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Enclosing Solutions of Linear Interval Equations is NP-Hard

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Abstract — Zusammenfassung

Enclosing Solutions of Linear Interval Equations is NP-Hard. We show that if the conjecture $P \neq NP$ is true, then there does not exist a general polynomial-time algorithm for enclosing the solution set of a system of linear interval equations.

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Die Lösungseinschließung bei linearen Intervall-Gleichungen ist NP-hart. Unter Annahme der Vermutung $P \neq NP$ wird gezeigt, daß es keinen allgemeinen polynomialen Algorithmus gibt, der die Intervallhülle der Lösungsmenge eines Systems linearer Intervall-Gleichungen einschließt.

1. Introduction

For an $n \times n$ interval matrix

$$A^I = \{A; \underline{A} \leq A \leq \overline{A}\}$$

and an n-dimensional interval vector

$$b^I = \{b; \underline{b} \leq b \leq \overline{b}\}$$

(componentwise inequalities), the solution set of the system of linear interval equations

$$A^I x = b^I \tag{1}$$

is defined by

$$X(A^I, b^I) := \{x;\ Ax = b \text{ for some } A \in A^I, b \in b^I\}.$$

Due to the complicated (generally nonconvex) structure of the solution set, it is customary to estimate the range of the solution of (1) by finding an interval vector $x^I = \{x; \underline{x} \leq x \leq \overline{x}\}$ satisfying

$$X(A^I, b^I) \subset x^I$$

(provided $X(A^I, b^I)$ is bounded). Such an interval vector is called an *enclosure* of the solution set. Various enclosure methods exist to date (see the monographies by

Alefeld and Herzberger [1] or Neumaier [3] for a survey); however, none of them solves the problem in full generality in polynomial time. As an example, consider the interval Gaussian elimination ([1], [3]) which solves the system (1) by Gaussian elimination performed in interval arithmetic. If it can be carried out till the end, then it gives an enclosure of the solution set in polynomial time; however, it may fail at some stage (because each interval coefficient eligible for pivot contains zero) even in case that $X(A^I, b^I)$ is bounded (see Reichmann [5] for an example). Other polynomial-time methods known exhibit a similar behavior: each of them works for some subclass of problems of type (1) only.

In this paper we show that an existence of a fully general polynomial-time enclosure algorithm (which would yield an enclosure of $X(A^I, b^I)$ provided it is bounded and would issue an error message if $X(A^I, b^I)$ is unbounded) is very unlikely since it would imply that $P = NP$, i.e., that each problem solvable by a nondeterministic polynomial-time algorithm could also be solved by a polynomial-time algorithm, which runs contrary to the current common belief that $P \neq NP$ (see Garey and Johnson [2] for basic concepts of the complexity theory). Hence, there is a strong evidence that each effective enclosure method for the problem (1) is confined to work for a particular subclass of problems only.

2. The Result

Throughout this section, A^I always denotes an $n \times n$ interval matrix and b^I an n-dimensional interval vector. A^I is said to be regular if each $A \in A^I$ is nonsingular, otherwise (i.e. if it contains a singular matrix) it is called singular. First we prove an auxiliary result.

Proposition. *Let A^I contain at least one nonsingular matrix. Then the following assertions are equivalent*:

(i) *A^I is regular,*
(ii) *$X(A^I, b^I)$ is bounded for some b^I,*
(iii) *$X(A^I, b^I)$ is bounded for each b^I.*

Proof: (i) ⇒ (iii) follows from Cramer's rule by continuity of the determinant; (iii) ⇒ (ii) is obvious; (ii) ⇒ (i) is proved by contradiction. By assumption, A^I contains a nonsingular matrix A_0; assume to the contrary that it also contains a singular matrix A_1. For each $t \in [0, 1]$ define

$$A_t = A_0 + t(A_1 - A_0),$$

then $A_t \in A^I$ in view of the convexity of A^I. Furthermore, let

$$\tau = \min\{t \in [0, 1];\ A_t \text{ is singular}\};$$

such a τ exists since the set of singular matrices contained in A^I is closed. Hence $\tau \in (0, 1]$ and A_τ is singular. Let b^I be arbitrary nonempty. Take a $b \in b^I$ and choose a sequence $\{t_j\}$ such that $t_j \in [0, \tau)$ for each j, $t_j \to \tau$. Since A_{t_j} is nonsingular, the equation

$$A_{t_j} x_j = b \tag{2}$$

has a solution $x_j \in R^n$. If the sequence $\{x_j\}$ is unbounded, then $X(A^I, b^I)$ is unbounded and we are done. If $\{x_j\}$ is bounded, then it contains a convergent subsequence, $x_{j_k} \to x$. Then, taking the limit in (2), we obtain

$$A_\tau x = b.$$

But since A_τ is singular, there exists an $x_0 \neq 0$ with $A_\tau x_0 = 0$. Then we have

$$A_\tau(x + \lambda x_0) = b$$

for each $\lambda \in R^1$, thus $x + \lambda x_0 \in X(A^I, b^I)$ for each $\lambda \in R^1$ which gives that $X(A^I, b^I)$ is unbounded; this proves (ii) $\Rightarrow$ (i). ■

The assumption that A^I contains at least one nonsingular matrix cannot be omitted: in the one-dimensional example $[0, 0]x_1 = [1, 1]$, where it is violated, A^I is singular but the solution set is empty, hence bounded.

Before giving the main result, let us recall that P denotes the class of problems solvable by polynomial-time algorithms, whereas NP denotes those solvable by nondeterministic polynomial-time algorithms [2]. The conjecture that $P \neq NP$, although unproved so far, is commonly believed to be true.

Theorem. *If $P \neq NP$, then there does not exist a polynomial-time algorithm which for each A^I square, b^I would*

— *yield an enclosure of $X(A^I, b^I)$ provided $X(A^I, b^I)$ is bounded,*
— *issue an error message if $X(A^I, b^I)$ is unbounded.*

Proof: Assuming that such an algorithm exists, we can employ it for checking regularity of interval matrices as follows. Given A^I, first take an $A \in A^I$ and check it for singularity (this can be done in polynomial time, see [7]). If A is singular, then A^I is singular; otherwise, choose any b^I and apply the algorithm to enclose $X(A^I, b^I)$. If an enclosure is found, then A^I is regular; if an error message is issued, then A^I is singular (Proposition, (i) $\Leftrightarrow$ (ii)). Thus, we have a polynomial-time algorithm for checking regularity of interval matrices. However, this problem has been proved to be NP-hard (Poljak and Rohn [4]), hence an existence of a polynomial-time algorithm for an NP-hard problem implies that $P = NP$ (Garey and Johnson [2]), which is a contradiction. ■

As we have seen in the proof, the problem of checking regularity of interval matrices can be polynomially reduced to the enclosure problem. Since the former one is NP-hard, the latter one is NP-hard as well. We note that a recent relevant result [6] shows that computing the *optimal* (i.e., the narrowest) enclosure is NP-hard even if the regularity of A^I is assumed.

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