On Chebyshev Polynomials of Matrices

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joint work with

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Chebyshev polynomials of a compact set

- Chebyshev polynomials on the interval $[-1;1]$ [Chebyshev 1859].
- Generalized by [Georg Faber 1920] to the idea of the Chebyshev polynomials of Ω, where Ω is a compact set in the complex plane $\mathbb{C}$: These polynomials $T_m^{\Omega}(z)$ solve the problem

$$\min_{p \in \mathcal{M}_m(\Omega)} \| \, p(z) \, \|_\infty$$

where $\mathcal{M}_m$ is the class of monic polynomials of degree m.

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Example:

Ω is an interval, a set of discrete points, the unit circle, etc.

Chebyshev polynomials of normal matrices

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be normal, i.e.,

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^*, \quad \mathbf{Q}^*\mathbf{Q} = \mathbf{I}\,.$$

Let $\|\cdot\|$ be the spectral norm and consider the problem

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Then

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where $\Omega = \{\lambda_1, \ldots, \lambda_n\}$.

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where $\Omega = \{\lambda_1, \ldots, \lambda_n\}$.

The problem for $\mathbf{A}$ is solved by the Chebyshev polynomial of Ω.

Chebyshev polynomials of general matrices

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be a general matrix. We consider the problem

$$\min_{p \in \mathcal{M}_m} \| p(\mathbf{A}) \| .$$

- Introduced in [Greenbaum, Trefethen 1994].
- Unique solution $T_m^A(z) \in \mathcal{M}_m$ exists if $m < d(\mathbf{A})$, [Greenbaum, Trefethen 1994; Liesen, T. 2009].
- $T_m^A(z)$ is called the mth Chebyshev polynomial of $\mathbf{A}$, or the mth ideal Arnoldi polynomial of $\mathbf{A}$.
- Previous work on these polynomials in [Toh PhD thesis 1996], [Toh, Trefethen 1998], [Trefethen, Embree 2005] .
- Here: [Faber, Liesen, T. 2010].

Motivation

[Toh, Trefethen 1998] „Chebyshev polynomials of matrices are never far away from any discussion of convergence of Krylov subspace iterations in numerical linear algebra".

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GMRES and Arnoldi approximation problems:

[Greenbaum, Trefethen 1994]

$$\min_{p \in \pi_m} \| p(\mathbf{A}) b \| \quad \text{(GMRES)}, \qquad \min_{q \in \mathcal{M}_m} \| q(\mathbf{A}) b \| \quad \text{(Arnoldi)},$$

$$b \approx \{\mathbf{A}b, \ldots, \mathbf{A}^m b\}, \qquad \mathbf{A}^m b \approx \{b, \ldots, \mathbf{A}^{m-1} b\}.$$

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$$b \approx \{\mathbf{A}b, \ldots, \mathbf{A}^m b\}, \qquad \mathbf{A}^m b \approx \{b, \ldots, \mathbf{A}^{m-1} b\}.$$

One may remove b from the discussion and pose the following "ideal" approximation problems:

$$\min_{p \in \pi_m} \| p(\mathbf{A}) \| \quad \text{(Ideal GMRES)}, \qquad \min_{q \in \mathcal{M}_m} \| q(\mathbf{A}) \| \quad \text{(Ideal Arnoldi)},$$

$$I \approx \{\mathbf{A}, \ldots, \mathbf{A}^m\}, \qquad \mathbf{A}^m \approx \{I, \ldots, \mathbf{A}^{m-1}\}$$

(Chebyshev polynomial of $\mathbf{A}$)

Motivation Example

Let $\lambda \in \mathbb{C}$. Consider an n by n Jordan block

$$\mathbf{J}_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{C}^{n \times n}.$$

Question: How do the ideal GMRES and Chebyshev polynomials of $\mathbf{J}_\lambda$ look like?

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Question: How do the ideal GMRES and Chebyshev polynomials of $\mathbf{J}_\lambda$ look like?

- Ideal GMRES polynomial of $\mathbf{J}_\lambda$ - a very difficult problem [T., Liesen, Faber 2007].
- Chebyshev polynomial of $\mathbf{J}_\lambda$ [Liesen, T. 2009]:

$$T_m^{\mathbf{J}_\lambda}(z) = (z-\lambda)^m .$$

Outline

1. General results
2. Examples
3. Matrices and sets in the complex plane

Outline

Shifts and scaling

Theorem [Faber, Liesen, T. 2010]

For $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\alpha \in \mathbb{C}$ the following hold:

$$\min_{p\in\mathcal{M}_m} \|p(\,\mathbf{A} + \alpha\,\mathbf{I}\,)\| = \min_{p\in\mathcal{M}_m} \|p(\mathbf{A})\|\,,$$

$$\min_{p\in\mathcal{M}_m} \|p(\,\alpha\,\mathbf{A}\,)\| = |\alpha|^m \min_{p\in\mathcal{M}_m} \|p(\mathbf{A})\|\,.$$

- Shift invariance: Not surprising, because the polynomials are normalized at infinity.
- Paper contains explicit relations between the coefficients of $T_m^{\mathbf{A}}(z)$, $T_m^{\mathbf{A}+\alpha\mathbf{I}}(z)$, and $T_m^{\alpha\mathbf{A}}(z)$.

Example - shift of a matrix

Let $a, b \in \mathbb{R}$ be given. Consider the block-diagonal matrix $\mathbf{A}$ with two $n \times n$ Jordan blocks,

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{J}_a & 0 \\ 0 & \mathbf{J}_b \end{bmatrix}.$$

Define

$$\alpha \equiv \frac{a+b}{2}.$$

Then

$$\mathbf{A} - \alpha \mathbf{I} = \begin{bmatrix} \mathbf{J}_\lambda & 0 \\ 0 & \mathbf{J}_{-\lambda} \end{bmatrix} \quad \text{where} \quad \lambda \equiv \frac{a-b}{2},$$

and the previous theorem implies

$$\min_{p \in \mathcal{M}_m} \|p(\mathbf{A})\| = \min_{p \in \mathcal{M}_m} \|p(\mathbf{A} - \alpha \mathbf{I})\|.$$

Symmetry with respect to the origin

The Chebyshev polynomials of real intervals that are symmetric with respect to the origin are alternating between *even* and *odd*, i.e.

$$T_m^{[-a,a]}(z) \;=\; (-1)^m \, T_m^{[-a,a]}(-z)\,.$$

Analogous result for Chebyshev polynomials of $\mathbf{A}$?

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Theorem [Faber, Liesen, T. 2010]

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and a positive integer $m < d(\mathbf{A})$ be given. If there exists a unitary matrix $\mathbf{P}$ such that either

$$\mathbf{P}^*\mathbf{A}\mathbf{P} \;=\; -\mathbf{A} \quad \text{or} \quad \mathbf{P}^*\mathbf{A}\mathbf{P} = -\mathbf{A}^T\,,$$

then

$$T_m^{\mathbf{A}}(z) \;=\; (-1)^m \, T_m^{\mathbf{A}}(-z)\,.$$

Example

$$\mathbf{A} \;=\; \begin{bmatrix} \mathbf{J}_\lambda & \\ & \mathbf{J}_{-\lambda} \end{bmatrix}, \qquad \mathbf{P} \;=\; \begin{bmatrix} & \mathbf{I}^{\pm} \\ \mathbf{I}^{\pm} & \end{bmatrix},$$

where $\mathbf{I}^{\pm} = \mathrm{diag}(1, -1, 1, \ldots, (-1)^{n-1})$. Then

$$\mathbf{J}_{-\lambda} \;=\; -\mathbf{I}^{\pm}\mathbf{J}_\lambda\mathbf{I}^{\pm} \quad \Rightarrow \quad \mathbf{P}^*\mathbf{A}\mathbf{P} \;=\; -\mathbf{A}\,.$$

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$$\mathbf{J}_{-\lambda} \;=\; -\mathbf{I}^\pm \mathbf{J}_\lambda \mathbf{I}^\pm \quad \Rightarrow \quad \mathbf{P}^* \mathbf{A} \mathbf{P} \;=\; -\mathbf{A}\ .$$

Moreover

$$T_m^{\mathbf{A}}(\mathbf{A}) \;=\; \begin{bmatrix} T_m^{\mathbf{A}}(\mathbf{J}_\lambda) & \\ & T_m^{\mathbf{A}}(\mathbf{J}_{-\lambda}) \end{bmatrix},$$

and

$$\|\, T_m^{\mathbf{A}}(\mathbf{J}_{-\lambda}) \,\| \;=\; \|\, \mathbf{I}^\pm T_m^{\mathbf{A}}(-\mathbf{J}_\lambda) \mathbf{I}^\pm \,\| \;=\; \|\, T_m^{\mathbf{A}}(\mathbf{J}_\lambda) \,\|,$$

i.e., the Chebyshev polynomial of $\mathbf{A}$ attains the same norm on each of the two diagonal blocks.

An alternation theorem

- Chebyshev polynomials for compact sets are characterized by alternation properties.
- Example: $T_m(z)$ for $[a, b] \subset \mathbb{R}$ has at least $m+1$ alternations.

An Alternation Theorem for Matrices [Faber, Liesen, T. 2010]

Consider a block-diagonal matrix $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_h)$ where $d(\mathbf{A}_j) \leq k$, $j = 1, \ldots, h$. Then the matrix

$$T_{k \cdot \ell}^{\mathbf{A}}(\mathbf{A}) = \mathrm{diag}(\mathbf{B}_1, \ldots, \mathbf{B}_h) \qquad \ell = 1, 2, \ldots,$$

has at least $\ell + 1$ diagonal blocks $\mathbf{B}_j$ such that

$$\| \mathbf{B}_j \| = \| T_{k \cdot \ell}^{\mathbf{A}}(\mathbf{A}) \| .$$

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$$\| \mathbf{B}_j \| = \| T_{k \cdot \ell}^{\mathbf{A}}(\mathbf{A}) \| .$$

Example: If $\mathbf{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n \times n}$, then $T_m^{\mathbf{A}}(\mathbf{A})$ has at least $m + 1$ diagonal entries with the same maximal absolute value.

Example

$$\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4)$$

where each $\mathbf{A}_j = \mathbf{J}_{\lambda_j}$ is a 3×3 Jordan block. The four eigenvalues are $-3,\ -0.5,\ 0.5,\ 0.75$, and $k = d(\mathbf{A}_j) = 3$.

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m	$\\|T_m^{\mathbf{A}}(\mathbf{A}_1)\\|$	$\\|T_m^{\mathbf{A}}(\mathbf{A}_2)\\|$	$\\|T_m^{\mathbf{A}}(\mathbf{A}_3)\\|$	$\\|T_m^{\mathbf{A}}(\mathbf{A}_4)\\|$
1	$\underline{2.6396}$	1.4620	2.3970	$\underline{2.6396}$
2	$\underline{4.1555}$	$\underline{4.1555}$	3.6828	$\underline{4.1555}$
3	$\underline{9.0629}$	5.6303	7.6858	$\underline{9.0629}$
4	$\underline{14.0251}$	$\underline{14.0251}$	11.8397	$\underline{14.0251}$
5	$\underline{22.3872}$	20.7801	17.6382	$\underline{22.3872}$
6	$\underline{22.6857}$	$\underline{22.6857}$	20.3948	$\underline{22.6857}$
7	$\underline{26.3190}$	$\underline{26.3190}$	$\underline{26.3190}$	$\underline{26.3190}$

Outline

Perturbed Jordan block

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \nu & & & 0 \end{bmatrix} = \nu(\mathbf{J}_0^T)^{n-1} + \mathbf{J}_0 \in \mathbb{C}^{n\times n},$$

$\nu \in \mathbb{C}$ is a complex parameter (studied by [Greenbaum 2009]).

We have $d(\mathbf{A}) = n$ for any $\nu \in \mathbb{C}$.

Perturbed Jordan block [Faber, Liesen, T. 2010]

For $1 \le m \le n-1$ and any $\nu \in \mathbb{C}$:

$$T_m^{\mathbf{A}}(z) = z^m \,.$$

Special bidiagonal matrices

Given are $\lambda_1, \ldots, \lambda_\ell \in \mathbb{C}$ and $n \geq 1$. Consider

$$\mathbf{A} = \begin{bmatrix} \mathbf{D} & \mathbf{E} & & \\ & \mathbf{D} & \ddots & \\ & & \ddots & \mathbf{E} \\ & & & \mathbf{D} \end{bmatrix} \in \mathbb{C}^{\ell \cdot h \times \ell \cdot h},$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_\ell \end{bmatrix} \in \mathbb{C}^{\ell \times \ell}, \quad \mathbf{E} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} \in \mathbb{R}^{\ell \times \ell},$$

[Reichel, Trefethen 1992] related the pseudospectra of $\mathbf{A}$ to their symbol $f_{\mathbf{A}}(z) = \mathbf{D} + z\mathbf{E}$.

Special bidiagonal matrices

Special bidiagonal matrices [Faber, Liesen, T. 2010]

Consider the matrix $\mathbf{A}$ defined above. Let $\chi_{\mathbf{D}}(z)$ be the characteristic polynomial of $\mathbf{D}$,

$$\chi_{\mathbf{D}}(z) = (z - \lambda_1) \cdot \ldots \cdot (z - \lambda_\ell)\,.$$

Then

$$T^{\mathbf{A}}_{k\cdot\ell}(z) = \left(\chi_{\mathbf{D}}(z)\right)^k\,, \qquad k = 1, 2, \ldots, h-1\,.$$

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Question: Can we find a set $S \subset \mathbb{C}$ such that

$$T^{\mathbf{A}}_{k\cdot\ell}(z) = T^{S}_{k\cdot\ell}(z)$$

for this matrix $\mathbf{A}$?

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Chebyshev polynomials $T_m^{\Omega}(z)$ of compact sets $\Omega \subset \mathbb{C}$

...unique polynomials that solve the problem

$$\min_{p \in \mathcal{M}_m} \max_{z \in \Omega} |p(z)| .$$

Chebyshev polynomials of Ω and Ψ [Kamo, Borodin 1994]

Let T_k^{Ω} be the kth Chebyshev polynomial of the infinite compact set $\Omega \subset \mathbb{C}$, let $p(z)$ be a monic polynomial of degree ℓ, and let

$$\Psi \equiv p^{-1}(\Omega) = \{z \in \mathbb{C} : p(z) \in \Omega\}$$

be the pre-image of Ω under the polynomial map p. Then

$$T_{k \cdot \ell}^{\Psi}(z) = T_k^{\Omega}(p(z)) .$$

Chebyshev polynomials for lemniscates

$$\mathbf{A} = \begin{bmatrix} \mathbf{D} & \mathbf{E} & & \\ & \mathbf{D} & \ddots & \\ & & \ddots & \mathbf{E} \\ & & & \mathbf{D} \end{bmatrix} \in \mathbb{C}^{\ell \cdot h \times \ell \cdot h}.$$

- Let $p(z) = (z - \lambda_1) \cdot \cdots \cdot (z - \lambda_\ell)$.
- The lemniscatic region $\mathcal{L}(p) \equiv \{z \in \mathbb{C} \ : \ |p(z)| \leq 1\}$.
- $\Psi \ \equiv \ \mathcal{L}(p)$, $\Omega \ \equiv$ the unit circle.

Chebyshev polynomials of $\mathbf{A}$ and of $\mathcal{L}(p)$ [Faber, Liesen, T. 2010]

$$T_{k\cdot\ell}^{\mathcal{L}(p)}(z) \ = \ (p(z))^k \ = \ T_{k\cdot\ell}^{\mathbf{A}}(z), \qquad k = 1, 2, \ldots, h-1 \,.$$

Moreover,

$$\max_{z \in \mathcal{L}(p)} |T_{k\cdot\ell}^{\mathcal{L}(p)}(z)| \ = \ \|T_{k\cdot\ell}^{\mathbf{A}}(\mathbf{A})\| \,.$$

Summary

- We considered Chebyshev polynomials of matrices and showed general properties (shifts and scaling, alternation).
- We can relate Chebyshev polynomials for lemniscatic regions to those for certain block-Toeplitz matrices.

Summary

- We considered Chebyshev polynomials of matrices and showed general properties (shifts and scaling, alternation).
- We can relate Chebyshev polynomials for lemniscatic regions to those for certain block-Toeplitz matrices.

Open question: Is it possible to translate the problem

$$\min_{p \in \mathcal{M}_m} \| p(\mathbf{A}) \|$$

into the problem

$$\min_{p \in \mathcal{M}_m} \max_{z \in \Omega} |p(z)|$$

where Ω is a set in the complex plane associated with $\mathbf{A}$?

Related papers

- V. Faber, J. Liesen and P. Tichý,
  [On Chebyshev polynomials of matrices, accepted for publication in SIMAX (2010).]
- K-C. Toh, N. L. Trefethen,
  [The Chebyshev polynomials of a matrix, SIMAX 20 (1999), no. 2, 400–419]
- A. Greenbaum and N. L. Trefethen,
  [GMRES/CR and Arnoldi/Lanczos as matrix approximation problems, SISC 15 (1994), no. 2, 359–368]

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http://www.cs.cas.cz/tichy
http://www.math.tu-berlin.de/~liesen

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