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An Existence Theorem for Systems of Nonlinear Equations

A well-known theorem of analysis states that a real function $F(x)$, continuous in $[a, b] \subset R^1$ and satisfying the conditions $F(a) \leqq 0$, $F(b) \geqq 0$, vanishes in at least one point of $[a, b]$. This result can be extended to the n-dimensional Euclidean space R^n as follows (see [1], [2]): *if the n functions F_j, $j = 1, \dots, n$, are continuous in an n-dimensional rectangle $I = \{x \mid a_i \leqq x_i \leqq b_i,\ i = 1, \dots, n\}$ and satisfy the conditions*

(α) $F_j(x) \leqq 0$ *for each $x \in I$ with $x_j = a_j$,*

(β) $F_j(x) \geqq 0$ *for each $x \in I$ with $x_j = b_j$*

for $j = 1, \dots, n$, then there exists at least one point of I where $F_1, \dots, F_n$ vanish simultaneously. In this paper, we show that, after an appropriate modification of conditions (α), (β), this theorem remains valid if we replace a rectangle by a compact convex set.

Before giving the main result, we introduce some notations. Let C be a compact convex set in R^n. For $j = 1, \dots, n$, we denote by C_j^-, C_j^+ the sets

$$C_j^- = \{x \mid x \in C,\ x - te_j \notin C \text{ for any } t > 0\},$$

$$C_j^+ = \{x \mid x \in C,\ x + te_j \notin C \text{ for any } t > 0\}$$

where e_j is the unit vector whose j-th entry is 1 and the others are 0. In the proof of the following theorem, we use the Euclidean norm $\|x\| = \sqrt{\sum_j x_j^2}$ and the function of a real variable sign x, whose value is equal to 1 for $x > 0$, -1 for $x < 0$ and 0 for $x = 0$.

Theorem: *Let C be a nonempty compact convex set in R^n. For $j = 1, \dots, n$, let F_j be a function which is continuous in C, nonpositive in C_j^- and nonnegative in C_j^+. Then there exists a point $z \in C$ such that $F_j(z) = 0$, $j = 1, \dots, n$.*

Proof: We shall construct by induction continuous mappings $D_j: C \to C$, $j = 0, 1, \dots, n$. For the mapping D_0, defined by

$$D_0(x) = x, \qquad x \in C, \tag{1}$$

there is nothing to prove. Let $1 \leqq j \leqq n$ and let a mapping D_{j-1} with the above property be defined. Before defining D_j, we introduce an auxiliary mapping r_j by

$$r_j(x) = x - d_j(x) \operatorname{sign} F_j(x)\, e_j, \qquad x \in C, \tag{2}$$

where $d_j(x)$ is the distance of x from the set $T_j = \{y \mid y \in C,\ F_j(y) = 0\}$, which is obviously nonempty and compact. We prove that r_j is a continuous mapping of C into C. To this end, consider an $x \in C$. Then one of the three cases (a), (b), (c) occurs:

(a) $F_j(x) = 0$. Then $d_j(x) = 0$, hence $r_j(x) = x$. Since d_j is continuous in C, for each $\varepsilon > 0$ there exists a neighbourhood U of x such that for any $y \in U \cap C$ we have $\|y - x\| < \varepsilon/2$ and $d_j(y) < \varepsilon/2$, which implies $\|r_j(y) - r_j(x)\| < \varepsilon$. Hence r_j is continuous at x.

(b) $F_j(x) > 0$. In this case, there exists a neighbourhood V of x such that r_j has the form $r_j(y) = y - d_j(y)\, e_j$ for $y \in V \cap C$, so that r_j is continuous at x. To prove $r_j(x) \in C$, denote by u the intersection of the half-ray $R_x^- = \{x - te_j \mid t \geqq 0\}$ with C_j^-. Since $F_j(u) \leqq 0$, the segment $\overline{x, u}$ contains a point w with $F_j(w) = 0$, i.e. $w \in T_j$. Then, we have $\|r_j(x) - x\| = d_j(x) \leqq \|w - x\|$, which combined with $r_j(x) \in R_x^-$ gives $r_j(x) \in \overline{w, x}$, hence $r_j(x) \in C$.

(c) $F_j(x) < 0$. Arguing as above and using the half-ray $R_x^+ = \{x + te_j \mid t \geqq 0\}$, we again obtain the desired result.

Hence the mapping D_j defined by

$$D_j(x) = r_j(D_{j-1}(x)), \qquad x \in C, \tag{3}$$

is a continuous mapping of C into itself. Proceeding as described for $j = 1, \dots, n$, at the last induction step we obtain a continuous mapping $D_n: C \to C$ satisfying the identity

$$D_n(x) = x - \sum_{j=1}^{n} d_j(D_{j-1}(x)) \operatorname{sign} F_j(D_{j-1}(x))\, e_j \tag{4}$$

for any $x \in C$, as it can be easily derived from (1), (2), (3) by induction. According to the Brouwer fixed-point theorem ([3], p. 63), there exists a $z \in C$ such that $D_n(z) = z$, which in view of (4) and of the compactness of all the T_j's implies $F_j(D_{j-1}(z)) = 0$ for $j = 1, \dots, n$. Then (2), (3) give $D_j(z) = D_{j-1}(z)$ for $j = 1, \dots, n$, hence $D_j(z) = z$ for each j, which yields $F_j(z) = 0$, $j = 1, \dots, n$, Q.E.D.

Remark: If $C = \{x \mid a_i \leqq x_i \leqq b_i,\ i = 1, \dots, n\}$, is a rectangle, then $C_j^- = \{x \mid x \in C,\ x_j = a_j\}$ and $C_j^+ = \{x \mid x \in C,\ x_j = b_j\}$, so that the result quoted at the beginning of the paper is an easy consequnce of the theorem.

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References

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BUCHBESPRECHUNGEN

Zienkiewicz, O. C., The Finite Element Method. 3. Edition. London. McGraw-Hill Book Company (UK) Limited. 1977. XV, 787 S.,

Die 3. Auflage des Standardwerkes von O. C. Zienkiewicz zur Finite-Elemente-Methode (FEM) wurde gegenüber der 2. Auflage, die 1971 unter dem Titel „The Finite Element Method in Engineering Science" erschien, überarbeitet und wesentlich erweitert, so daß sowohl vom Inhalt als auch von der Form eigentlich ein neues Buch entstanden ist. Das Grundanliegen der 1. Auflage, die Leistungsfähigkeit der FEM an vielen Beispielen überzeugend nachzuweisen und Ingenieure aller Fachdisziplinen, aber auch Mathematiker, Physiker u. a. m. in die FEM-Ideologie einzuführen und sie zur Anwendung der FEM für die Lösung ihrer speziellen Aufgaben anzuregen, wird auch von der 3. Auflage erfüllt. Einige Kapitel zur Lösung von Standardaufgabenklassen der Elastizitätstheorie, wie ebene Spannungs- und Verzerrungszustände, axialsymmetrische Spannungszustände, dreidimensionale Spannungszustände, Biegung dünner Platten, dünne und dicke Schalen, Rotationsschalen usw., blieben weitgehend unverändert, jedoch wurden auch hierfür die Literaturangaben aktualisiert.

Die zahlreichen Überarbeitungen und Erweiterungen der 2. Auflage und die Aufnahme mehrerer neuer Kapitel verdeutlichen die stürmische Entwicklung, die die FEM in den letzten Jahren genommen hat. Das Kapitel 3 gibt einen guten Überblick über einen verallgemeinerten theoretischen Zugang zur FEM auf der Grundlage der Methode der gewichteten Residuen (MWR) sowie allgemeiner Variationsformulierungen. Dabei wird auch die auf effektive Anwendung von Penalty-Funk-