The multiplicative Schwarz method for matrices with a special block structure

Petr Tichý

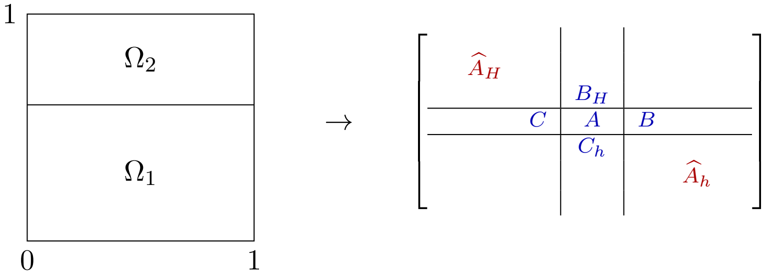
joint work with

Carlos Echeverría and Jörg Liesen

Algoritmy 2020

Motivation

Discretization of PDE's using finite differences



How to solve efficiently the corresponding linear systems?

Related papers

- **C. Echeverría, J. Liesen, P. Tichý, D. Szyld**, Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on a Shishkin mesh, Electron. Trans. Numer. Anal., 2018.
- **C. Echeverría, J. Liesen, and P. Tichý**, Analysis of the multiplicative Schwarz method for matrices with a special block structure, arXiv.org/abs/1912.09107

Solving linear systems with the system matrix

$$\left[\begin{array}{c|c|c}
\widehat{A}_H & B_H & \\
\hline
C & A & B \\
\hline
 & C_h & \widehat{A}_h
\end{array}\right]$$

Linear solver

and geometry of the problem

$$\mathcal{A} = \left[\begin{array}{c|c|c} \widehat{A}_H & B_H & \\ \hline C & A & B \\ \hline & C_h & \widehat{A}_h \end{array}\right] \in \mathbb{R}^{N(2m+1)\times N(2m+1)}$$

- Systems with submatrices easily solvable (Toeplitz).
- Use the **multiplicative Schwarz method**.
- Restriction op. $R_1 = \left[\begin{array}{cc} I_{N(m+1)} & 0 \end{array}\right]$, $R_2 = \left[\begin{array}{cc} 0 & I_{N(m+1)} \end{array}\right]$.

Multiplicative Schwarz method

- Given $x^{(k)}$, then $x = x^{(k)} + y$ and y satisfies

$$\mathcal{A}y = b - \mathcal{A}x^{(k)} \equiv r^{(k)}.$$

- **Restriction** to the **first domain**

$$(R_1 \mathcal{A} R_1^T)\, y_1 \;=\; R_1 r^{(k)}$$

and **prolongation**

$$x^{(k+\frac{1}{2})} = x^{(k)} + R_1^T y_1.$$

- Do the same with $x^{(k+\frac{1}{2})}$ on the **second domain**

$$x^{(k+1)} = x^{(k+\frac{1}{2})} + R_2^T y_2.$$

Multiplicative Schwarz method

as a stationary method

- Consistent stationary method

$$x^{(k)} = \overbrace{(I - P_2)(I - P_1)}^{T} x^{(k-1)} + v$$

$$\downarrow$$

$$x - x^{(k)} = T^k \, (x - x^{(0)})$$

- Bounds

$$\|x - x^{(k)}\| \le \|T^k\| \, \|x - x^{(0)}\| \le \|T\|^k \, \|x - x^{(0)}\|$$

- We would like to show that

$$\|T^k\| = \rho^k, \qquad \rho \le ?$$

Schwarz method as a preconditioner

- Consistent scheme

$$x^{(k+1)} = T\,x^{(k)} + v$$

- **Preconditioned** system

$$(I - T)x = v$$

- If T has **rank** ℓ, then

$$\dim\left(\mathcal{K}_k(I - T, r_0)\right) \leq \ell + 1$$

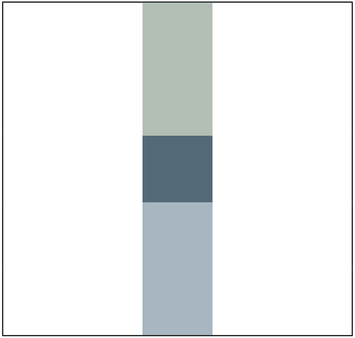
and GMRES converges in **at most** $\ell + 1$ **iterations**.

Structure of T for

$$\left[\begin{array}{c|c|c} \widehat{A}_H & B_H & \\ \hline C & A & B \\ \hline & C_h & \widehat{A}_h \end{array}\right]$$

$$x^{(k+1)} = T\,x^{(k)} + v$$

Structure of T



$$T = (I - P_2)(I - P_1)$$

$$\|T^{k+1}\| \;\leq\; \rho^k \,\|T\|$$

$$\rho = \|Z_{11}^{(h)} C_h \Pi^{(2)} Z_{mm}^{(H)} B_H \Pi^{(1)}\|$$

$$\widehat{A}_h^{-1} = [Z_{ij}^{(h)}], \quad \Pi^{(2)} = \left(A - B Z_{11}^{(h)} C_h\right)^{-1} C$$

How to bound **norms of blocks** of inverses of $\widehat{A}_h$ and $\widehat{A}_H$?

Block tridiagonal case

$$
\left[\begin{array}{c|c|c}
\widehat{A}_H & B_H & \\
\hline
C & A & B \\
\hline
 & C_h & \widehat{A}_h
\end{array}\right]
$$

Block tridiagonal case

New results of [Echeverría, Liesen, Nabben, 2018]

$$\widehat{A}_h = \begin{bmatrix} A_h & B_h & & \\ C_h & \ddots & \ddots & \\ & \ddots & \ddots & B_h \\ & & C_h & A_h \end{bmatrix}, \qquad \widehat{A}_H = \dots$$

$\widehat{A}_h$ is **row block diagonally dominant** if

$$\|A_h^{-1} B_h\| + \|A_h^{-1} C_h\| \leq 1$$

How to bound $\|Z_{ij}^{(h)}\|$? [Echeverría, Liesen, Nabben, 2018]

Bounding ρ

for $\mathcal{A}$ **row and column** block **diagonally dominant**

Using [Echeverría, Liesen, Nabben, 2018] we have shown

$$\rho \;\leq\; \frac{\eta_h \|A^{-1}C\|}{1-\eta_h\|A^{-1}B\|}\,\frac{\eta_H \|A^{-1}B\|}{1-\eta_H\|A^{-1}C\|}$$

where $\|\cdot\|$ is any induced matrix norm and

$$\eta_h \;=\; \min\left\{\frac{\|A_h^{-1}C_h\|}{1-\|A_h^{-1}B_h\|}, \frac{\|A_h^{-1}\|\|C_h\|}{1-\|C_hA_h^{-1}\|}\right\}$$

Bounds now contain only inverses of **individual blocks**.

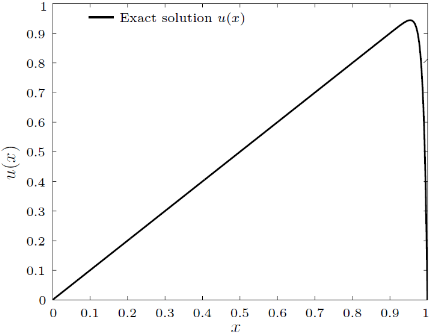
$$\|x - x^{(k)}\| \;\leq\; \rho^k\, \|T\|\, \|x - x^{(0)}\|\,.$$

Application to singularly perturbed convection-diffusion equation

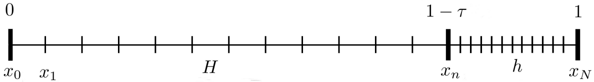
$$-\varepsilon\,\Delta u + \alpha\,u_y + \beta\,u = f$$

One-dimensional case

$$-\varepsilon\, u'' + \alpha\, u' + \beta\, u = f$$



Shishkin mesh $\to$ uniform convergence



The standard upwind difference scheme

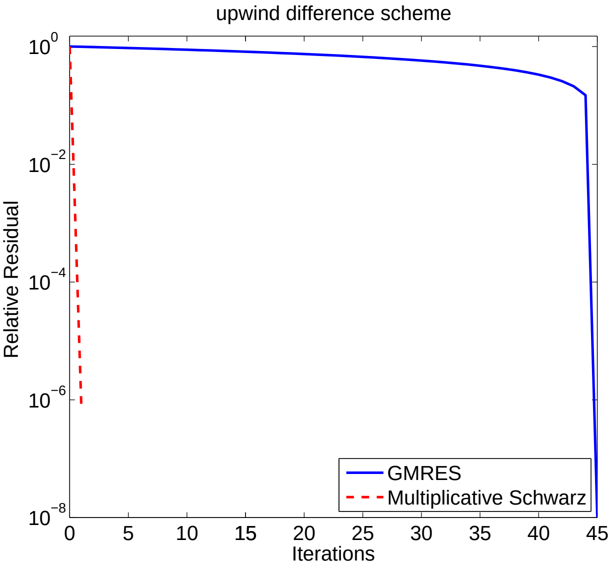
$$\mathcal{A} = \left[\begin{array}{c|c|c} A_H & b_H & \\ \hline c & a & b \\ \hline & c_h & A_h \end{array}\right]$$

nonsymmetric M-matrix

$$\mathcal{A} = Y \Lambda Y^{-1}$$

$\varepsilon = 10^{-8}$	original	scaled
$\kappa(\mathcal{A})$	10^{10}	10^{3}
$\kappa(Y)$	10^{17}	10^{19}

GMRES versus Schwarz



Multiplicative Schwarz method

results [Echeverría, Liesen, Tichý, Szyld, 2018]

We have shown for $\|\cdot\| = \|\cdot\|_\infty$ that

$$\|x - x^{(k+1)}\| \leq \rho^k \|T\| \|x - x^{(0)}\|$$

where

$$\rho < \frac{\varepsilon}{\varepsilon + \frac{\alpha}{N}}$$

and

$$\|T\| \leq \rho \quad \text{or} \quad \|T\| \leq 1,$$

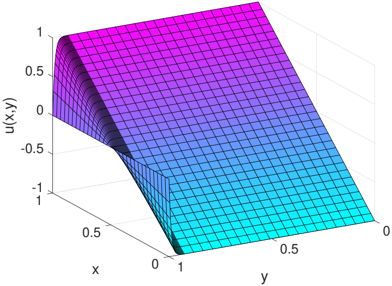
depending on the order of domains in the definition of T.

- Rank-one structure of T
- Schwarz as a preconditioner
- Preconditioned GMRES $\rightarrow$ at most 2 iterations

Two-dimensional case

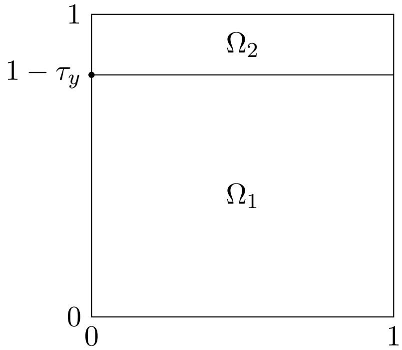
A problems with **one** boundary layer

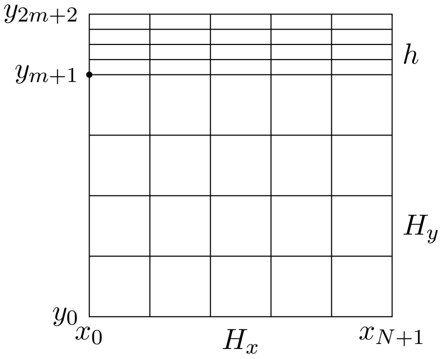
$$-\varepsilon\,\Delta u \;+\; u_y \;=\; 0$$



$$u(x,y) = (2x-1)\left(\frac{1-e^{(y-1)/\varepsilon}}{1-e^{-1/\varepsilon}}\right)$$

Shishkin mesh





Use the standard upwind difference scheme.

Application to the convection-diffusion equation

Discretization on the Shishkin mesh

$$-\varepsilon\,\Delta u + \alpha\,u_y + \beta\,u = f$$

$$\mathcal{A} = \left[\begin{array}{cccc|c|cccc}
A_H & B_H & & & & & & & \\
C_H & \ddots & \ddots & & & & & & \\
 & \ddots & \ddots & B_H & & & & & \\
 & & C_H & A_H & B_H & & & & \\
\hline
 & & & C & A & B & & & \\
\hline
 & & & & C_h & A_h & B_h & & \\
 & & & & & C_h & \ddots & \ddots & \\
 & & & & & & \ddots & \ddots & B_h \\
 & & & & & & & C_h & A_h
\end{array}\right]$$

$C_H,\ C,\ C_h,\ B_H,\ B,\ B_h,$ .... scalar multiples of I

$A_H,\ A,\ A_h$ .... tridiagonal and Toeplizt

$\mathcal{A}$ .... row and column block diagonally dominant

Application to the convection-diffusion equation
discretized on the Shishkin mesh

In [Echeverría, Liesen, Tichý, 2020] we have shown for $\|\cdot\| = \|\cdot\|_\infty$ that

$$\|x - x^{(k+1)}\| \;\leq\; \rho^k \, \|T\| \, \|x - x^{(0)}\|$$

where

$$\rho \;<\; \frac{\varepsilon}{\varepsilon + \frac{\alpha}{m}}$$

and

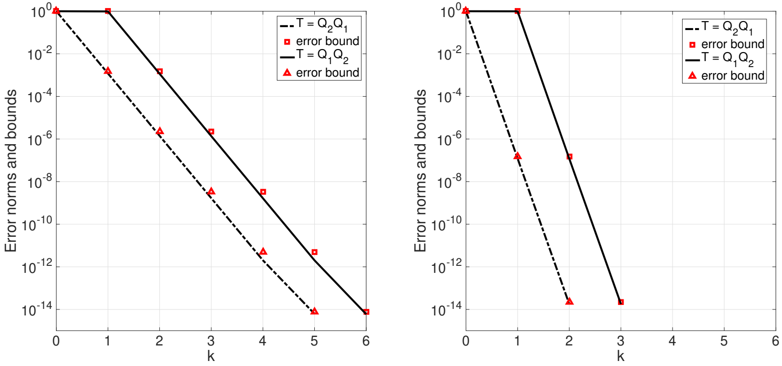
$$\|T\| \;\leq\; \rho \quad \text{or} \quad \|T\| \;\leq\; 1\,,$$

depending on the order of domains in the definition of T.

- Low-rank structure of T
- Schwarz as a preconditioner
- Preconditioned GMRES $\rightarrow$ at most $N + 1$ iterations

Tightness of the bound

$$N = 30, \quad m = 20, \quad \mathcal{A} \in \mathbb{R}^{1230\times 1230}, \quad \alpha = 1, \quad \beta = 0$$



Convergence of multiplicative Schwarz and error bounds
for $\varepsilon = 10^{-4}$ (left) and $\varepsilon = 10^{-8}$ (right)

Conclusions

- Generalization of results [Echeverría, Liesen, Tichý, Szyld, 2018].

- We analyzed **convergence** of the multiplicative Schwarz method applied to systems with a special block structure

$$\left[\begin{array}{c|c|c} \widehat{A}_H & B_H & \\ \hline C & A & B \\ \hline & C_h & \widehat{A}_h \end{array}\right]$$

- Detailed results for **block tridiagonal** matrices.

- For a particular problem $\rightarrow$ tight and **simple bounds**.

Related papers

- **C. Echeverría, J. Liesen, and P. Tichý**, Analysis of the multiplicative Schwarz method for matrices with a special block structure, accepted, Electron. Trans. Numer. Anal., 2020.
- **C. Echeverría, J. Liesen, and R. Nabben**, Block diagonal dominance of matrices revisited: bounds for the norms of inverses and eigenvalue inclusion sets, Linear Algebra Appl. 553, 2018.
- **C. Echeverría, J. Liesen, P. Tichý, and D. Szyld**, Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on . . . , Electron. Trans. Numer. Anal. 48, 2018.
- **H-G. Roos, M. Stynes, L. Tobiska**, Robust numerical methods for singularly perturbed differential equations, Springer-Verlag, Berlin, 2008.
- **M. Stynes**, Steady-state convection-diffusion problems, Acta Numer. 14, 2005.

Thank you for your attention!

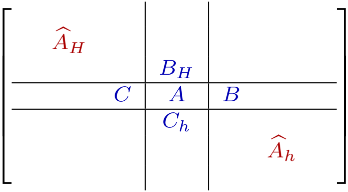
Open problems

Practical implementation issues

To use the iterative scheme

$$x^{(k)} = T\,x^{(k-1)} + v$$

we need to solve **linear systems with submatrices** of



- Schur complement and fast Toeplitz solvers?
- Problems with non-constant coefficients?
- Inexact solvers?

Additive Schwarz method

$$x^{(k+1)} = Tx^{(k)} + v, \quad T \equiv I - (P_1 + P_2)$$

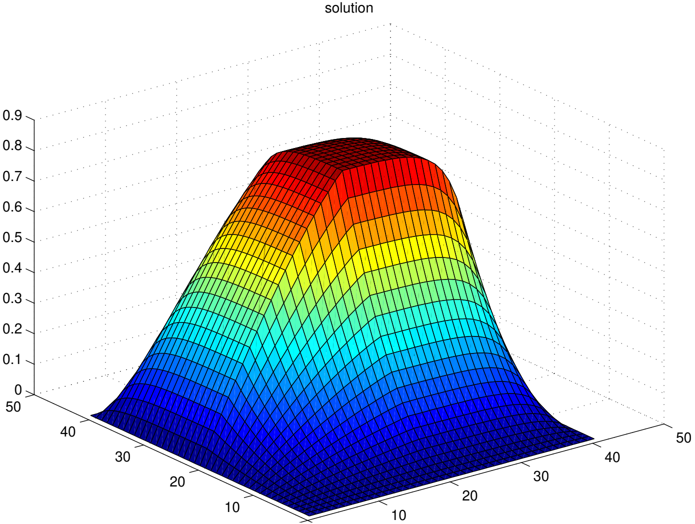
where

$$T = - \begin{bmatrix} 0_{N(m-1)} & & & P^{(1)}_{1:m-1} & \\ & & & P^{(1)}_m & \\ & \Pi^{(2)} & I_N & \Pi^{(1)} & \\ & P^{(2)}_1 & & & \\ & P^{(2)}_{2:m} & & & 0_{N(m-1)} \end{bmatrix}.$$

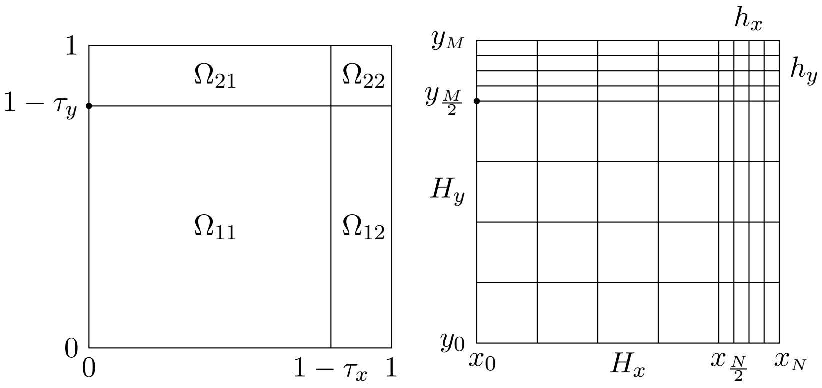
- $\rho(T) \geq 1$
- $I - T$ is nonsingular, T is low rank
- can be used as a preconditioner

Two boundary layers

$$-\varepsilon\,\Delta u \;+\; \alpha_1\,u_x \;+\; \alpha_2\,u_y \;+\; \beta\,u = f$$



Shishkin mesh



- Definition of the multiplicative Schwarz method?
- Structure of $\mathcal{A}$?
- Is T low-rank?