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A Theorem of the Alternatives for the Equation $Ax + B|x| = b$

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The following theorem is proved: given square matrices A, D of the same size, D nonnegative, then either the equation $Ax+B|x|=b$ has a unique solution for each B with $|B|\leq D$ and for each b, or the equation $Ax+B_0|x|=0$ has a nontrivial solution for some matrix B_0 of a very special form, $|B_0|\leq D$; the two alternatives exclude each other. Some consequences of this result are drawn. In particular, we define a λ to be an absolute eigenvalue of A if $|Ax|=\lambda|x|$ for some $x\neq 0$, and we prove that each square real matrix has an absolute eigenvalue.

Keywords: Nonlinear equation; Existence; Uniqueness; Interval matrix; Eigenvalue

1 INTRODUCTION

Theorems of the alternatives are assertions stating that for each instance of the data, exactly one of the two (or, sometimes, more) alternatives (i), (ii) holds. They are not much frequent because the assertion can always be reformulated in a more usual form saying that (i) holds if and only if (ii) does not hold; but a formulation using alternatives may help to reveal some kind of formal similarity between the two assertions.

As a typical, and nontrivial, example, consider Farkas lemma [1] which says that a system

$$Ax = b \tag{1}$$

(with a general matrix $A \in \mathbb{R}^{m\times n}$) has a nonnegative solution if and only if each $p \in \mathbb{R}^m$ with $A^Tp \geq 0$ satisfies $b^Tp \geq 0$. It can be (and often is) stated in the form of a theorem of the alternatives: for each $A \in \mathbb{R}^{m\times n}$ and each $b \in \mathbb{R}^m$, exactly one of the two alternatives holds: (i) the system (1) has a nonnegative solution, and (ii) the system $A^Tp \geq 0$, $b^Tp < 0$ has a solution. In this version, the common feature is revealed: both assertions concern solvability of system of linear equations or inequalities.

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As the main result of this article we present a theorem of the alternatives for a nonlinear equation of the type

$$Ax + B|x| = b. \tag{2}$$

The theorem says that given two matrices $A, D \in \mathbb{R}^{n\times n}$, $D \geq 0$, then exactly one of the two alternatives holds: (i) the Eq. (2) has a unique solution for each B with $|B| \leq D$ and for each $b \in \mathbb{R}^n$, and (ii) the equation $Ax + B_0|x| = 0$ has a nontrivial solution for some B_0 of the form $B_0 = \lambda\, \mathrm{diag}(y)D$, where $\lambda \in [0, 1]$ and y is a ± 1-vector (so that $|B_0| \leq D$). Here and in the sequel, we use the following notations: for a matrix $B = (b_{ij})$ the absolute value is defined by $|B| = (|b_{ij}|)$, similarly for vectors; matrix or vector inequalities are understood componentwise; $y \in \mathbb{R}^n$ is called a ± 1-vector if $y_j \in \{-1, 1\}$ for $j = 1, \ldots, n$; and

$$\mathrm{diag}(y) = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_n \end{pmatrix}.$$

Hence, unlike most theorems of the alternatives, the result concerns nonlinear systems.

The proof of the main theorem, given in Section 2, is not long, but it relies on three nontrivial results published in [2,5,6], the last two of them concerning regularity or singularity of interval matrices. These two notions are the main tools used behind the scene in the proof, although not explicitly mentioned in formulation of the theorem.

In Section 3 we show that the alternative (ii) of the main theorem can be reformulated in two equivalent ways. As a consequence of it, we present a sufficient condition for an Eq. (2) with given A and B (i.e., no matrix D prescribed here) to have a unique solution for each $b \in \mathbb{R}^n$. In the next theorem we show that for given rational data A and D, the problem of determining which one of the alternatives (i), (ii) holds is NP-hard. A brief account of absolute eigenvalues, an offspring of the above results, concludes the article.

Finally, a few words should be said about why to study equations of the form (2). First, if $A + B$ is nonsingular, then, as shown in the proof of the main result, the Eq. (2) can be rewritten in the equivalent form

$$x^+ = (A + B)^{-1}(A - B)x^- + (A + B)^{-1}b,$$

where $x^+ = (|x| + x)/2$ and $x^- = (|x| - x)/2$, which is a linear complementarity problem [2]. Hence, (2) offers another way of formulating linear complementarity problems. Second, equations of type (2) arise quite naturally in solving systems of interval linear equations [5], and in fact this is the field where the inspiration for this article came from.

2 THEOREM OF THE ALTERNATIVES

The following theorem is the main result of this article:

Theorem 1 *Let $A, D \in \mathbb{R}^{n\times n}$, $D \geq 0$. Then exactly one of the following alternatives holds*:

(i) *for each $B \in \mathbb{R}^{n\times n}$ with $|B| \leq D$ and for each $b \in \mathbb{R}^n$ the equation*

$$Ax + B|x| = b \tag{3}$$

has a unique solution,

(ii) *there exist $\lambda \in [0, 1]$ and a ± 1-vector y such that the equation*

$$Ax + \lambda\,\mathrm{diag}(y)D|x| = 0 \tag{4}$$

has a nontrivial solution.

Proof Given A, $D \in \mathbb{R}^{n\times n}$, $D \geq 0$, consider the set

$$\mathbf{A} = \{A'; |A' - A| \leq D\} = \{A'; A - D \leq A' \leq A + D\},$$

which is called an interval matrix [5]. $\mathbf{A}$ is said to be regular if each $A' \in \mathbf{A}$ is nonsingular, and it is called singular otherwise (i.e., if it contains a singular matrix). We shall prove that (a) regularity of $\mathbf{A}$ implies (i), (b) singularity of $\mathbf{A}$ implies (ii), and (c) both (i) and (ii) cannot hold simultaneously. This will prove that exactly one of the alternatives (i), (ii) holds.

(a) Let $\mathbf{A}$ be regular and let $|B| \leq D$ and $b \in \mathbb{R}^n$. Then using the nonnegative vectors $x^+ = (|x| + x)/2$ and $x^- = (|x| - x)/2$, we have that $x = x^+ - x^-$ and $|x| = x^+ + x^-$, and we may rewrite the Eq. (3) into the equivalent form

$$x^+ = (A + B)^{-1}(A - B)x^- + (A + B)^{-1}b. \tag{5}$$

Since $|B| \leq D$, both matrices $A + B$ and $A - B$ belong to $\mathbf{A}$, hence $(A + B)^{-1}$ exists and, moreover, $(A + B)^{-1}(A - B)$ is a P-matrix by Theorem 1.2 in [5]. Hence the linear complementarity problem (5) has a unique solution [2], and the equivalent Eq. (3) has a unique solution as well.

(b) Let $\mathbf{A}$ be singular. Then the value

$$\lambda = \min\{\varepsilon \geq 0;\ \text{the interval matrix } [A - \varepsilon D, A + \varepsilon D] \text{ is singular}\}$$

belongs to $[0, 1]$ because $\mathbf{A} = [A - D,\ A + D]$ is singular, and Theorem 2.2 in [6] asserts that there exist ± 1-vectors y, z and an $x \neq 0$ such that

$$(A - \lambda\,\mathrm{diag}(y)D\,\mathrm{diag}(z))x = 0, \tag{6}$$

$$\mathrm{diag}(z)x \geq 0 \tag{7}$$

hold. Then (7) implies that $\operatorname{diag}(z)x = |x|$, and substituting this quantity into (6) we obtain

$$Ax - \lambda \operatorname{diag}(y)D|x| = 0,$$

so that it suffices to put $y := -y$ to conclude that the Eq. (4) has a nontrivial solution.

(c) Finally we show that (i) and (ii) cannot hold simultaneously. For, if (4) has a nontrivial solution x for some $\lambda \in [0, 1]$ and some ± 1-vector y, and if we put $B = \lambda \operatorname{diag}(y)D$, then $|B| = \lambda D \leq D$, hence (4) is of the form (3) for some B, but (4) has at least two solutions x and 0, which contradicts (i). ■

We add some comments on and some consequences of this result in the next section.

3 CONSEQUENCES

First we shall show that the assertion (ii) of Theorem 1 (identical with (α) below) can be recast in two equivalent ways:

THEOREM 2 *For A, $D \in \mathbb{R}^{n\times n}$, $D \geq 0$, the following assertions are equivalent*:

(α) *there exist $\lambda \in [0, 1]$ and a ± 1-vector y such that the equation*

$$Ax + \lambda \operatorname{diag}(y)D|x| = 0 \tag{8}$$

has a nontrivial solution,

(β) *there holds*

$$|Ax| = \lambda D|x| \tag{9}$$

for some $\lambda \in [0, 1]$ and $x \neq 0$,

(γ) *the inequality*

$$|Ax| \leq D|x| \tag{10}$$

has a nontrivial solution.

Proof We prove (α) ⇒ (β) ⇒ (γ) ⇒ (α). (α) ⇒ (β): If (α) holds, then, since $|\operatorname{diag}(y)| = I$, we have

$$|Ax| = |-\lambda \operatorname{diag}(y)D|x|| = \lambda D|x|,$$

where $\lambda \in [0, 1]$ and $x \neq 0$, which is (β). (β) ⇒ (γ): since $\lambda \in [0, 1]$, we obtain directly $|Ax| \leq D|x|$, which is (γ). (γ) ⇒ (α): if (10) holds for some $x \neq 0$, then the interval matrix $[A - D,\ A + D]$ is singular by Lemma 2.1 in [6], and from the part (b) of the proof of Theorem 1 it follows that (ii) holds, which gives (α). ■

Putting together the results of Theorems 1 and 2, we obtain this existence and uniqueness theorem for a single equation $Ax + B|x| = b$:

THEOREM 3 *Let $A, B \in \mathbb{R}^{n\times n}$ and let the inequality*

$$|Ax| \leq |B||x|$$

have the trivial solution only. Then the equation

$$Ax + B|x| = b$$

has a unique solution for each $b \in \mathbb{R}^n$.

Proof Put $D = |B|$. Then the assertion (γ) of Theorem 2 does not hold, hence neither does (α), which is the assertion (ii) of Theorem 1. Hence (i) holds, which gives that the equation $Ax + C|x| = b$ has a unique solution for each $b \in \mathbb{R}^n$ and each C satisfying $|C| \leq D = |B|$, thus in particular also for $C = B$. ■

Finally we show that the problem of deciding which one of the two alternatives (i), (ii) of Theorem 1 holds is NP-hard:

THEOREM 4 *The problem of checking whether the assertion* (i) *of Theorem* 1 *holds is NP-hard for rational square matrices* A, D *with* $D \geq 0$.

Proof We have seen in the proof of Theorem 1, parts (a) and (b), that regularity of the interval matrix $\mathbf{A} = [A - D, A + D]$ implies (i), whereas its singularity implies (ii). Hence $\mathbf{A}$ is regular if and only if (i) holds. Since the problem of checking regularity of interval matrices is NP-hard [4], the same is true for checking validity of (i) as well. ■

At the end of the article we shall briefly mention another interesting algebraic property which follows from our previous results. Given a square matrix A, let us call a λ an absolute eigenvalue of A if it satisfies

$$|Ax| = \lambda|x|$$

for some $x \neq 0$. It follows from the definition that an absolute eigenvalue of A is always nonnegative. It is well known that real matrices may have no real eigenvalues. The following rather surprising result shows that absolute eigenvalues always exist:

THEOREM 5 *Each square real matrix has an absolute eigenvalue.*

Proof Take an arbitrary vector $x > 0$ and let

$$\varepsilon = \max_i \frac{|Ax|_i}{x_i}.$$

Then we have $|Ax| \leq \varepsilon x = \varepsilon|x|$, which is the inequality (10) for $D = \varepsilon I$. Hence, by the equivalence (β) $\Leftrightarrow$ (γ) of Theorem 2, we have that there holds

$$|Ax| = \lambda D|x| = \lambda\varepsilon|x|$$

for some $x \neq 0$, which shows that $\lambda\varepsilon$ is an absolute eigenvalue of A. ■

If λ is a real eigenvalue of A, then $|\lambda|$ is an absolute eigenvalue of A because $Ax = \lambda x$ implies $|Ax| = |\lambda||x|$. In particular, if $A \geq 0$, then the spectral radius $\varrho(A)$ is an absolute eigenvalue of A [3], hence $\varrho(A)$ is an absolute eigenvalue of A. Absolute eigenvalues are likely to deserve further study; but we shall not follow this line here.

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