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INVERSE INTERVAL MATRIX*

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Dedicated to Professor U. Kulisch on the occasion of his 60th birthday.

Abstract. The inverse interval matrix is defined as the narrowest interval matrix containing the inverses of all the matrices from a given interval matrix. Both theoretical and practical results concerning computation of the inverse interval matrix are presented. In particular, explicit formulas are given for the inverse of an interval matrix with radius of rank one.

Key words. interval matrix, inverse matrix, inverse stability

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Introduction. An interval matrix is defined as a set of matrices of the form

$$A^I = [\underline{A}, \bar{A}] := \{A; \underline{A} \le A \le \bar{A}\}$$

for some $\underline{A}, \bar{A}$ satisfying $\underline{A} \le \bar{A}$. Throughout this paper we shall assume that A^I is square $n \times n$ and regular, i.e., that each $A \in A^I$ is nonsingular (we follow the terminology of [4] and [5]). Since trivial examples show that the set $\{A^{-1}; A \in A^I\}$ need not be an interval matrix [4], it is reasonable to define the inverse of A^I as the narrowest interval matrix containing the set $\{A^{-1}; A \in A^I\}$, i.e., as the interval matrix $[\underline{B}, \bar{B}]$ whose bounds are given by

$$\underline{B}_{ij} = \min \{(A^{-I})_{ij};\, A \in A^I\} \tag{0.1}$$

and

$$\bar{B}_{ij} = \max \{(A^{-1})_{ij};\, A \in A^I\} \tag{0.2}$$

$(i, j = 1, \ldots, n)$.

In contrast to the problem of solving systems of linear interval equations, the problem of inverting interval matrices has been given much less attention. This is explained by the fact that methods for enclosing the solution set of a system of linear interval equations, as referenced in Alefeld and Herzberger [1] or Neumaier [4], can be used for enclosing the inverse interval matrix by solving the family of systems $A^I x = e_j$, where e_j is the jth column of the unit matrix $E (j = 1, \ldots, n)$. As far as is known to the author, explicit results on $\underline{B}, \bar{B}$ are only available for interval matrices of a very special inverse sign pattern (cf. [5, Thm. 6.5], which generalizes previous results by Barth and Nuding [2] and Garloff [3]).

The results of this paper are based on ideas which were briefly mentioned (but not elaborated on) in the last remark of our paper [5, p. 76]. In § 1 we introduce a finite set of matrices $A_{yz} \in A^I$, where y and z are n-dimensional parameter ± 1-vectors and we prove that each $\underline{B}_{ij}$ (or $\bar{B}_{ij}$) is achieved at the ijth coefficient of some A_{yz}^{-1}. In § 2 we show that the optimum yielding values of y and z can be specified if the coefficients of A^{-1} do not change their signs over A^I, and we give some necessary and/or sufficient conditions for this property to hold. Finally in § 3 we give closed form expressions for $\underline{B}_{ij}$, $\bar{B}_{ij} (i, j = 1, \ldots, n)$ provided the matrix $\bar{A} - \underline{A}$ is of rank one. As an application of the latter result we obtain formulae for estimating the inverse of a real matrix by the inverse of the rounded matrix.

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We shall use the following notation. For a matrix $A=(A_{ij})$ we introduce its absolute value as $|A|=(|A_{ij}|)$ and denote its transpose by A'. The same notation applies to vectors which are always considered one-column matrices. For a vector $x\in R^n$, we define its sign vector sgn x by $(\operatorname{sgn} x)_i=1$ if $x_i\ge 0$ and $(\operatorname{sgn} x)_i=-1$ otherwise. Conv X denotes the convex hull of X.

1. General results. Let $A^I=[\underline{A},\bar{A}]$ be an $n\times n$ interval matrix. It turns out to be more appropriate to use instead of the matrices $\underline{A}$ and $\bar{A}$ the center matrix

$$A_c=\tfrac{1}{2}(\underline{A}+\bar{A})$$

and the radius matrix

$$\Delta=\tfrac{1}{2}(\bar{A}-\underline{A}).$$

Obviously, Δ is nonnegative and we have $\underline{A}=A_c-\Delta$, $\bar{A}=A_c+\Delta$. Let us introduce the set of ± 1-vectors

$$Y=\{y\in R^n;\ |y_j|=1 \text{ for } j=1,\dots,n\},$$

and for each $q\in R^n$ let us denote by T_q the $n\times n$ diagonal matrix defined by $(T_q)_{ii}=q_i$ and $(T_q)_{ij}=0$ for $i\ne j$, $i,j=1,\dots,n$. We shall characterize the inverse interval matrix by means of matrices A_{yz} defined by

$$A_{yz}=A_c-T_y\Delta T_z \tag{1.1}$$

for $y,z\in Y$. Clearly, for each $i,j\in\{1,\dots,n\}$ we have $(A_{yz})_{ij}=(A_c)_{ij}-y_i\Delta_{ij}z_j=(A_c-\Delta)_{ij}=\underline{A}_{ij}$ if $y_iz_j=1$, and $(A_{yz})_{ij}=(A_c+\Delta)_{ij}=\bar{A}_{ij}$ if $y_iz_j=-1$; hence each A_{yz} is a vertex of A^I if it is considered a polyhedron in R^{n^2}. However, generally the set of matrices $\{A_{yz};\ y,z\in Y\}$ does not exhaust all the vertices of A^I since there are at most 2^{2n-1} matrices of the form (1.1) (since $A_{-y,-z}=A_{yz}$ for each $y,z\in Y$) while the number of vertices of A^I is equal to 2^{n^2} in the worst case $\Delta>0$.

First we prove a general representation theorem which shows that for each $A\in A^I$, its inverse can be expressed as a kind of convex combination of the matrices A_{yz}^{-1}, $y,z\in Y$. A similar result was given in [5, Thm. 6.1] where, instead of A_{yz}^{-1}, we used rather obscure matrices B_y obtained there as solutions of certain nonlinear matrix equations.

THEOREM 1.1. *Let A^I be regular. Then for each $A\in A^I$ there exist nonnegative diagonal matrices L_{yz}, $y,z\in Y$, satisfying $\sum_{y,z\in Y}L_{yz}=E$ such that*

$$A^{-1}=\sum_{y,z\in Y}A_{yz}^{-1}L_{yz} \tag{1.2}$$

holds.

Proof. According to [5, Thm. 2.2], for each $y\in Y$ and each $j\in\{1,\dots,n\}$ there exists a $z_{yj}\in Y$ such that each solution of the linear interval system $A^Ix=e_j$ belongs to Conv $\{A_{yz_{yj}}^{-1}e_j;\ y\in Y\}$. Hence for a given $A\in A^I$ we have

$$A^{-1}e_j=\sum_{y\in Y}\lambda_{yj}A_{yz_{yj}}^{-1}e_j$$

for some $\lambda_{yj}\ge 0$ satisfying $\sum_{y\in Y}\lambda_{yj}=1$. Now, for $y,z\in Y$ define a diagonal matrix L_{yz} by

$$(L_{yz})_{ij}=\lambda_{yj} \quad \text{if } i=j \text{ and } z=z_{yj}$$

$$(L_{yz})_{ij}=0 \quad \text{otherwise,}$$

then L_{yz} is a nonnegative diagonal matrix with $(\sum_{y,z\in Y} L_{yz})_{jj} = \sum_{y\in Y} \lambda_{yj} = 1$ for each j; hence $\sum_{y,z\in Y} L_{yz} = E$. Thus we have

$$\sum_{y,z\in Y} A_{yz}^{-1} L_{yz} e_j = \sum_{y,z\in Y} (L_{yz})_{jj} A_{yz}^{-1} e_j = \sum_{y\in Y} \lambda_{yj} A_{yz_{yj}}^{-1} e_j = A^{-1} e_j$$

for each j, which gives (1.2). □

Let $A \in A^I$ and $i, j \in \{1, \ldots, n\}$. Then from (1.2) we have

$$(A^{-1})_{ij} = \sum_{y,z\in Y} (A_{yz}^{-1})_{ij} (L_{yz})_{jj}$$

where $\sum_{y,z\in Y} (L_{yz})_{jj} = 1$; hence $(A^{-1})_{ij}$ is a convex combination of the numbers $(A_{yz}^{-1})_{ij}$, $y, z \in Y$. This immediately implies that the bounds of the inverse interval matrix $[\underline{B}, \bar{B}]$ given by (0.1), (0.2) satisfy

$$\underline{B}_{ij} = \min \{(A_{yz}^{-1})_{ij} ;\ y, z \in Y\} \tag{1.3}$$

and

$$\bar{B}_{ij} = \max \{(A_{yz}^{-1})_{ij} ;\ y, z \in Y\} \qquad (i, j = 1, \ldots, n). \tag{1.4}$$

Hence the inverse interval matrix is completely described by a finite set of inverse matrices A_{yz}^{-1}, $y, z \in Y$. In the next theorem we give some property of the vectors y and z for which the optimal value is achieved in (1.3) or (1.4).

THEOREM 1.2. *Let A^I be regular and $i, j \in \{1, \ldots, n\}$. Then we have*:

(i) $\underline{B}_{ij} = (A_{yz}^{-1})_{ij}$ *for some $y, z \in Y$ satisfying*

$$y_k (A_{yz}^{-1})_{ik} \le 0 \qquad (k = 1, \ldots, n),$$
$$z_h (A_{yz}^{-1})_{hj} \ge 0 \qquad (h = 1, \ldots, n),$$

(ii) $\bar{B}_{ij} = (A_{yz}^{-1})_{ij}$ *for some $y, z \in Y$ satisfying*

$$y_k (A_{yz}^{-1})_{ik} \ge 0 \qquad (k = 1, \ldots, n),$$
$$z_h (A_{yz}^{-1})_{hj} \ge 0 \qquad (h = 1, \ldots, n).$$

Proof. For given $i, j \in \{1, \ldots, n\}$, $\underline{B}_{ij}$ is the minimum value of the ith coordinate of solutions of the system of linear interval equations $A^I x = e_j$. Hence [5, Thm. 4.2] applies to obtain (i). A similar reasoning for the maximum coordinate gives (ii). □

The results of Theorem 1.2 cannot be effectively applied unless we know something about the sign pattern of the matrices A_{yz}^{-1}. We shall study such a special case in the next section.

2. Inverse stability. A square interval matrix A^I is called inverse stable [5] if it is regular and satisfies $|A^{-1}| > 0$ for each $A \in A^I$. Due to the continuity of the coefficients of the inverse matrix, this means that for each $i, j \in \{1, \ldots, n\}$, either $(A^{-1})_{ij}$ is negative for each $A \in A^I$, or $(A^{-1})_{ij}$ is positive for each $A \in A^I$; in other words, each A^{-1} is of the same sign pattern. In Theorems 2.1 and 2.2 we give some conditions for inverse stability. The first result is theoretical, but it shows that inverse stability can again be characterized in terms of the finite set of matrices A_{yz}^{-1}, $y, z \in Y$.

THEOREM 2.1. *A^I is inverse stable if and only if all the matrices A_{yz}, $y, z \in Y$ are nonsingular and their inverses have all coefficients nonzero and are of the same sign pattern.*

Proof. The "only if" part is a consequence of the definition of inverse stability. In the proof of the "if" part we must first show that A^I is regular. So let $j \in \{1, \ldots, n\}$. Let us denote by $z(j)$ the sign vector of the jth column of A_{yz}^{-1} (which is constant over $y, z \in Y$ by the assumption) and, furthermore, let

$$x_{yj} = A_{yz(j)}^{-1} e_j$$

for $y \in Y$. Then we have $T_{z(j)}x_{yj} > 0$; hence $A_{yz(j)}x_{yj} = A_c x_{yj} - T_y \Delta |x_{yj}| = e_j$, which implies

$$T_y(A_c x_{yj} - e_j) = \Delta |x_{yj}|$$

for each $y \in Y$. Now let $A \in A^I$. Since $|T_y(A - A_c)x_{yj}| \leq \Delta |x_{yj}|$, we have $T_y(Ax_{yj} - e_j) = T_y(A_c x_{yj} - e_j) + T_y(A - A_c)x_{yj} \geq \Delta |x_{yj}| - \Delta |x_{yj}| = 0$; hence

$$T_y A x_{yj} \geq T_y e_j$$

holds for each $y \in Y$. Then [6, Thm. 2] shows that the system

$$Ax = e_j$$

has a solution. Since j was arbitrary in $\{1, \ldots, n\}$, this shows that A is nonsingular. Now, applying Theorem 1.1, we obtain from (1.2) that A^{-1} is of the same sign pattern as the matrices A_{yz}^{-1}, $y, z \in Y$, which proves that A^I is inverse stable. □

In Theorem 2.2 to follow we give a verifiable sufficient condition for inverse stability. It involves a matrix R specified only by some inequality; the result is formulated in this way to avoid an explicit use of A_c^{-1}. For practical purposes we recommend setting R equal to the *computed* value of A_c^{-1}.

THEOREM 2.2. *Suppose R is an $n \times n$ matrix such that the matrix*

$$G_R = |E - RA_c| + |R|\Delta$$

satisfies

$$2G_R|R| < |R|. \tag{2.1}$$

Then A^I is inverse stable and the sign pattern of each inverse matrix is identical with that of R. Moreover, R is nonsingular and $\rho(G_R) < \frac{1}{2}$.

Proof. Let r be an arbitrary column of $|R|$. Then $2G_R r < r$, where r is positive by (2.1); hence $\rho(G_R) < \frac{1}{2}$ due to a well-known result to be found, e.g., in Neumaier [4, Cor. 3.2.3]. Moreover, $(E - G_R)^{-1}$ exists and is nonnegative. Now, for each $A \in A^I$ we can write

$$RA = E - (E - RA). \tag{2.2}$$

Since

$$|E - RA| = |E - RA_c + R(A_c - A)| \leq G_R,$$

we have $\rho(E - RA) < \frac{1}{2}$; hence RA is nonsingular, which implies that R is nonsingular and A^I is regular. Hence from (2.2) we have

$$A^{-1} = \left(\sum_{j=0}^{\infty} (E - RA)^j \right) R$$

and consequently

$$|A^{-1} - R| \leq \sum_{j=1}^{\infty} |E - RA|^j |R| \leq \sum_{j=1}^{\infty} G_R^j |R| = (E - G_R)^{-1} G_R |R|. \tag{2.3}$$

Since (2.1) implies

$$G_R|R| < (E - G_R)|R|,$$

premultiplying this inequality by the nonnegative matrix $(E - G_R)^{-1}$ yields

$$(E - G_R)^{-1} G_R |R| < |R|$$

which, combined with (2.3), gives

$$|A^{-1} - R| < |R|.$$

Thus if $R_{ij} > 0$, then $A_{ij}^{-1} > 0$ and if $R_{ij} < 0$, then $A_{ij}^{-1} < 0$ (the case $R_{ij} = 0$ cannot occur due to (2.1)) which proves that A^I is inverse stable. □

Now, for an inverse stable interval matrix A^I, denote by $y(i)$ the sign vector of the ith row of A_c^{-1} and by $z(j)$ the sign vector of the jth column of A_c^{-1}. The introduction of inverse stable interval matrices is then justified by the following result, which specifies the matrices A_{yz}^{-1} at which the exact bounds on the inverse matrix coefficients are achieved.

THEOREM 2.3. *Let A^I be inverse stable. Then for each $i, j \in \{1, \dots, n\}$ we have*

$$\underline{B}_{ij} = (A^{-1}_{-y(i),z(j)})_{ij} \tag{2.4}$$

and

$$\bar{B}_{ij} = (A^{-1}_{y(i)z(j)})_{ij}. \tag{2.5}$$

Proof. According to Theorem 1.2 we have

$$\underline{B}_{ij} = (A_{yz}^{-1})_{ij}$$

for some $y, z \in Y$ satisfying

$$y_k(A_{yz}^{-1})_{ik} \le 0 \qquad (k = 1, \dots, n),$$

$$z_h(A_{yz}^{-1})_{hj} \ge 0 \qquad (h = 1, \dots, n).$$

Since A^I is inverse stable, we obtain $y_k = -\mathrm{sgn}(A_{yz}^{-1})_{ik} = -(y(i))_k$ and $z_h = \mathrm{sgn}(A_{yz}^{-1})_{hj} = (z(j))_h$ for $k, h = 1, \dots, n$; hence $y = -y(i)$ and $z = z(j)$, which yields (2.4). In a similar way we obtain (2.5). □

In this way we have reduced the number of inverse matrices to be computed from 2^{2n-1} to $2n^2$. In the last section we shall examine a special class of interval matrices for which $\underline{B}_{ij}$ and $\bar{B}_{ij}$ can be given by closed form formulae involving only *one* matrix inversion.

3. Interval matrices with rank one radius. In this section we shall investigate interval matrices $A^I = [A_c - \Delta, A_c + \Delta]$ with the radius matrix of the form

$$\Delta = qp^t \tag{3.1}$$

for some nonnegative vectors $q, p \in R^n$. Thus, with exception of the trivial cases $q = 0$ or $p = 0$, the matrix Δ is of rank one. It turns out that in this case it is possible to obtain explicit formulae for $\underline{B}$ and $\bar{B}$ as a result of the following description of matrices A_{yz}^{-1} which shows that if Δ is of rank one, then $A_{yz}^{-1} - A_c^{-1}$ is also of rank one for each $y, z \in Y$.

THEOREM 3.1. *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a regular interval matrix with Δ of the form* (3.1). *Then for each $y, z \in Y$ we have*

$$A_{yz}^{-1} = A_c^{-1} + \frac{q_y p_z^t}{1 - p_z^t A_c q_y} \tag{3.2}$$

where

$$q_y = A_c^{-1} T_y q$$

and

$$p_z = (A_c^{-1})^t T_z p.$$

Proof. Let $y, z \in Y$. Set $\alpha = 1 - p_z' A_c q_y$, then regularity of A^I implies that $\alpha > 0$ [5, Thm. 5.2]. Since

$$T_y q p' T_z q_y p_z' = T_y q (p' T_z A_c^{-1} T_y q) p_z' = (1-\alpha) T_y q p_z',$$

we have

$$(A_c - T_y q p' T_z)\left(A_c^{-1} + \frac{1}{\alpha} q_y p_z'\right) = E + \frac{1}{\alpha} T_y q p_z' - T_y q p_z' - \frac{1-\alpha}{\alpha} T_y q p_z' = E,$$

which proves (3.2). □

Now, combining this result with Theorem 2.3, we obtain explicit formulae for $\underline{B}_{ij}$ and $\bar{B}_{ij}$.

THEOREM 3.2. *Let A^I be inverse stable with Δ of the form* (3.1). *Then for each $i, j \in \{1, \ldots, n\}$ we have*

$$\underline{B}_{ij} = (A_c^{-1})_{ij} - \frac{\bar{q}_i \bar{p}_j}{1+\lambda_{ij}}, \tag{3.3}$$

$$\bar{B}_{ij} = (A_c^{-1})_{ij} + \frac{\bar{q}_i \bar{p}_j}{1-\lambda_{ij}}, \tag{3.4}$$

where

$$\bar{q} = |A_c^{-1}| q, \qquad \bar{p} = |A_c^{-1}|' p$$

and

$$\lambda_{ij} = y(i)' T_q (A_c^{-1})' T_p z(j). \tag{3.5}$$

Proof. Since the assumptions of Theorems 2.3 and 3.1 are met, we obtain

$$\underline{B}_{ij} = (A_c^{-1})_{ij} - \frac{(q_{y(i)})_i (p_{z(j)})_j}{1+\lambda_{ij}} \tag{3.6}$$

and

$$\bar{B}_{ij} = (A_c^{-1})_{ij} + \frac{(q_{y(i)})_i (p_{z(j)})_j}{1-\lambda_{ij}}, \tag{3.7}$$

where $\lambda_{ij} = p_{z(j)}' A_c q_{y(i)} = y(i)' T_q (A_c^{-1})' T_p z(j)$. Since $(q_{y(i)})_i = \sum_k (A_c^{-1})_{ik} (y(i))_k q_k = \sum_k |A_c^{-1}|_{ik} q_k = \bar{q}_i$ and, similarly, $(p_{z(j)})_j = \bar{p}_j$, from (3.6), (3.7) we obtain (3.3) and (3.4). □

Notice that the numbers λ_{ij} from (3.5) can also be described as elements of the matrix

$$\Lambda = S T_q (A_c^{-1})' T_p S$$

where S is the sign matrix of A_c^{-1} (i.e., $S_{ij} = 1$ if $(A_c^{-1})_{ij} > 0$ and $S_{ij} = -1$ otherwise).

As an application of this result, consider a real (possibly not exactly known) matrix A and the matrix A_c constructed by rounding all elements of A to a fixed number of d decimal places. Then with $p = e = (1, 1, \ldots, 1)'$ and $q = \beta e$, $\beta = \frac{1}{2} 10^{-d}$, we obtain from (3.3), (3.4) the bounds on A^{-1} in terms of A_c^{-1}, provided the respective interval matrix $[A_c - \beta ee', A_c + \beta ee']$ is inverse stable, which may be tested by means of Theorem 2.2. The particular form of (3.3) and (3.4) for this special case was derived in another way in [7, p. 10]; cf. also [8] for other methods for bounding the inverse interval matrix.

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