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Strong Solvability of Interval Linear Programming Problems

J. Rohn, Prague

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Abstract — Zusammenfassung

Strong Solvability of Interval Linear Programming Problems. Necessary and sufficient conditions for a linear programming problem whose parameters (both in constraints and in the objective function) are prescribed by intervals are given under which any linear programming problem with parameters being fixed in these intervals has a finite optimum.

Starke Lösbarkeit von Problemen der linearen Optimierung mit Intervallkoeffizienten. Es werden notwendige und hinreichende Bedingungen für das Problem der linearen Optimierung mit Intervallkoeffizienten angegeben, bei denen jedes Problem der linearen Optimierung, dessen Koeffizienten in gegebenen Intervallen fixiert werden, eine optimale Lösung besitzt.

1. Introduction

In this paper we examine necessary and sufficient conditions for a problem

$$\max \{c^T x \mid A x = b,\ x \geq 0\} \tag{1}$$

to have a finite optimum for any $A \in A^I$, $b \in b^I$, $c \in c^I$, where A^I, b^I, c^I are matrix and vector intervals, respectively. Such a problem can arise e.g. in case that (1) is a model of a real-world situation which must be repeatedly solved for various values of parameters ranging in known intervals. If a model has the above property, which we call strong solvability, then one can be sure that an optimal solution will be obtained at each computation.

The above formulated problem consists of two subproblems: to find conditions under which

(a) $A x = b$ has a nonnegative solution for any $A \in A^I$, $b \in b^I$,
(b) the problem (1) has a finite optimum for any $A \in A^I$, $b \in b^I$, $c \in c^I$ provided (a) holds.

These subproblems will be discussed separately in the next two sections.

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2. Strong Feasibility

Let $A^I=[\underline{A}, \bar{A}]=\{A \mid \underline{A} \leq A \leq \bar{A}\}$, $b^I=[\underline{b}, \bar{b}]=\{b \mid \underline{b} \leq b \leq \bar{b}\}$, where $\underline{A}$, $\bar{A}$ are two m by n matrices and $\underline{b}$, $\bar{b}$ are m-dimensional vectors, $\underline{A} \leq \bar{A}$, $\underline{b} \leq \bar{b}$. For any $A \in A^I$, $b \in b^I$, a system

$$A\,x=b \tag{2}$$

is called a subsystem of an interval linear system $A^I x=b^I$. A subsystem (2) is called an *extremal subsystem* of $A^I x=b^I$ if for each i, $i=1, \ldots, m$, its i-th equation has either the form $(\underline{A}\,x)_i=\bar{b}_i$ or the form $(\bar{A}\,x)_i=\underline{b}_i$. A subsystem (2) is called feasible if it has a nonnegative solution. An interval linear system $A^I x=b^I$ is called *strongly feasible* if each its subsystem is feasible. The strong feasibility can be characterized in terms of extremal subsystems:

Theorem 1: *A system $A^I x=b^I$ is strongly feasible if and only if all its extremal subsystems are feasible.*

Proof: The "only if" part of the theorem is obvious. We shall give two proofs of the "if" part, an existencial one and a constructive one.

Existential proof: Let $A \in A^I$, $b \in b^I$. According to the Farkas theorem [1], to prove that the subsystem $A\,x=b$ is feasible, it will suffice to show that $A^T y \geq 0$ implies $b^T y \geq 0$ for any (m-dimensional) y. Thus let $A^T y \geq 0$ for some y. Define a subsystem

$$A_0\,x=b_0 \tag{3}$$

as follows: for $i=1, \ldots, m$, the i-th equation of (3) has the form $(\bar{A}\,x)_i=\underline{b}_i$ if $y_i \geq 0$ and it has the form $(\underline{A}\,x)_i=\bar{b}_i$ if $y_i<0$. Then (3) is an extremal subsystem and $A_0^T y \geq A^T y$, $b^T y \geq b_0^T y$. Hence $A_0^T y \geq 0$ and the Farkas theorem as applied to (3) gives $b_0^T y \geq 0$, thus $b^T y \geq 0$, which completes the existencial proof.

Constructive proof: Let $A \in A^I$, $b \in b^I$. A subsystem whose i-th equation has the form $(A\,x)_i=b_i$ for $i=1, \ldots, r$ and has either the form $(\underline{A}\,x)_i=\bar{b}_i$ or the form $(\bar{A}\,x)_i=\underline{b}_i$ for $i=r+1, \ldots, m$, will be called an r-subsystem, $r=0, 1, \ldots, m$. We shall construct by induction on r a feasible solution to any r-subsystem. At the last step, we shall obtain a feasible solution to $A\,x=b$, which is the only m-subsystem. For $r=0$ it is nothing to prove since each 0-subsystem is an extremal subsystem which is feasible due to the assumption. Thus let $0<r \leq m$ and let to any $(r-1)$-subsystem a feasible solution has been constructed. Take an arbitrary r-subsystem

$$A^1\,x=b^1 \tag{4}$$

and replace its r-th equation once by $(\underline{A}\,x)_r=\bar{b}_r$, once by $(\bar{A}\,x)_r=\underline{b}_r$. This gives rise to two $(r-1)$-subsystems that have nonnegative solutions x', x'', respectively. Consider a real-valued function of one real variable:

$$f_r(\lambda)=(A\,(\lambda\,x'+(1-\lambda)\,x''))_r-b_r, \quad \lambda \in [0, 1].$$

We have

$$f_r(0)=(A\,x'')_r-b_r \leq (\bar{A}\,x'')_r-\underline{b}_r=0$$

$$f_r(1)=(A\,x')_r-b_r \geq (\underline{A}\,x')_r-\bar{b}_r=0,$$

so that there exists a $\lambda_0 \in [0, 1]$ with $f_r(\lambda_0)=0$. Put $x=\lambda_0 x' + (1-\lambda_0) x''$. Then the above means that x satisfies the r-th equation of (4). Since for any i, $i \neq r$, the i-th equation of (4) is identical with those of both $(r-1)$-subsystems, which are satisfied by x' and x'', it is also satisfied by their convex combination x. Hence x is a feasible solution to (4) and the inductive step is completed. Thus each r-subsystem is feasible $(r=0, 1, \ldots, m)$, hence $A x = b$ is also feasible, Q.E.D.

Note: It can be easily seen that the number of mutually different extremal subsystems is equal to 2^p, where p is the number of rows of the matrix $(A^I \mid b^I)$ that contain at least one nondegenerate interval coefficient. Hence Theorem 1 shows that 2^p subsystems must be examined, as regards their feasibility, before the strong feasibility of a system $A^I x = b^I$ can be stated.

A subsystem (2) is called positively feasible if it has a positive solution (whose all entries are positive). A system $A^I x = b^I$ is called *strongly positively feasible* if each its subsystem is positively feasible. We have a similar result:

Theorem 2: *A system $A^I x = b^I$ is strongly positively feasible if and only if all its extremal subsystems are positively feasible.*

Proof: Again, only the "if" part needs to be proved. An existencial proof can be carried out in a similar way as in the previous proof when using the Stiemke's theorem [1] instead of the Farkas one. Further, the construction given in the constructive part of the proof of Theorem 1 can be also used in this case, since at each step a new solution is constructed as a convex combination of two previous ones. Thus if they are positive, then the new solution remains to be positive, Q.E.D.

3. Strong Solvability

A problem

$$\max \{c^T x \mid A x = b,\ x \geq 0\} \tag{5}$$

is called *strongly solvable in A^I, b^I, c^I* if for any $A \in A^I$, $b \in b^I$, $c \in c^I$ the problem (5) has a finite optimum.

Theorem 3: *Let $A^I x = b^I$ be strongly feasible. Then the following assertions are mutually equivalent:*

(*i*) *the problem* (5) *is strongly solvable in A^I, b^I, c^I,*

(*ii*) *the problem*

$$\max \{\bar{c}^T x \mid \underline{A}\, x \leq \bar{b},\ \bar{A}\, x \geq \underline{b},\ x \geq 0\} \tag{6}$$

has a finite optimum,

(*iii*)
$$\max \{\bar{c}^T x \mid \underline{A}\, x \leq 0,\ \bar{A}\, x \geq 0,\ x \geq 0\} = 0. \tag{7}$$

Proof: We shall prove (ii) ⇒ (i) ⇒ (iii) ⇒ (ii).

(ii) ⇒ (i): Consider a problem (5) with some $A \in A^I$, $b \in b^I$, $c \in c^I$. If x is a feasible solution of (5), then it satisfies the constraints of the problem (6) (see [2]), hence

$c^T x \leq \bar{c}^T x \leq \bar{c}^T x_0$, x_0 being an optimal solution of (6). Hence the objective function is bounded from above and (5) has a finite optimum.

(i)$\Rightarrow$(iii): Let $\underline{A}\, x \leq 0$, $\bar{A}\, x \geq 0$ for some nonnegative x. Then there exists a matrix $A_0 \in A^I$ with $A_0\, x = 0$ (see [2]). Consider the problem (5) with $A = A_0$, $b = \bar{b}$, $c = \bar{c}$. Since it has a finite optimum, the dual problem to it is feasible (in the linear programming terminology), hence the system

$$A_0^T\, u - A_0^T\, v + w = -\bar{c} \tag{8}$$

has a nonnegative solution u, v, w. Then the Farkas theorem as applied to (8) gives that $x \geq 0$, $A_0\, x = 0$ imply $\bar{c}^T x \leq 0$. Hence the optimal value of (7) is nonpositive but since $x = 0$ satisfies the constraints of (7), the optimal value is zero.

(iii)$\Rightarrow$(ii): Let x be a nonnegative vector with $\underline{A}\, x \leq 0$, $\bar{A}\, x \geq 0$. Then $\bar{c}^T x \leq 0$, which in the light of the Farkas theorem means that the system

$$\bar{A}^T u - \underline{A}^T v + w = -\bar{c}$$

has a nonnegative solution u, v, w. This shows that the dual problem to (6) is feasible, hence (6) has a finite optimum due to the duality theorem [1], Q.E.D.

Note: This result shows that as soon as the strong feasibility of a system $A^I x = b^I$ has been stated, only one LP problem (6) (with a doubled number of rows, however) needs to be solved in order to determine the strong solvability of the original problem.

References

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Dr. J. Rohn
Faculty of Mathematics and Physics
Charles University
Malostranské nám. 25
CS-11800 Praha 1
Czechoslovakia

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