Real First-Order Inequality Constraints Are Stable with Probability One— A Case for Constraint Programming

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Abstract. Let a real first-order inequality constraint be a formula in the first-order predicate language over the reals, with the predicate symbols $\geq$ and $>$, and function symbols such as $+$, $\times$, sin, ...
A closed real first-order constraint is stable iff its truth value does not change under small perturbations of its constants. Traditionally, such constraints have been solved using methods from computer algebra. These use symbolic arithmetic that allow them to solve even unstable inputs. Recently, methods based on interval computation/constraint satisfaction techniques appear. These cannot solve unstable inputs, but concentrate on being fast for stable inputs, instead. In this paper we present a strong case for such methods by proving that in a realistic stochastic model, real first-order constraints are stable with probability one.

1 Introduction

Real first-order inequality constraints occur in numerous applications [7]. They have been traditionally solved by methods from computer algebra [3, 4], but recently methods based on interval arithmetic and constraint satisfaction appear [5, 6, 8, 1, 9].

Computer algebra methods take a huge effort by employing symbolic arithmetic with real algebraic numbers [11], which allows them to deal with inputs that are not stable, that is, inputs whose truth value changes under small perturbations of its constants [10]. The methods based on interval arithmetic/constraint satisfaction avoid this effort, by concentrating on stable inputs. This allows them to use floating-point arithmetic, which is usually several orders of magnitude faster than arithmetic with real algebraic numbers.

In this paper we give a strong theoretical argument for using constraint satisfaction techniques for solving real first-order inequality constraints: We develop a realistic stochastic model and show that, in this model, stable first-order constraints occur with probability one.

The structure of the paper is as follows: Section 2 describes the stochastic model, Section 3 recalls some characterizations of unstable first-order constraints, Section 4 proves the main theorem, and Section 5 contains an extended conclusion with a detailed discussion of the consequences of the result, and of our corresponding ongoing research program. Except for basic predicate logic and probability theory, the paper is self-contained.

2 A Stochastic Model for First-Order Inequality Constraints

A real first-order inequality constraint (from now on, short: *constraint*) is a formula in the first-order predicate language over the structure of the real numbers, where—in addition to the usual logical symbols—we allow the predicate symbols $\geq$ and $>$, and function symbols such as $+$, $\times$, sin, $\ldots$. We give the usual meaning in the structure of real numbers to all predicate and function symbols, and denote by $[\![\phi]\!]$ the according semantics of a constraint or term ϕ (for closed constraints this is a truth-value, and for terms a function over the reals).

We assume that all atomic constraints are of the form $f \geq 0$ or $f > 0$. Furthermore we divide the set of constraints into equivalence classes (the *supports*) of constraints that are the same up to constants. For example, $\forall \mathtt{x}\ \mathtt{x} - \mathtt{0y} \leq \mathtt{0}$ has the same support as $\forall \mathtt{x}\ \mathtt{x} - \mathtt{1y} \leq \mathtt{0}$. We address the constants appearing at the same position in each constraint of a support by *S-constant addresses* (this can be either be the path from the root to this position in a tree representation of the constraints, or just a unique integer value). For the rest of the paper let S be an arbitrary, but fixed support, containing only closed constraints.

Definition 1. *A* left-constant assignment for S *is an assignment of real numbers to all S-constant addresses on the left-hand side of predicate symbols. A* right-constant assignment for S *is an assignment of real numbers to the S-constant addresses on the right-hand side of predicate symbols. A* constant assignment for S *is a left-constant assignment in S plus a right-constant assignment in S.*

There is a one-to-one correspondence between S-constant assignments and constraints in S. For each constant assignment C, we denote its constraint in S by $\overline{C}^S$.

Now we associate with each constant address an according probability space $(\mathbb{R}, \mathcal{B}, P)$ (where $\mathcal{B}$ is the set of Borel sets in $\mathbb{R}$). Here we assume that the distribution function of the probability measure P is continuous (i.e., no point masses). We arrive at a probability space on left-constant assignments, right-constant assignments and constant assignments, respectively, by taking the product of the probability spaces of the according constant addresses. The probability space on constant assignments corresponds to a probability space on the constraints in S. Intuitively, given a set of constraints $A \subseteq S$, the according probability assesses how often a constraint occurring in a practical applications is an element of A.

Of course it is impossible to determine exactly the probability measures on the constant addresses that correspond to reality. This is why we use no specific knowledge about them except for the above "no-point-mass" assumption.

3 Unstable Constraints

In this paper we summarize special cases of some older results [10] necessary for proving the main theorem of this paper. From now on we assume an ordered set of variables V where each variable x in V ranges over a compact set R_x. For two real numbers x and y, we call $|x-y|$ their *distance*.

Definition 2. *A closed constraint ϕ in S is* stable *iff there exists a $\delta > 0$ such that for all ϕ' in S such that the distance between constants in ϕ and ϕ' with the same address is smaller than δ, ϕ and ϕ' have the same truth value.*

For characterizing stable constraints, we use a continuous notion of truth:

Definition 3. *For every constraint ϕ, let $[\![\phi]\!]^\circ$, the* degree of truth *of ϕ, be the following function in $\mathbb{R}^{|V|} \to \mathbb{R}$:*

- $[\![\theta \; \mathtt{r} \; \mathtt{0}]\!]^\circ(a_1, \ldots, a_{|V|}) = [\![\theta]\!](a_1, \ldots, a_{|V|})$, *where r is a predicate symbol and θ a term.*
- $[\![\phi_1 \wedge \phi_2]\!]^\circ(a_1, \ldots, a_{|V|}) = \min\{[\![\phi_1]\!]^\circ(a_1, \ldots, a_{|V|}), [\![\phi_2]\!]^\circ(a_1, \ldots, a_{|V|})\}$
- $[\![\phi_1 \vee \phi_2]\!]^\circ(a_1, \ldots, a_{|V|}) = \max\{[\![\phi_1]\!]^\circ(a_1, \ldots, a_{|V|}), [\![\phi_2]\!]^\circ(a_1, \ldots, a_{|V|})\}$
- $[\![\forall \mathtt{x} \; \phi']\!]^\circ(a_1, \ldots, a_{|V|}) = \inf\{[\![\phi']\!]^\circ(a_1, \ldots, a_{k-1}, u, a_{k+1}, \ldots, a_{|V|}) | u \in R_{\mathtt{x}}\}$, *where $\mathtt{x}$ is the k-th variable in V.*
- $[\![\exists \mathtt{x} \; \phi']\!]^\circ(a_1, \ldots, a_{|V|}) = \sup\{[\![\phi']\!]^\circ(a_1, \ldots, a_{k-1}, u, a_{k+1}, \ldots, a_{|V|}) | u \in R_{\mathtt{x}}\}$, *where $\mathtt{x}$ is the k-th variable in V.*

One can view this definition as the interpretation of constraints in an infinitely-many-valued logic. The degree of truth of a closed constraint is a constant function, and so we just call the assigned real number its degree of truth.

In the rest of the paper we will use the following characterization of stable constraints [10]:

Property 1. A closed constraint ϕ is *unstable* iff the degree of truth of ϕ zero.

Furthermore, adding a real number to the constants on the right-hand side of predicate symbols, lowers the degree of truth by that number [10]:

Property 2. Let (C_l, C_r) be a constant assignment and let p be a real constant. Then $\overline{(C_l, \lambda c.C_r(c) + p)}^S$ has degree of truth $[\![\overline{(C_l, C_r)}^S]\!]^\circ - p$.

4 Probability of Stability

Lemma 1. *Let C_l be a left-constant assignment. Then, in the probability space of right-constant assignments, the event that for an assignment C_r, the constraint $\overline{(C_l, C_r)}^S$ is unstable, has probability zero.*

Proof. By Properties 1 and 2, for each right-constant assignment C_r such that $\overline{(C_l, C_r)}^S$ is unstable, and for each non-zero real number p, the constant assignment $\overline{(C_l, \lambda c.C_r(c) + p)}^S$ has degree of truth $-p$, which means that it is stable. As a consequence, in the probability space of right-constant assignments of the form $\lambda c.C_r(c)+p$, with p any real number, the event of all C_r such that $\overline{(C_l, C_r)}^S$ is unstable, has probability zero.

Now we can divide the set of a right-constant assignments into equivalence classes of elements that just differ by a constant. By the above observation in each equivalence class the unstable ones have probability zero and so by Fubini's theorem, the total probability is also zero. □

Now, by Fubini's theorem, we get:

Theorem 1. *For a constant assignment C, $\overline{C}^S$ is unstable with probability zero.*

So in each support, real first-order inequality constraints are unstable with probability zero, and stable with probability one.

5 Conclusion

In this paper we have proven that closed real first-order inequality constraints are unstable with probability zero. This has various important consequences. For discussing these, we first ponder, whether the assumption that the distribution of the probability measures of constant addresses does not have point masses, is realistic.

Let a *problem domain* be a set of constraints that arise in "similar" applications (e.g., the set of constraints that occur in robust control [7]). Here we make the following (rough) distinction: We call a problem domain *concrete* if each of its elements corresponds to a specific real-world situation (like in robust control), otherwise we call it *abstract* (e.g., a certain class of theorems in geometric theorem proving).

In the former case, constants usually correspond to specific continuous real-word phenomena (e.g., the stiffness of a certain steel bar, the gross national product), so it is reasonable to assume that no single real number is privileged by a point-mass probability. In the latter case, constants are chosen by the person who formulates the theorem to be proven. Here, very often one is interested in theorems that are as strong as possible, that is, on the border between truth and falsehood, and thus in unstable theorems. Therefore it is reasonable to assume that the no-point-mass-assumption applies to concrete problem domains, but not necessarily to abstract problem domains.

As a consequence, the result of our paper confirms, that it is very important that more effort be spent on algorithms for concrete problem domains that are *efficient for stable inputs* (possibly containing trigonometric functions). Here the area of constraint programming can provide important tools:

Solving first-order constraints can be viewed as search for true elements (for existential quantifiers) and false elements (for universal quantifiers). However, as search is costly, we want to avoid it as much as possible by removing elements from the search space for which it is easy to prove that they are true/false. This is exactly, what consistency techniques can do very well. That is why we have used them in the branch and prune approach developed in another paper [9]: Branching represents search, and pruning uses consistency techniques to reduce the search space spanned by the quantifiers. Timings with a first prototype show, that for stable inputs the new method is usually orders of magnitude faster than classical symbolic methods. Currently we are working on a stable implementation, and on extending the approach to real first-order constraints that may contain equality symbols.

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