INTERVAL SOLUTIONS OF LINEAR INTERVAL EQUATIONS

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Summary. It is shown that if the concept of an interval solution to a system of linear interval equations given by Ratschek and Sauer is slightly modified, then only two nonlinear equations are to be solved to find a modified interval solution or to verify that no such solution exists.

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In this paper we shall deal with the following problem. Given a square interval matrix $A^I = [A^-, A^+] = \{A; A^- \leqq A \leqq A^+\}$, where $A^- = (a_{ij}^-)$ and $A^+ = (a_{ij}^+)$ are $n \times n$ matrices, and an interval vector $b^I = [b^-, b^+] = \{b; b^- \leqq b \leqq b^+\}$ with $b^- = (b_i^-)$, $b^+ = (b_i^+) \in R^n$, find an interval n-vector $x^I = [x^-, x^+]$ such that

$$\sum_{j=1}^{n} [a_{ij}^-, a_{ij}^+] \cdot [x_j^-, x_j^+] = [b_i^-, b_i^+] \quad (i = 1, \ldots, n) \tag{1}$$

holds, where the operations involved are performed in interval arithmetic and are generally defined by

$$[\alpha^-, \alpha^+] \circ [\beta^-, \beta^+] = \{\alpha \circ \beta;\ \alpha \in [\alpha^-, \alpha^+],\ \beta \in [\beta^-, \beta^+]\}$$

for $\circ \in \{+, -, \cdot, /\}$, which amounts to

$$[\alpha^-, \alpha^+] + [\beta^-, \beta^+] = [\alpha^- + \beta^-,\ \alpha^+ + \beta^+]$$

$$[\alpha^-, \alpha^+] - [\beta^-, \beta^+] = [\alpha^- - \beta^+,\ \alpha^+ - \beta^-]$$

$$[\alpha^-, \alpha^+] \cdot [\beta^-, \beta^+] = [\min\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\},\ \max\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\}]$$

$$[\alpha^-, \alpha^+]/[\beta^-, \beta^+] = [\alpha^-, \alpha^+] \cdot \frac{1}{[\beta^-, \beta^+]},$$

where

$$\frac{1}{[\beta^-, \beta^+]} = \left[\frac{1}{\beta^+}, \frac{1}{\beta^-}\right] \quad \text{provided} \quad 0 \notin [\beta^-, \beta^+]$$

(for interval arithmetic, see e.g. [4]). This concept of solution was formulated for interval systems with arbitrary $m \times n$ interval matrices by Ratschek and Sauer [7] and solved there for the case $m = 1$. It seems that a general solution to (1) is not yet known; cf. also Nickel [5]. In this paper we shall show that systems of type (1) with square regular interval matrices can be solved if we impose an additional restriction upon the concept of a solution in the following sense:

Definition. Given A^I (square) and b^I, an interval vector x^I is called a ***strong solution*** if it satisfies (1) and if there exist $A', A'' \in A^I$ and $x', x'' \in x^I$ such that $A'x' = b^-$, $A''x'' = b^+$ hold.

Let us first introduce

$$A_c = \tfrac{1}{2}(A^- + A^+),$$
$$\Delta = \tfrac{1}{2}(A^+ - A^-);$$

so that $\Delta \geqq 0$ and $A^- = A_c - \Delta$, $A^+ = A_c + \Delta$. We shall show that the problem of finding a strong solution is closely connected with solving the nonlinear equations

(2.1) $$A_c x - \Delta|x| = b^-,$$

(2.2) $$A_c x + \Delta|x| = b^+$$

where $x = (x_j)$ is a real (not interval) vector and the absolute value is defined as $|x| = (|x_j|)$. We shall need some results about solutions to (2.1), (2.2). A square interval matrix A^I is called regular if each $A \in A^I$ is nonsingular.

Theorem 1. *Let A^I be regular. Then the equations* (2.1), (2.2) *have unique solutions x_1 and x_2, respectively.*

For the proof of this result, see [8], Theorem 1.2. To solve (2.1) and (2.2), we may observe that $|x| = Dx$, where D is a diagonal matrix with $D_{jj} = 1$ if $x_j \geqq 0$ and $D_{jj} = -1$ otherwise. Then (2.1) can be written as a system of linear equations $(A_c - \Delta D)\, x = b^-$, where D must be found such that $Dx (= |x|) \geqq 0$. This is the underlying idea of the following algorithm:

Algorithm 1 (for solving (2.1), (2.2)).

Step 0. Set $D = E$ (unit matrix).

Step 1. Solve $(A_c - \Delta D)\, x = b^-$ (for (2.2): $(A_c + \Delta D)\, x = b^+$).

Step 2. If $Dx \geqq 0$, set $x_1 := x$ (or, $x_2 := x$) and terminate.

Step 3. Otherwise find $k = \min\{j;\ D_{jj}x_j < 0\}$.

Step 4. Set $D_{kk} := -D_{kk}$ and go to *Step* 1.

The algorithm is general, as the following result shows:

Theorem 2. *Let A^I be regular. Then Algorithm 1 is finite, passing through Step 1 at most 2^n times.*

The proof of this theorem can be again found in [8]. Another possibility, though not general, for solving (2.1) (similarly, (2.2)) consists in reformulating (2.1) as a fixed-point equation

$$x = A_c^{-1} \Delta |x| + A_c^{-1} b^-$$

which may be solved iteratively by

$$x^0 = A_c^{-1} b^- ,$$
$$x^{i+1} = A_c^{-1} \Delta |x^i| + A_c^{-1} b^- \quad (i = 0, 1, \ldots),$$

but convergence of $\{x^i\}$ to x_1 can be established only under the assumption that $\varrho(|A_c^{-1}| \Delta) < 1$, which is not always the case with regular interval matrices; nevertheless, if Δ is of small norm, then the iterative method is to be preferred.

Returning now back to our original problem of finding a strong solution, we shall show in the next theorem that if strong solutions exist at all, then one of them can be easily expressed by means of the above vectors x_1, x_2. Since generally neither $x_1 \leqq x_2$, nor $x_1 \geqq x_2$ holds, we introduce $\min\{x_1, x_2\}$ as the vector with components $\min\{(x_1)_j, (x_2)_j\}$ $(j = 1, \ldots, n)$, and similarly for $\max\{x_1, x_2\}$.

Theorem 3. *Let A^I be regular and let (1) have a strong solution. Then $x^I = [x^-, x^+]$, given by*

$$x^- = \min\{x_1, x_2\}, \qquad (3)$$
$$x^+ = \max\{x_1, x_2\},$$

is also a strong solution.

Proof. Let $\tilde{x}^I$ be a strong solution. Then there exist $A', A'' \in A^I$ and $x', x'' \in \tilde{x}^I$ such that $A'x' = b^-$, $A''x'' = b^+$ hold. Due to the definition of interval operations, the resulting left-hand side interval vector in (1) contains all vectors of the form Ax', $A \in A^I$. On the other hand, according to the theorem by Oettli and Prager [6], we have $\{Ax';\ A \in A^I\} = [A_c x' - \Delta|x'|, A_c x' + \Delta|x'|]$. Since $A'x' = b^-$, we conclude that

$$A_c x' - \Delta|x'| = b^-$$

holds, implying $x' = x_1$ in view of the uniqueness of the solution to (2.1) stated in Theorem 1. In a similar way we would obtain that $x'' = x_2$. Now, for x^I given by (3), the interval vector with the components

$$\sum_{j=1}^{n} [a_{ij}^-, a_{ij}^+] \cdot [x_j^-, x_j^+] \quad (i = 1, \ldots, n)$$

is contained in b^I since $x^I \subset \tilde{x}^I$, but also contains b^- and b^+ since $x_1, x_2 \in x^I$; hence it equals b^I, so that (1) holds and x^I is a strong solution. Q.E.D.

Summing up the results, we can formulate the following algorithm for solving our problem:

Algorithm 2 (finding a strong solution)

Step 1. Solve (2.1), (2.2) (by Algorithm 1 or iteratively) to find x_1, x_2.

Step 2. Construct x^I by (3).

Step 3. If x^I satisfies (1), stop: x^I is a strong solution.

Step 4. Otherwise stop: no strong solution exists.

The algorithm works provided A^I is regular, which is the case e.g. if the spectral radius of $|A_c^{-1}| \Delta$ is less than 1 (Beeck [2]), a condition widely satisfied in practice.

We add two small examples with regular matrices to illustrate the possible outcomes.

Example 1 (Hansen [3]). Let

$$A^- = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$$

and $b^- = (0, 60)^T$, $b^+ = (120, 240)^T$. Solving (2.1), (2.2), we obtain

$$x_1 = (0, 30)^T, \quad x_2 = (\tfrac{120}{7}, \tfrac{480}{7})^T,$$

and

$$x^I = ([0, \tfrac{120}{7}], [30, \tfrac{480}{7}])^T$$

satisfies (1), therefore it is a strong solution.

Example 2 (Barth and Nuding [1]). Let

$$A^- = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$$

and $b^- = (-2, -2)^T$, $b^+ = (2, 2)^T$. Here x^I does not satisfy (1), so that no strong solution exists.

A preliminary version of this paper appeared in [9].

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Souhrn

INTERVALOVÁ ŘEŠENÍ SOUSTAV LINEÁRNÍCH INTERVALOVÝCH ROVNIC

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Je zavedeno modifikované intervalové řešení soustavy lineárních intervalových rovnic, k jehož výpočtu je třeba vyřešit dvě soustavy nelineárních rovnic.

Резюме

ИНТЕРВАЛЬНЫЕ РЕШЕНИЯ СИСТЕМ ЛИНЕЙНЫХ ИНТЕРВАЛЬНЫХ УРАВНЕНИЙ

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В статье показано, как можно вычислить модифицированное интервальное решение системы линейных интервальных уравнений путём решения двух систем нелинейных уравнений.

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