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SHORT COMMUNICATION

On unique solvability of the absolute value equation

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Abstract It is proved that the singular value condition $\sigma_{\max}(|B|) < \sigma_{\min}(A)$ implies unique solvability of the absolute value equation $Ax + B|x| = b$ for each right-hand side b. This is a generalization of an earlier result by Mangasarian and Meyer proved for the special case of $B = -I$.

Keywords Absolute value equation · Unique solution · Singular values

1 Introduction

In this note we consider unique solvability on the absolute value equation

$$Ax + B|x| = b, \tag{1}$$

where $A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and the absolute value of a vector $x = (x_i)$ is defined by $|x| = (|x_i|)$. We show that the condition

$$\sigma_{\max}(|B|) < \sigma_{\min}(A),$$

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where $\sigma_{\max}, \sigma_{\min}$ denote the maximal and minimal singular value, respectively, implies unique solvability of the equation (1) for each right-hand side b. This generalizes an earlier result by Mangasarian and Meyer who showed in [6], Proposition 3, (i) that the equation

$$Ax - |x| = b$$

is uniquely solvable for each $b \in \mathbb{R}^n$ if $\sigma_{\min}(A) > 1$. This is a special case of our result for $B = -I$ (I is the unit matrix) since $\sigma_{\max}(|-I|) = 1$.

The absolute value equation (1) seems to be a useful tool in optimization since it subsumes the linear complementarity problem and thus also linear programming and convex quadratic programming. It has been recently studied by Mangasarian [3–5], Mangasarian and Meyer [6], Prokopyev [7], Rohn [8,9] and Schäfer [10]. In particular, Prokopyev [7] proved that checking unique solvability of the absolute value equation (1) is NP-hard.

2 The result

In the proof of the main result we shall essentially utilize a theorem of the alternatives from [8].

Theorem 1 *Let $A, D \in \mathbb{R}^{n\times n}$, $D \geq 0$. Then exactly one of the following alternatives holds:*

(i) *for each $B \in \mathbb{R}^{n\times n}$ with $|B| \leq D$ and for each $b \in \mathbb{R}^n$ the equation*

$$Ax + B|x| = b \tag{2}$$

has a unique solution,

(ii) *there exist $\lambda \in [0, 1]$ and a ± 1-vector y such that the equation*

$$Ax + \lambda\, \mathrm{diag}(y) D|x| = 0 \tag{3}$$

has a nontrivial solution.

Our main result is then formulated as follows.

Theorem 2 *Let $A, B \in \mathbb{R}^{n\times n}$ satisfy*

$$\sigma_{\max}(|B|) < \sigma_{\min}(A). \tag{4}$$

Then for each right-hand side $b \in \mathbb{R}^n$ the absolute value equation

$$Ax + B|x| = b \tag{5}$$

has a unique solution.

Proof The proof proceeds by contradiction. Assume to the contrary that the equation (5) does not have a unique solution. Then Theorem 1 with $D := |B|$ implies existence of a $\lambda \in [0, 1]$ and of a ± 1-vector y such that the equation

$$Ax + \lambda \mathrm{diag}(y)|B||x| = 0 \tag{6}$$

has a nontrivial solution x which can be normalized so that $\|x\|_2 = 1$. Then (6) implies that

$$|Ax| = \left|-\lambda \mathrm{diag}(y)|B||x|\right| \le |B||x|,$$

hence

$$x^T A^T Ax \le |(Ax)^T(Ax)| \le |Ax|^T|Ax| \le (|B||x|)^T(|B||x|) = |x|^T|B|^T|B||x|$$

and

$$\begin{aligned}\sigma_{\min}(A) &= \min_{\|z\|_2=1} z^T A^T Az \le x^T A^T Ax \le |x|^T|B|^T|B||x| \\ &\le \max_{\|z\|_2=1} z^T|B|^T|B|z = \sigma_{\max}(|B|),\end{aligned}$$

which contradicts (4).

3 Computing the unique solution

The software package VERSOFT, freely available at [2], contains a function VERABSVALEQN.M (VERified ABSolute VALue EQuatioN) which under the condition (4) yields the unique solution of (1). Its syntax is

```
x=verabsvaleqn(A,B,b)
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For more details, see [1].

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