemma 3.5. *Let* $\mathbf{A}$ *be a* BL*-algebra and* $a \leq b$ *two idempotent elements of* $\mathbf{A}$. *Then* $(b \Rightarrow a) \Rightarrow a = 1^{\mathbf{A}}$ *in* $\mathbf{A}$ *iff* $a < b \leq 1^{\mathbf{A}}$ *or* $a = 1^{\mathbf{A}}$.

The following is straightforward (using [7] if needed):

- $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras are the equivalent algebraic semantics of $\mathrm{BL}_{\mathrm{EP}(*)}$
- $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras form a variety

- Strong completeness: for theory T and any formula φ in the language of $\mathrm{BL}_{\mathrm{EP}(*)}$, $T \vdash_{\mathrm{BL}_{\mathrm{EP}(*)}} \varphi$ iff $T \models_{\mathbf{A}} \varphi$ for each (linearly ordered) $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra $\mathbf{A}$.
- Each BL-algebra can be expanded to a $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra by setting $e(c_r) = 1$ for each e and for each $r \in \mathrm{EP}(*), r > 0$.
- $\mathrm{BL}_{\mathrm{EP}(*)}$ is a conservative expansion of BL.

We now prove finite strong standard completeness for $\mathrm{BL}_{\mathrm{EP}(*)}$, using partial embeddability of $\mathrm{BL}_{\mathrm{EP}(*)}$-chains into standard $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras. Note that strong standard completeness cannot be hoped for as even the fragment of the logic in the language of BL typically does not enjoy it. Also, $\mathrm{BL}_{\mathrm{EP}(*)}$ is not strong enough to yield canonical completeness; the reason is that the axioms of $\mathrm{BL}_{\mathrm{EP}(*)}$ do not reflect the isomorphism type of each of the components of $*$, as can be observed by considering an $*$ where $\mathrm{EP}(*) = \{0, 1\}$.

Lemma 3.6. *Let $*$ be a continuous t-norm with endpoints $\mathrm{EP}(*)$. The class of $\mathrm{BL}_{\mathrm{EP}(*)}$-chains is partially embeddable into the class of standard $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras.*

Proof. Let $\mathbf{A}$ be a $\mathrm{BL}_{\mathrm{EP}(*)}$-chain and $A_0 = \{a_1 < \cdots < a_m\}$ a finite set of elements from A. We may assume $\mathbf{A}$ is saturated, so $\mathbf{A} = \bigoplus_{j \in J} \mathbf{A}_j$. For each $i = 1, \ldots, m$, choose c_i and d_i in such a way that if a_i is an idempotent then let $c_i = d_i = a_i$, otherwise let $[c_i, d_i]$ be the component $\mathbf{A}_j$ containing a_i. Let $V = \{r^{\mathbf{A}} : r \in \mathrm{EP}(*)\} \cup \{c_i, d_i\}_{i=1}^m \cup \{0^{\mathbf{A}}, 1^{\mathbf{A}}\}$. Then V is a countable set and, as an ordered subset of A, it is order-embeddable into $[0,1]$ via some f that sends $0^{\mathbf{A}}$ to 0 and $1^{\mathbf{A}}$ to 1. Now define a standard BL-algebra $[0,1]_{*_1}$ as follows: for each $b \in V$, let $f(b)$ idempotent of $*_1$; for each $b_1, b_2 \in V$, if $[b_1, b_2]$ is an MV-component (G-component, Π-component) in $\mathbf{A}$, then let $[f(b_1), f(b_2)]$ be a (standard) MV-component (ditto respectively) in $[0,1]_{*_1}$ (thus $f(b_1)$ and $f(b_2)$ are idempotent elements of $*_1$); if for any $y \in [0,1]$, $y = \lim f(x_n), x_n \in V$, then let y be an idempotent of $*_1$; for any two consecutive idempotents y, y' of $*_1$ obtained in the above manner, if the component $[y, y']$ has not been determined, then let it be an MV-component. Let $r^{[0,1]_{*_1}} = f(r^{\mathbf{A}})$. Then $*_1$ is a well-defined continuous t-norm and, with the interpretations of constants, it is a standard $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra. Moreover, there is an embedding of the partial algebra on $0, a_1, \ldots, a_m, 1$ into $[0,1]_{*_1}$, obtained component-wise. □

Corollary 3.7. (FSSC for $\mathrm{BL}_{\mathrm{EP}(*)}$) *Let $*$ be a continuous t-norm and $\mathrm{EP}(*)$ its endpoints. Let T be a finite theory and φ a formula in the language* *of* $\mathrm{BL}_{\mathrm{EP}(*)}$. *Then* $T \vdash_{\mathrm{BL}_{\mathrm{EP}(*)}} \varphi$ *iff* $T \models_{\mathbf{A}} \varphi$ *for each standard* $\mathrm{BL}_{\mathrm{EP}(*)}$*-algebra* $\mathbf{A}$.

4 Logic of components

For a given continuous t-norm $*$, we work in a propositional language obtained by expanding the language of BL with new truth constants for the elements of $\mathrm{EP}(*)$ (as explained in the previous section). Our aim is to give an axiomatization in the expanded language that, under the canonical interpretation of constants, would yield completeness with respect to the given algebra $[0,1]_*$. The union of non-singleton components of $*$ is dense in $[0,1]$, and hence, in order to identify a standard BL-algebra up to an isomorphism, it is sufficient to specify the isomorphism type of each of its components. In this section, we suggest a way of describing the latter by means of a suitable translation of formulas.

Assume $*$ is a continuous t-norm with endpoints $\mathrm{EP}(*)$. For each $r, s \in \mathrm{EP}(*)$ such that $r < s$, we define a translation function operating on formulas of the language of BL. (Note that we do not assume $[r,s]$ is a component of $*$.) The result of the translation of a formula φ will be denoted $\varphi^{[c_r,c_s]}$. The translation function is defined by induction on formula structure in the following manner:

$$\begin{aligned}
\overline{0}^{[c_r,c_s]} &= c_r \\
\overline{1}^{[c_r,c_s]} &= c_s \\
p^{[c_r,c_s]} &= (p \vee c_r) \wedge c_s \\
(\varphi \& \psi)^{[c_r,c_s]} &= \varphi^{[c_r,c_s]} \,\&\, \psi^{[c_r,c_s]} \\
(\varphi \to \psi)^{[c_r,c_s]} &= (\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]}) \wedge c_s
\end{aligned}$$

Lemma 4.1. *Let* $*$ *be a continuous t-norm,* $\mathrm{EP}(*)$ *its endpoints,* $r < s \in \mathrm{EP}(*)$*, and* φ*,* ψ *formulas in the language of* BL. *Then the following formulas are provable in* $\mathrm{BL}_{\mathrm{EP}(*)}$*:*
(1) $c_r \wedge \varphi \to c_r \,\&\, \varphi$
(2) $c_r \to \varphi^{[c_r,c_s]}$, $\varphi^{[c_r,c_s]} \to c_s$
(3) $\varphi^{[c_r,c_s]} \equiv ((\varphi^{[c_r,c_s]} \vee c_r) \wedge c_s)$
(4) $(\varphi \wedge \psi)^{[c_r,c_s]} \equiv \varphi^{[c_r,c_s]} \wedge \psi^{[c_r,c_s]}$
(5) $(\varphi \vee \psi)^{[c_r,c_s]} \equiv \varphi^{[c_r,c_s]} \vee \psi^{[c_r,c_s]}$

Proof. All proofs carried out in $\mathrm{BL}_{\mathrm{EP}(*)}$.
(1) $c_r \wedge \varphi \to \varphi$, which is, by definition of $\wedge$,
$c_r \,\&\, (c_r \to \varphi) \to \varphi$; then also
$c_r \,\&\, (c_r \,\&\, (c_r \to \varphi)) \to c_r \,\&\, \varphi$, and, using idempotence of c_r, we get
$c_r \,\&\, (c_r \to \varphi) \to c_r \,\&\, \varphi$, which is, by definition of $\wedge$,
$c_r \wedge \varphi \to c_r \,\&\, \varphi$.
(2), (3) by induction on formula structure.
(4) $(\varphi \wedge \psi)^{[c_r,c_s]}$ is, by definition of $\wedge$,
$(\varphi \,\&\, (\varphi \to \psi))^{[c_r,c_s]}$, which is, by definition of the translation functions,
$\varphi^{[c_r,c_s]} \,\&\, ((\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]}) \wedge c_s)$, which distributes to
$(\varphi^{[c_r,c_s]} \,\&\, (\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]})) \wedge (\varphi^{[c_r,c_s]} \,\&\, c_s)$, which, using (2) and (3), is equivalent to
$\varphi^{[c_r,c_s]} \wedge \psi^{[c_r,c_s]}$.
(5) From (2) we get $\varphi^{[c_r,c_s]} \to c_s$ and $\psi^{[c_r,c_s]} \to c_s$, hence $\varphi^{[c_r,c_s]} \vee \psi^{[c_r,c_s]} \to c_s$.
Also, $((\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]}) \to \psi^{[c_r,c_s]}) \to (((\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]}) \wedge c_s) \to \psi^{[c_r,c_s]})$.
This concludes the proof of the right-to-left implication.
On the other hand, using (1), the left-hand side is equivalent to
$(((\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]}) \,\&\, c_s) \to \psi^{[c_r,c_s]}) \,\&\, c_s$, i.e.,
$c_s \,\&\, (c_s \to ((\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]}) \to \psi^{[c_r,c_s]})$, which clearly entails
$((\varphi^{[c_r,c_s]} \to \psi^{[c_r,c_s]}) \to \psi^{[c_r,c_s]})$. □

Theorem 4.2. *Let $*$ be a continuous t-norm and $r < s \in \mathrm{EP}(*)$. Let $\mathbf{A}$ be a $\mathrm{BL}_{\mathrm{EP}(*)}$-chain, so $r^{\mathbf{A}}$, $s^{\mathbf{A}}$ are the values of c_r, c_s in $\mathbf{A}$, respectively. Let φ, ψ be* BL*-formulas. Then $\varphi^{[c_r,c_s]} \equiv \psi^{[c_r,c_s]}$ is a tautology of $\mathbf{A}$ iff $\varphi \equiv \psi$ is a tautology of $\mathbf{A}|[r^{\mathbf{A}}, s^{\mathbf{A}}]$.*

Proof. Denote $\mathbf{B} = \mathbf{A}|[r^{\mathbf{A}}, s^{\mathbf{A}}]$. For each evaluation $e_{\mathbf{A}}$ in $\mathbf{A}$, define an evaluation $e_{\mathbf{A}\downarrow\mathbf{B}}$ in $\mathbf{B}$ by $e_{\mathbf{A}\downarrow\mathbf{B}}(p) = (e_{\mathbf{A}}(p) \vee r^{\mathbf{A}}) \wedge s^{\mathbf{A}}$. It is obvious that for each evaluation e in $\mathbf{B}$, there is an evaluation $e_{\mathbf{A}}$ in $\mathbf{A}$ s. t. $e = e_{\mathbf{A}\downarrow\mathbf{B}}$. Moreover, for any BL-formula φ and any evaluation $e_{\mathbf{A}}$ in $\mathbf{A}$, we have $e_{\mathbf{A}}(\varphi^{[c_r,c_s]}) = e_{\mathbf{A}\downarrow\mathbf{B}}(\varphi)$. If $\varphi^{[c_r,c_s]} \equiv \psi^{[c_r,c_s]}$ holds in $\mathbf{A}$, take an evaluation e in $\mathbf{B}$, find a suitable $e_{\mathbf{A}}$ s. t. $e = e_{\mathbf{A}\downarrow\mathbf{B}}$, and we get $e(\varphi) = e_{\mathbf{A}\downarrow\mathbf{B}}(\varphi) = e_{\mathbf{A}}(\varphi^{[c_r,c_s]}) = e_{\mathbf{A}}(\psi^{[c_r,c_s]}) = e_{\mathbf{A}\downarrow\mathbf{B}}(\psi) = e(\psi)$. Conversely, if $\varphi \equiv \psi$ in $\mathbf{B}$, then for each $e_{\mathbf{A}}$ we have $e_{\mathbf{A}}(\varphi^{[c_r,c_s]}) = e_{\mathbf{A}\downarrow\mathbf{B}}(\varphi) = e_{\mathbf{A}\downarrow\mathbf{B}}(\psi) = e_{\mathbf{A}}(\psi^{[c_r,c_s]})$. □

Recall the result of [9]: each logic given by a standard BL-algebra is finitely axiomatizable. Hence, to each standard BL-algebra $[0,1]_*$, one can associate a single BL-formula φ_* in such a way that the axiomatic extension given by $\mathrm{BL} \cup \{\varphi_*\}$ corresponds to the subvariety $\mathrm{Var}([0,1]_*)$ of $\mathbb{BL}$ (the members of this variety let be called φ_*-algebras).

In particular, we denote $\text{Ł}(\varphi) = \neg\neg\varphi \to \varphi$ the formula that yields Łukasiewicz logic as an axiomatic extension of BL, thus determining the subvariety $\mathbb{MV}$ of $\mathbb{BL}$. Analogously for the formula $\text{G}(\varphi) = \varphi \to \varphi \& \varphi$ with respect to Gödel logic, and the formula $\Pi(\varphi, \psi, \chi) = (\varphi \to \chi) \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$ with respect to product logic.

Corollary 4.3. *Let* $[0,1]_\circ$ *be a standard* BL*-algebra given by a continuous t-norm* $\circ$ *and let* $\varphi_\circ$ *be such that* $\text{BL} \cup \{\varphi_\circ\}$ *corresponds to the variety* $\text{Var}([0,1]_\circ)$*. Let* $*$ *be an arbitrary continuous t-norm,* $\text{EP}(*)$ *its endpoints,* $r < s \in \text{EP}(*)$*, and* $\mathbf{A}$ *any* $\text{BL}_{\text{EP}(*)}$*-chain. Then* $\varphi_\circ^{[c_r, c_s]} \equiv c_s$ *is a tautology of* $\mathbf{A}$ *iff* $(\mathbf{A}|[r^{\mathbf{A}}, s^{\mathbf{A}}]) \restriction \text{BL}$ *is an* $\varphi_\circ$*-chain.*

We refine the calculus $\text{BL}_{\text{EP}(*)}$ with axiom schemata specifying the isomorphism type of each of the components of $*$.

Definition 4.4. *Let* $*$ *be a continuous t-norm with endpoints* $\text{EP}(*)$*. The logic* $\text{BL}_{\text{COMP}(*)}$ *has the axioms of* $\text{BL}_{\text{EP}(*)}$ *plus, for each* $r < r^+ \in \text{EP}(*)$ *and each triple of* BL*-formulas* φ*,* ψ*, and* χ*:*

$(\text{COMP}(\text{Ł})^r)$ $\quad (\text{Ł}(\varphi))^{[c_r, c_r^+]} \equiv c_r^+$ *iff* $[r, r^+]$ *is an MV-component of* $[0,1]_*$

$(\text{COMP}(\text{G})^r)$ $\quad (\text{G}(\varphi))^{[c_r, c_r^+]} \equiv c_r^+$ *iff* $[r, r^+]$ *is a G-component of* $[0,1]_*$

$(\text{COMP}(\Pi)^r)$ $\quad (\Pi(\varphi, \psi, \chi))^{[c_r, c_r^+]} \equiv c_r^+$ *iff* $[r, r^+]$ *is a* Π*-component of* $[0,1]_*$

The deduction rule is modus ponens.

For any $*$, the logic $\text{BL}_{\text{COMP}(*)}$ is an axiomatic expansion of BL (and an axiomatic extension of $\text{BL}_{\text{EP}(*)}$). A $\text{BL}_{\text{COMP}(*)}$-algebra is a $\text{BL}_{\text{EP}(*)}$-algebra such that all axioms of $\text{BL}_{\text{COMP}(*)}$ are its tautologies. It is immediate that $\text{BL}_{\text{COMP}(*)}$-algebras form a variety, and that they constitute the equivalent algebraic semantics of the logic $\text{BL}_{\text{COMP}(*)}$.

Theorem 4.5. (Strong completeness for $\text{BL}_{\text{COMP}(*)}$**)** *Let* T *be a theory and* φ *a formula in the language of* $\text{BL}_{\text{EP}(*)}$*. Then* $T \vdash_{\text{BL}_{\text{COMP}(*)}} \varphi$ *iff* $T \models_{\mathbf{A}} \varphi$ *for each (linearly ordered)* $\text{BL}_{\text{COMP}(*)}$*-algebra* $\mathbf{A}$*.*

Unlike $\text{BL}_{\text{EP}(*)}$, the logic $\text{BL}_{\text{COMP}(*)}$ is generally not conservative over BL; that is to say, not unless BL happens to be the logic of $[0,1]_*$. Indeed, consider the standard BL-algebra $[0,1]_{\text{Ł}\oplus\text{G}}$, and the corresponding logic $\text{BL}_{\text{COMP}(\text{Ł}\oplus\text{G})}$. The formula $(\neg\neg\varphi \to \varphi) \vee (\varphi \to \varphi \& \varphi)$ is obviously valid in the canonical chain and in all $\text{BL}_{\text{COMP}(\text{Ł}\oplus\text{G})}$-chains, hence provable by completeness theorem, but it is not a BL-tautology.

The following is proved in exactly the same way as Lemma 3.6.

Lemma 4.6. *Let $*$ be a continuous t-norm with endpoints* $\mathrm{EP}(*)$. *The class of* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-chains is partially embeddable into the class of standard* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-algebras.*

Corollary 4.7. (FSSC for $\mathrm{BL}_{\mathrm{COMP}(*)}$) *Let $*$ be a continuous t-norm with endpoints* $\mathrm{EP}(*)$. *Let T be a finite theory and φ a formula, all in the language of* $\mathrm{BL}_{\mathrm{COMP}(*)}$. *Then $T \vdash_{\mathrm{BL}_{\mathrm{COMP}(*)}} \varphi$ iff $T \models_{\mathbf{A}} \varphi$ for each standard* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-algebra* $\mathbf{A}$.

Now we discuss canonical completeness. If $*$ is an ordinal sum of finitely many components, then each standard $\mathrm{BL}_{\mathrm{COMP}(*)}$-chain is a homomorphic image of the canonical $\mathrm{BL}_{\mathrm{COMP}(*)}$-chain (as is easily observed by taking into account that the type of each component, delimited by two consecutive constants, is fully determined by the axioms, cf. Corollary 4.3). It follows that any formula valid in the canonical algebra is valid in all standard algebras. This yields canonical completeness of $\mathrm{BL}_{\mathrm{COMP}(*)}$ for finite ordinal sums.

Theorem 4.8. *Let $*$ be a continuous t-norm with finitely many components, let* $\mathrm{EP}(*)$ *be its endpoints. The logic* $\mathrm{BL}_{\mathrm{COMP}(*)}$ *is complete w.r.t. its canonical algebra.*

But it is equally obvious that finite strong canonical completeness of $\mathrm{BL}_{\mathrm{COMP}(*)}$ fails: take a continuous t-norm $*$ with three components, delimited by 0, 1/3, 2/3, 1 (the isomorphism type of components does not matter). Then, while $c_{2/3} \models c_{1/3}$ in the canonical $\mathrm{BL}_{\mathrm{COMP}(*)}$-algebra, one can present other standard $\mathrm{BL}_{\mathrm{COMP}(*)}$-algebras in which this fails to hold: e.g., it does not hold in the algebra obtained from the canonical one by sending $[2/3, 1]$ to 1, because then the constant $c_{2/3}$ is interpreted by the value 1 in the new algebra, while the constant $c_{1/3}$ is not. Using FSSC for $\mathrm{BL}_{\mathrm{COMP}(*)}$, we get $c_{2/3} \not\vdash c_{1/3}$.

If a continuous t-norm $*$ is an ordinal sum of infinitely many components, it is not true that every standard $\mathrm{BL}_{\mathrm{COMP}(*)}$-algebra is a homomorphic image of the canonical one. As a matter of fact, both standard and general $\mathrm{BL}_{\mathrm{COMP}(*)}$-chains may contain components whose delimiting endpoints are not denoted by any propositional constants in the language. To achieve canonical completeness for infinite sums, we have to add more axioms. For the following definition, recall that for any continuous t-norm $\circ$, we take $\varphi_\circ$ to denote the single BL-formula which (with the axioms of BL) provides a complete axiomatization of the BL-algebra $[0,1]_\circ$. Now if $*$ is a continuous t-norm and c, d two of its idempotents, then $[0,1]_*|[c,d]$ is a BL-algebra that is an isomorphic copy of some standard BL-algebra (unique up to isomorphism) that we denote $[0,1]_{*[c,d]}$.

Definition 4.9. (The logic $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$) *Let $*$ be a continuous t-norm with endpoints* $\mathrm{EP}(*)$*. The logic* $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$ *has the axioms of* $\mathrm{BL}_{\mathrm{EP}(*)}$ *plus, for each $r < s \in \mathrm{EP}(*)$, the formula $(\varphi_{[0,1]_{*[r,s]}})^{[c_r,c_s]} \equiv c_s$. The deduction rule is modus ponens.*

Note that the logic $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$ extends the logic $\mathrm{BL}_{\mathrm{COMP}(*)}$, since the axioms of $\mathrm{BL}_{\mathrm{COMP}(*)}$ are a special case of the above for $s = r^+$. Therefore, it enjoys its properties of algebraizability and (general) strong completeness. We focus on the canonical completeness of $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$.

Lemma 4.10. *Let $*$ be a continuous t-norm with endpoints* $\mathrm{EP}(*)$*. Each* $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$*-chain is partially embeddable into a homomorphic image of the canonical* $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$*-algebra* $[0,1]_*$.

Proof. Let $\mathbf{A}$ be a $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$-chain. Denote $F = \{r \in \mathrm{EP}(*) : r^{\mathbf{A}} = 1^{\mathbf{A}}\}$. Let $[0,1]_{*_1}$ be the homomorphic image of $[0,1]_*$ determined by F on $[0,1]$ (i.e., by sending all $[r, r^+]$, $r \in F$, to 1). Let $A_0 = \{0^{\mathbf{A}} = a_0 < \cdots < a_m = 1^{\mathbf{A}}\}$ be a finite set of elements from A. Denote $A_1 = \{0^{\mathbf{A}} = a_{j_0} < a_{j_1} < \cdots < a_{j_k} = 1^{\mathbf{A}}\}$, $k \le m$, the subset of A_0 s. t. for each $i = 0 \dots k$, there is an $r_i \in \mathrm{EP}(*)$ such that $a_{j_i} = r^{\mathbf{A}}$); in other words, A_1 is the set of elements of A_0 that interpret some propositional constant in $\mathbf{A}$. Let f be a partial mapping from $\mathbf{A}$ to $[0,1]$, assigning to each $a_{j_i} \in A_1$ the corresponding value $r_i^{[0,1]_{*_1}}$ in the $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$-chain $[0,1]_{*_1}$. The axioms of $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$ guarantee that, for each $i = 0 \dots k-1$ we have that $(\mathbf{A}|[a_{j_i}, a_{j_{i+1}}]) \restriction \mathrm{BL}$ belongs to the variety generated by $[0,1]_*|[r_i^{[0,1]_{*_1}}, r_{i+1}^{[0,1]_{*_1}}] \restriction \mathrm{BL}$ (cf. Corollary 4.3). Therefore, for each $i = 0 \dots k-1$, $[a_{j_i}, a_{j_{i+1}}]$ is partially embeddable into $[r_i^{[0,1]_{*_1}}, r_{i+1}^{[0,1]_{*_1}}]$ by Theorem 7.3. The statement follows. □

Theorem 4.11. *Let $*$ be a continuous t-norm with endpoints* $\mathrm{EP}(*)$*. The logic* $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$ *is complete with respect to its canonical algebra.*

Proof. If a formula φ is not provable in $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$, then by general completeness theorem it is not valid in some $\mathrm{BL}_{\mathrm{COMP}(*)}{}^{\star}$-chain. By Lemma 4.10, it is not valid in a homomorphic image of the canonical algebra; hence, it is not valid in the canonical algebra. □

5 Complexity issues

We look at computational complexity of the logics discussed in this paper; in particular, of the sets of tautologies of the $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras given by $*$, for a continuous t-norm $*$. Let us stress at this point that, given a continuous t-norm $*$, we work with propositional formulas in the language of $\mathrm{BL}_{\mathrm{EP}(*)}$; thus, notions such as "tautology" assume the expanded language. As to connectives of the language of BL, we restrict ourselves to the basic ones, that is, $\overline{0}$, $\&$, $\rightarrow$.

Results presented here point out the gap discernible already in the previous sections: for finite ordinal sums, the results are straightforward and follow the example of logics without constants, i.e., the sets of their tautologies are coNP-complete; this is the first result presented in this section. On the other hand, some continuous t-norms that are infinite ordinal sums give rise to non-recursive sets of tautologies. Even worse: the expanded language determined by some continuous t-norms $*$—that is to say, the set (of rationals) $\mathrm{EP}(*)$ and the corresponding set of propositional constants $\mathcal{C}_*$—is non-recursive (cf. Lemma 7.2), which leaves no room indeed for nice complexity results in general. For this reason, in the case of infinite ordinal sums, we restrict our investigation to presenting two examples. First, we point out an infinite ordinal sum whose set of tautologies (in the language of $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras) is coNP-complete. Second, we show that an arbitrary subset of $\mathbb{N}$ is m-reducible to the set of tautologies of a suitably chosen infinite ordinal sum (in the language of $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras); hence, tautologies given by continuous t-norms in the expanded language can be placed arbitrarily high in the arithmetical hierarchy, or can be non-arithmetical. Both of these examples use continuous t-norms where components are ordered by ω.

Let $*$ be a continuous t-norm and $\mathbf{A}$ be an arbitrary $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra. We define $\mathrm{TAUT}(\mathbf{A}) = \{\varphi : \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1^{\mathbf{A}})\}$ where φ runs through propositional $\mathrm{BL}_{\mathrm{EP}(*)}$-formulas and $e_{\mathbf{A}}$ runs through evaluations in $\mathbf{A}$. The formulas satisfying the condition are referred to as tautologies of the algebra $\mathbf{A}$.

Theorem 5.1. *Let $*$ be a continuous t-norm which is a finite ordinal sum of* Ł-, G-, *and* Π*-components, and* $\mathbf{A}$ *the* $\mathrm{BL}_{\mathrm{EP}(*)}$*-algebra given by* $*$ *on* $[0,1]$*. Then the set* $\mathrm{TAUT}(\mathbf{A})$ *is coNP-complete.*

Proof. $\mathrm{TAUT}(\mathbf{A})$ is trivially coNP-hard as the tautologies of the standard BL-algebra $[0,1]_*$ in the language without constants (a coNP-complete set) can be reduced to it (using identity). Containment in coNP is obtained as a variant of the proof in [13]; we repeat parts of it here for the reader's convenience.

Let n be the number of components in $\mathbf{A}$, and assume $\{i/n\}_{0\leq i\leq n}$ are the endpoints of $\mathbf{A}$. For any formula φ, let $|\varphi|$ denote the number of occurrences of propositional variables or constants, and denote $m = 2|\varphi| - 1$ (so m is the number of subformulas in φ). Fix an enumeration of all subformulas of φ, assuming φ is assigned an index 1.

We present a nondeterministic acceptor of the complement of TAUT($\mathbf{A}$), running in time polynomial in m. This entails coNP-containment of TAUT($\mathbf{A}$).

`nameSubformulas()` Introduce variables $x_1 \dots, x_m$, and assign the variable x_i to the subformula φ_i of φ (x_1 is assigned to φ).

Set $V = \{i/n\}_{0 \le i \le n} \cup \{x_1, \dots, x_m\}$.

`guessOrder()` Guess a linear ordering $\preceq$ of elements of V, such that $x_1 \prec 1$.

`checkOrder()` Check that $\preceq$ satisfies basic natural conditions: first, that it preserves the ordering of constants $\{i/n\}_{0 \le i \le n}$ on the real unit interval; second, for $i = 0, \dots, n$, any variable assigned to the constant $c_{i/n}$ must be $\approx$-equal to i/n, the variable denoting the i-th endpoint.

We say that variables x_j s. t. $i/n \preceq x_j \preceq (i+1)/n$ *belong to* i.

`checkExternal()` Check external soundness of $\preceq$: for φ_i, φ_j subformulas of φ ($1 \le i, j \le m$),

– if $\varphi_i \mathbin{\&} \varphi_j$ is a subformula φ_k of φ for some $k \in \{1, \dots, m\}$ and, for some $l \in \{0, \dots, n\}$, we have $x_i \preceq l/n \preceq x_j$, then $x_k \approx x_i$;

– if $\varphi_i \to \varphi_j$ is a subformula φ_k of φ for some $k \in \{1, \dots, m\}$ and $x_i \preceq x_j$, then $x_k \approx n/n$;

– if $\varphi_i \to \varphi_j$ is a subformula φ_k of φ for some $k \in \{1, \dots, m\}$ and for some $l \in \{0, \dots, n\}$, we have $x_j \prec l/n \preceq x_i$, then $x_k \approx x_j$.

`checkInternal()` Check internal soundness of $\preceq$ for each interval $[i/n, i+1/n]$, $i = 0, \dots, n-1$, in $\preceq$. Consider variables belonging to i. Construct a system $\mathcal{S}_i$ of equations and inequalities; $\mathcal{S}_i$ is initially empty. For each subformula φ_l which is $\varphi_j \mathbin{\&} \varphi_k$, if x_j and x_k are in i, check x_l is also in i and put equation $x_j * x_k = x_l$ into $\mathcal{S}_i$. For each subformula φ_l which is $\varphi_j \to \varphi_k$, such that $x_k \prec x_j$, if x_j and x_k are in i, check x_l is also in i and put equation $x_j \Rightarrow x_k = x_l$ into $\mathcal{S}_i$.

Further, put all equations and inequalities defined by $\preceq$ for the variables in i into $\mathcal{S}_i$. Check whether the system $\mathcal{S}_i$ has a solution in the i-th component of $\mathbf{A}$.

`end`

It is shown in [13] that the last check can be performed (nondeterministically) in polynomial time w.r.t. the size of $\mathcal{S}$, for all three isomorphism types of components. This concludes the proof. $\square$

Now we examine continuous t-norms with infinitely many endpoints.

Theorem 5.2. *Let $*$ be a continuous t-norm which is an infinite sum of infinitely many Ł-components and a singleton, with endpoints $(2^i - 1)/2^i$ for $i \in \mathbb{N}$. Let* $\mathbf{A}$ *be the* $\mathrm{BL}_{\mathrm{EP}(*)}$*-algebra given by $*$ on $[0,1]$. Then the set* $\mathrm{TAUT}(\mathbf{A})$ *is coNP-complete.*

Proof. Modify the algorithm from the preceding proof. Depending on the input formula, the algorithm will start with an ordinal sum which is a sufficiently large initial part of $\mathbf{A}$. It must contain all constants occurring in φ, so if $(2^i - 1)/2^i$ for some i is the largest constant in φ, then the algorithm will work with a subsum that includes the whole of $[0, (2^i - 1)/2^i]$ (that is, i components, linearly many in the binary representation of the largest constant) and sufficiently many more components of $*$ (no matter which ones as they are all of type Ł) to harbour all the subformulas of φ that might come out greater than the largest constant; this number is linear in m. $\square$

However, the following statement holds for ordinal sums whose endpoints are also ω-ordered, but with both Ł- and Π-components.

Theorem 5.3. *Let S be any subset of $\mathbb{N}$. Let $*$ be a continuous t-norm with endpoints $(2^i - 1)/2^i$ for $i \in \mathbb{N}$. Assume $*$ has two types of non-singleton components,* Ł *and* Π*, and for each $i \in \mathbb{N}$, we have $[(2^i - 1)/2^i, (2^{i+1} - 1)/2^{i+1}]$ is an Ł-component iff $i \in S$. Let* $\mathbf{A}$ *be the* $\mathrm{BL}_{\mathrm{EP}(*)}$*-algebra given by $*$. Then S is m-reducible to* $\mathrm{TAUT}(\mathbf{A})$.

Proof. The proof of this statement was inspired by [12]. Take a formula λ which is valid in $[0,1]_{\text{Ł}}$ but not in $[0,1]_{\Pi}$. Then one can reduce membership is S to tautologousness in $\mathbf{A}$ by asking, for a given $i \in S$, about the validity of of $\lambda^{[(2^i-1)/2^i,(2^{i+1}-1)/2^{i+1}]}$ in $\mathbf{A}$. $\square$

The last statement shows that tautologies of $\mathrm{BL}_{\mathrm{EP}(*)}$-algebras can be placed arbitrarily high in the arithmetical hierarchy and that they can be non-arithmetical.

6 Closing remarks

This paper is a contribution to research in logics expanding propositional Basic Fuzzy Logic BL with truth constants. Under the particular restrictions on the set of constants (namely, that they denote only non-singleton component delimiters), it obtains canonical completeness.

Possible directions for a follow-up include relaxing the conditions on the t-norm or on the set of constants. In particular, one may investigate (standard) ordinal sums of MTL-chains, with constants for non-singleton component delimiters. As for constants, one might want to combine the strong results there are on introducing constants into standard BL-algebras with only one component (Ł, G, or Π) with the present approach: given a continuous t-norm $*$, it is possible to introduce propositional constants for all component delimiters *and* for, say, rationals in some of the components.

In yet another direction, a first-order rendering of the logics described in this paper is still to be investigated.

7 Appendix

We present a few facts that form a useful background for this paper. The propositional and algebraic language considered in this section is that of BL.

Lemma 7.1. *Let* $[0,1]_*$ *be a continuous t-norm and let* $\mathrm{EP}(*)$ *be its endpoints. Then there is a continuous t-norm* $*'$ *isomorphic to* $*$ *and with* $\mathrm{EP}(*') \subseteq \mathbb{Q}$.

Proof. Take the set $(\mathrm{EP}(*) \cup \mathbb{Q}) \cap (0,1)$. As a dense, countable set, it is order-isomorphic to $\mathbb{Q} \cap (0,1)$ via some f_0. Extend f_0 to $f : [0,1] \longrightarrow [0,1]$ by taking, for $x \in [0,1]$, some $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim x_n = x$ and for each $n \in \mathbb{N}$ we have $x_n \in (\mathrm{EP}(*) \cup \mathbb{Q}) \cap (0,1)$. Set $f(x) = \lim f_0(x_n)$. Then f is a well-defined, increasing bijection of $[0,1]$ onto itself, hence continuous, and so is f^{-1}. Therefore, $*'$ defined by $x *' y = f(f^{-1}(x) * f^{-1}(y))$ is a continuous t-norm isomorphic to $*$, with rational endpoints; namely, all endpoints of $*'$ are among $f_0(c)$ for c an endpoint of $*$. □

The following lemma shows, among other things, that there are uncountably many classes of pairwise non-isomorphic continuous t-norms and that this is due to the ordering of the components as well as to their isomorphism type. In particular, there are uncountably many pairwise non-isomorphic continuous t-norms with only one type of non-singleton component.

Lemma 7.2. *There is a continuous t-norm* $*$ *such that, for each of its isomorphic copies* $*'$ *with* $\mathrm{EP}(*') \subseteq \mathbb{Q}$*, the set* $\mathrm{EP}(*')$ *is not recursive.*

Proof. The statement is obtained by a cardinality argument. Each countable ordinal number α is embeddable into $\mathbb{Q} \cap [0,1]$ via an order-preserving isomorphism f, and one can define a continuous t-norm on $\{f(\beta), \beta \in \alpha\}$ (using only one type of component, say Ł). It is obvious that distinct countable ordinals yield pairwise non-isomorphic continuous t-norms (in particular, the sets of idempotents of the resulting continuous t-norms are non-isomorphic). There are uncountably many countable ordinals, while the set of countable ordinals α such that at least one continuous t-norm whose idempotents are ordered by α has recursive endpoints in $\mathbb{Q}$ is countable. $\square$

The following result is useful in particular for proving standard completeness of logic in expanded languages.

Theorem 7.3. *Let* $\mathbf{A}$ *be a standard* BL*-algebra, and let* $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$ *be a* BL*-chain. Then* $\mathbf{B}$ *is partially embeddable into* $\mathbf{A}$*.*

Proof. The proof is based on the results in [9]. Assume $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$ is a BL-chain and $b_1 < \cdots < b_n \in B$. Consider $\mathbf{B} = \bigoplus_{i \in I} \mathbf{B}_i$ as an ordinal sum of Wajsberg hoops (cf. [1]) and let $\mathbf{B}'$ result from $\mathbf{B}$ by deleting all members of the sum except the initial one and each $\mathbf{B}_j$ such that $b_i \in B_j$ for some $i = 1 \ldots, n$, and some $j \in I$; so $\mathbf{B}'$ is a finite ordinal sum of Wajsberg hoops, namely $\mathbf{B}'_1 \oplus \cdots \oplus \mathbf{B}'_m$ for some $m \leq n+1$. Since $\mathbf{B}'$ is a BL-algebra that is a subalgebra of $\mathbf{B}$, we have $\mathbf{B}' \in \mathrm{Var}(\mathbf{A})$.
Define a finite ordinal sum of hoops $\mathbf{B}'' = \mathbf{B}''_1 \oplus \cdots \oplus \mathbf{B}''_m$, where $\mathbf{B}''_i$ is
– the standard cancellative hoop iff $\mathbf{B}'_i$ is an unbounded Wajsberg hoop;
– the two-element hoop **2** if $\mathbf{B}'_i$ is two-element hoop **2**;
– the standard Wajsberg algebra iff $\mathbf{B}'_i$ is a bounded Wajsberg hoop distinct from **2**.
Observe the partial subalgebra of $\mathbf{B}$ on $b_1 < \cdots < b_n$ is embeddable into $\mathbf{B}''$ componentwise. Now consider the identity $e_{\mathbf{B}''}$ (see [9], Definition 3.6). This identity is not valid in $\mathbf{B}''$ nor in $\mathbf{B}'$ (see [9], Lemma 3.7). Thus, it is not valid in $\mathrm{Var}(\mathbf{A})$ (as $\mathbf{B}' \in \mathrm{Var}(\mathbf{A})$). As $\mathrm{Var}(\mathbf{A}) = \mathrm{Var}(\mathit{Fin}(\mathbf{A}))$ by [9], Lemma 3.1, there is a $\mathbf{C} \in \mathit{Fin}(\mathbf{A})$ such that $e_{\mathbf{B}''}$ is not valid in $\mathbf{C}$. Then by [9], Lemma 3.7, $\mathbf{B}'' \in \mathit{Fin}(\mathbf{A})$. This concludes the proof, observing that all members of $\mathit{Fin}(\mathbf{A})$ are partially embeddable into $\mathbf{A}$. $\square$

Corollary 7.4. *If* $\mathbf{A}$ *is a standard* BL*-algebra, then* $\mathrm{Var}(\mathbf{A})$ *is generated by* $\mathbf{A}$ *as a quasivariety, and the logic given by* $\mathbf{A}$ *enjoys the finite strong standard completeness.*

Proof. Combine Theorems 3.8 and 3.2 of [6]. $\square$

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References

[1] P. Aglianò and F. Montagna. Varieties of BL-algebras I: general properties. *J. Pure Appl. Algebra*, 181:105–129, 2003.

[2] W. J. Blok and Don Pigozzi. Algebraizable logics. *Mem. Amer. Math. Soc.*, 77(396), 1989.

[3] Karel Chvalovský. On the independence of axioms in BL and MTL. submitted.

[4] Agata Ciabattoni, George Metcalfe, and Franco Montagna. Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions. *Fuzzy Sets and Systems*, 161:369 – 389, 2010.

[5] Roberto Cignoli, Francesc Esteva, Lluís Godo, and Antoni Torrens. Basic Fuzzy Logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4:106–112, 2000.

[6] Petr Cintula, Francesc Esteva, Joan Gispert, Lluís Godo, Franco Montagna, and Carles Noguera. Distinguished algebraic semantics for t-norm fuzzy logics: Methods and algebraic equivalencies. *Ann. Pure Appl. Logic*, 160:53–81, 2009.

[7] Francesc Esteva, Joan Gispert, Lluís Godo, and Carles Noguera. Adding truth-constants to continuous t-norm based logics: Axiomatization and completeness results. *Fuzzy Sets and Systems*, 158:597–618, 2007.

[8] Francesc Esteva, Lluís Godo, Petr Hájek, and Mirko Navara. Residuated fuzzy logic with an involutive negation. *Arch. Math. Logic*, 39:103–124, 2000.

[9] Francesc Esteva, Lluís Godo, and Franco Montagna. Equational characterization of subvarieties of BL-algebras generated by t-norm algebras. *Studia Logica*, 76(2):161–200, 2004.

[10] Petr Hájek. Basic fuzzy logic and BL-algebras. *Soft Computing*, 2:124–128, 1998.

[11] Petr Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 1998.

[12] Petr Hájek. Some hedges for continuous t-norm logics. *Neural Network World*, 12(2):159–164, 2002.

[13] Zuzana Haniková. A note on the complexity of propositional tautologies of individual t-algebras. *Neural Network World*, 12(5):453–460, 2002.

[14] Zuzana Honzíková. Axiomatizations of standard algebras for fuzzy propositional calculi by use of truth constants. *Journal of Electrical Engineering*, 51:68–75, 2000.

[15] Rostislav Horčík and Petr Cintula. Product Łukasiewicz logic. *Arch. Math. Logic*, 43(4):477–503, 2004.

[16] Enrico Marchioni and Franco Montagna. On triangular norms and uninorms definable in ŁΠ1/2. *Int. J. Approximate Reasoning*, 47:179–201, 2007.

[17] Franco Montagna. Storage operators and multiplicative quantifiers in many-valued logics. *J. Logic Computation*, 14(2):299–322, 2004.

[18] P. S. Mostert and A. L. Shields. On the structure of semigroups on a compact manifold with boundary. *Annals of Math.*, 65:117–143, 1957.

[19] Jan Pavelka. On fuzzy logic I, II, III. *Z. Math. Logik Grundlagen Math.*, 25:45–52, 119–134, 447–464, 1979.

[20] Petr Savický, Roberto Cignoli, Francesc Esteva, Lluís Godo, and Carles Noguera. On product logic with truth constants. *J. Logic Computation*, 16(2):205–225, 2006.

[21] Thomas Vetterlein. Analytic calculi for logics of ordinal multiples of standard t-norms. *J. Logic Computation*, 18(1):35–57, 2008.