ROBUST POLE CLUSTERING OF PARAMETRIC UNCERTAIN SYSTEMS USING INTERVAL METHODS

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Abstract: In this paper a new methodology to solve the pole clustering problem for parametric uncertain systems is introduced: The problem of clustering the closed loop poles into prescribed D-regions in the complex plane is stated as a quantified constraint problem that represents bounded uncertain parameters by intervals; and an engineering-oriented approach based on interval methods is developed to solve this quantified constraint problem. The result is a new, robust, reliable and design oriented method to deal with parametric uncertain systems.
The methodology presented in this paper allows to find a good controller that places the closed loop poles in the desired location in the complex plane. In case there is no solution, the method allows also to "tune" the problem, either enlarging the pole locations or reducing the uncertainty domain.
The approach presented in this paper can be used either for linear or non-linear systems and for any kind of parametric bounded uncertainty.
Several simple examples illustrate the uses, limits and scope of the methodology.

1. PROBLEM STATEMENT

In this paper, we consider a class of plants with structured parametric uncertainties described by the equations

$$\begin{aligned} \dot{x} &= A(\boldsymbol{q})x + B(\boldsymbol{q})u \\ y &= C(\boldsymbol{q})x + D(\boldsymbol{q})u, \end{aligned} \tag{1}$$

depending on a structured perturbation characterized by the parameter vector

$$\boldsymbol{q} = [q_1 \quad q_2 \quad \dots \quad q_l]^T, \tag{2}$$

where each parameter enters into the system description with arbitrary dependency.

Due to the physical interpretation of the uncertain parameters, each one can be considered independent from the other and their values lie between upper and lower bounds. Then, the uncertain domain can be defined as a hyperrectangle (*box*)

$$\boldsymbol{Q} = \{\boldsymbol{q} = [q_1 \quad q_2 \quad \dots \quad q_l]^T \mid q_i \in [\underline{q_i}, \overline{q_i}], i = 1, \dots, l\} \tag{3}$$

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It is known that, when all state variables are measurable and are available for feedback, and the system is totally controllable, then the poles of the closed loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

The classical approach to the pole placement problem consist of finding a compensator, $F(\mathbf{k})$, that assigns the closed loop nominal plant poles to some desired location. When this compensator is applied to an uncertain system (Söelemez 1999), the closed loop poles move away from the nominal location and can result in undesired dynamical responses or even instability of the closed loop system.

As the solution of the pole placement problem is not unique, it is possible that for a subset of all compensators the dynamic response of the closed loop system is acceptable under all possible perturbations. This acceptable response can be represented by a set of regions in the complex plane which the poles are supposed to stay inside. This regions are called D-regions or Γ-regions. In Figure 1, typical D-regions are drawn.

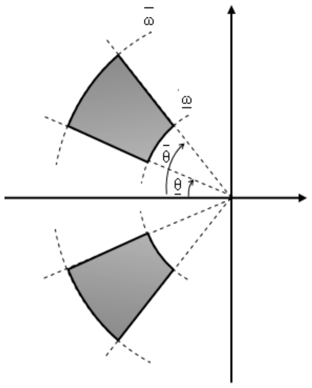

Fig. 1. Specified clustering regions

In the presence of parametric uncertainty, as it is described above, the *robust pole clustering* problem will be to find a control law $u = -\mathbf{K}x$ such that for every value of $\boldsymbol{q}$ belonging to the box $\boldsymbol{Q}$, the roots of the characteristic polynomial belongs to a prescribed set of stability regions called D-regions.

The way to describe the regions where the poles will be placed is by specifying a range for the desired damping ratio, $\zeta \in [\underline{\zeta}, \overline{\zeta}]$, and for the undamped natural frequency, $\omega \in [\underline{\omega}, \overline{\omega}]$. In the case of real poles, it will suffice to fix a range value for the real part $a \in [\underline{a}, \overline{a}]$.

In terms of the characteristic polynomial, the problem is stated as follows: Given the system described by (1), (2) and (3) and the state feedback law $u = -\mathbf{K}x$, the problem of robust pole placement consist of finding the gain matrix $\mathbf{K}$ such that:

$$
\begin{aligned}
\forall \boldsymbol{q} \in \boldsymbol{Q} \quad & \exists \omega_1 \in [\underline{\omega_1}, \overline{\omega_1}] \ldots \exists \omega_r \in [\underline{\omega_r}, \overline{\omega_r}] \\
& \exists \zeta_1 \in [\underline{\zeta_1}, \overline{\zeta_1}] \ldots \exists \zeta_r \in [\underline{\zeta_r}, \overline{\zeta_r}] \\
& \exists a_1 \in [\underline{a_1}, \overline{a_1}] \ldots \exists a_t \in [\underline{a_t}, \overline{a_t}] \\
\prod_{i=1}^{r} \prod_{j=1}^{t} & (s^2 + 2\zeta_i \omega_i s + \omega_i^2)(s + a_j) = |sI - (A - BK)|
\end{aligned}
\tag{4}
$$

where r is the number of complex poles and t is the number of real poles. Thus $n = 2r + t$ is the order of the system. In this paper we restrict ourselves to the case where $t = 1$ if r is odd, or $t = 0$ if r is even.

In Section 2 we introduce a methodology based on interval methods (Moore 1966, Jaulin *et al.* 2001) for solving the above problem; in Section 3 we apply the methodology to several examples; and in Section 4 we conclude the paper.

2. METHODOLOGY

Note that the equality sign in Equation 4 denotes equality on *polynomials*. So we can do coefficient comparison and replace it by several equalities on real numbers of the following form:

$$
\begin{aligned}
S_1(\zeta_1 \ldots \zeta_r, \omega_1 \ldots \omega_r, a_1 \ldots a_t) &= D_1(\mathbf{k}, \mathbf{q}), \\
&\ldots \\
S_{r+t}(\zeta_1 \ldots \zeta_r, \omega_1 \ldots \omega_r, a_1 \ldots a_t) &= D_{r+t}(\mathbf{k}, \mathbf{q})
\end{aligned}
\tag{5}
$$

Here the functions $S_1, \ldots, S_{r+t}$ are the coefficients of the left-hand side polynomial of Constraint 4 (which are again polynomials, in the variables ζ_i, ω_i and a_i), and $D_1, \ldots, D_{r+t}$ the coefficients of the right-hand side polynomial (with the uncertain parameters $\mathbf{q}$ and design parameters $\mathbf{k}$). Now we can write the whole problem (Constraint 4) as the following quantified constraint:

$$
\begin{aligned}
\forall \mathbf{q} \in \mathbf{Q} \quad & \exists \zeta_1 \in [\underline{\zeta_1}, \overline{\zeta_1}] \ldots \exists \zeta_r \in [\underline{\zeta_r}, \overline{\zeta_r}] \\
& \exists \omega_1 \in [\underline{\omega_1}, \overline{\omega_1}] \ldots \exists \omega_r \in [\underline{\omega_r}, \overline{\omega_r}] \\
& \exists a_1 \in [\underline{a_1}, \overline{a_1}] \ldots \exists a_t \in [\underline{a_t}, \overline{a_t}] \\
S_1(\zeta_1 \ldots \zeta_r, \omega_1 \ldots \omega_r, a_1 \ldots a_t) &= D_1(\mathbf{k}, \mathbf{q}), \\
&\ldots \\
S_{r+t}(\zeta_1 \ldots \zeta_r, \omega_1 \ldots \omega_r, a_1 \ldots a_t) &= D_{r+t}(\mathbf{k}, \mathbf{q})
\end{aligned}
\tag{6}
$$

Such quantified constraints are in principle solvable (Tarski 1951), but they represent an inherently difficult problem, both from the practical and from the theoretical (Weispfenning 1988, Davenport and Heintz 1988) point of view. Classically, mainly symbolic quantifier elimination algorithms (Collins 1975, Hong 1990, Loos and Weispfenning 1993) have been applied to such problems (Dorato *et al.* 1997, Jirstrand 1997, Nešić and Mareels 1998, Anai 1998, Anai and Weispfenning 2001, Dorato 2000). However, recently methods based on interval methods appear (Jaulin and Walter 1996, Benhamou and Goualard 2000, Ratschan 2002*a*). In this paper we also use interval techniques, and apply them to problem sizes that are currently too hard to solve for symbolic methods.

In order to solve this problem, we propose to divide the work into two main steps:

Step 1: Find a candidate $\mathbf{k^0}$ likely to fulfill Constraint 6
Step 2: *Prove* that Constraint 6 holds for $\mathbf{k^0}$.

Of course the notion of "likely to fulfill" in Step 1 is not precise—one can define it according to the problem at hand and the tools available. Here we use the following process to find such candidates: Replace the universal quantifier by a conjunction over a finite number of samples. Then, after giving the existentially quantified variables a different name in each branch of the conjunction, we can push this conjunction inside, drop the existential quantifiers and arrive at an underdetermined system of non-linear equations. It suffices now to find an (approximate) solution to this system. For this, one can use constraint satisfaction methods (Jaulin *et al.* 2001, Davis 1987), or (local or global (Horst and Pardalos 1995)) optimization methods (after introducing a suitable optimality criterion).

For solving Step 2 we use the observation that whenever we have a witness fulfilling the existentially quantified variables, it suffices to prove that

- it does not vanish under perturbations of $\mathbf{q}$, and
- it does not leave the bounds under perturbations of $\mathbf{q}$.

The first item is ensured as long as no ω_i gets negative. Here we need the assumption that $t = 1$ if r is odd and $t = 0$ if r is even: otherwise a real solution might change into a pair of complex ones—allowing a solution to vanish.

Assuming that all the intervals $[\underline{\omega_1}, \overline{\omega_1}], \ldots, [\underline{\omega_r}, \overline{\omega_r}]$ are non-negative we can now replace Step 2 by the following:

Step 2a: Prove that Constraint 6 holds for $\mathbf{k^0}$ after replacing $\forall$ by $\exists$
Step 2b: Prove that for all $\mathbf{q}\in\mathbf{Q}$, for all r on a border of $[\underline{\zeta_1}, \overline{\zeta_1}] \times \cdots \times [\underline{\zeta_r}, \overline{\zeta_r}] \times [\underline{\omega_1}, \overline{\omega_1}] \times \cdots \times [\underline{\omega_r}, \overline{\omega_r}] \times [\underline{a_1}, \overline{a_1}] \times \cdots \times [\underline{a_t}, \overline{a_t}]$, Equation 5 does not hold.

Step 2a can be done either symbolically or using software based on validated interval arithmetic. Certain tools based on validated interval arithmetic might provide such a proof already in Step 1.

Step 2b amounts to solving $2(r + t)$ universally quantified disequality-constraints. These can either be solved directly by software for solving quantified constraints (Ratschan 2000), or by guaranteed global constraint solvers after transforming $\forall\neg$ to $\neg\exists$, and using the solver to prove the non-existence of a solution.

The above discussion ensures the correctness of the presented methodology. Now we ask the dual question: How strong is this methodology? Does it always succeed?

Of course, for Step 1, this depends on the used definition of "likely to fulfill". Step 2a can always succeed for a $\mathbf{k_0}$, for which Constraint 6 holds, because the checked condition checked is implied by this constraint. For Step 2b we have to weaken the condition just a little bit:

Theorem 1. If

- Constraint 6 holds, with the intervals of the existentially quantified variables replaced by open intervals, and if
- $[\underline{\zeta_1}, \overline{\zeta_1}] = \cdots = [\underline{\zeta_r}, \overline{\zeta_r}]$ and $[\underline{\omega_1}, \overline{\omega_1}] = \cdots = [\underline{\omega_r}, \overline{\omega_r}]$

then the condition checked in Step 2b holds.

Proof. Assume that Constraint 6 holds, with the intervals of the existentially quantified variables replaced by open intervals. Then the witnesses for the existential quantifier do not touch the borders. So it remains to prove that no other solutions appear on the borders. This is true because the solutions are unique up to symmetry. □

Step 1 is itself the goal of most of the approaches to robust control for parametric systems. However, most of this methods fall in one of the following categories:

- Are applicable only for a class of uncertain plants (linear systems with affine dependency on the parameters, polynomials, etc)
- Does not provide a measure about the goodness of the designed controller

Moreover, all of them fail regarding reliability on the result. Only in some cases (for a class of uncertain plants or by overbounding the uncertainty) can those method assure that the resulting controller meets the requirements *for all* the family of plants.

The methodology proposed in the paper allows the user (the control engineer) to use any method for step 1 and provides, by Step 2, a way to overcome the weaknesses mentioned above, because Step 2

- provides reliability to the results (the problem is stated in such form that the results are guaranteed by definition),
- can be used as a tuning tool in the design procedure: one can enlarge (or reduce) the uncertainty bounds and/or the regions where the poles will be clustered, and
- is also an analysis tool since one can check the robust stability of a given controlled family of plants.

3. EXAMPLES

In this section, several design examples are presented. In the first example (Section 3.1) a result is found without changing the bounds. The second (Section 3.2) and the third example (Section 3.3) demonstrate what one can do when Step 2 fails for the parameter vector $\mathbf{k_0}$ found in Step 1. Example 3 can also be an example of stability analysis, as we compute regions in the uncertain parameters space where the controlled systems have the closed loop poles inside the prescribed regions.

Note that all three examples are too hard for solving them directly with currrent symbolic methods for solving quantified constraints (Collins 1975, Hong 1990, Loos and Weispfenning 1993).

3.1 Designing a Sliding Surface

Sliding mode control is an approach very often used to control systems under uncertainty. If the matching condition is accomplished, then the dynamics in sliding mode do not depend on the uncertain parameters.

In this example, the methodology is used to design the sliding surface for a sliding mode controller. The system to control corresponds to the crane example presented in (Ackermann 1993).

In this case, a sliding mode controller is proposed in order to make the system more robust against uncertainty. The uncertain system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_{43} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix}$$

$$a_{23} = \frac{m_L}{m_C} g, \qquad b_2 = \frac{1}{m_C}$$

$$a_{43} = -\frac{(m_L + m_C)g}{m_C l}, \qquad b_4 = -\frac{1}{m_C l}$$

As the matching condition is not accomplished, the dynamics of the sliding mode will depend on some of the uncertain parameters. By taking the sliding function

$$\sigma = \mathbf{C}\mathbf{x}; \quad \mathbf{C} = \begin{bmatrix} c_1 & c_2 & c_3 & 1 \end{bmatrix},$$

the dynamic in sliding mode ($\dot{\sigma} = 0, \sigma = 0$) is given by the following characteristic polynomial:

$$p_{sm} = s^3 + \frac{c_3 + lc_1}{c_2 l + 1} s^2 + \frac{c_2}{c_2 l + 1} s - 9.8 \frac{c_1}{c_2 l + 1}$$

The corresponding characteristic polynomial is:

$$p_D = s^3 + (2\zeta\omega_n + a)s^2 + (2\zeta\omega_n a + \omega_n^2)s + \omega_n^2$$

Designing the sliding surface consist of choosing the design parameters, c_1, c_2, c_3, such that

$$\forall l \in [8, 16] \; \exists \zeta \in [0.451] \; \exists \omega_n \in [0.6, 2] \; \exists a \in [0.5, 2]$$
$$\begin{aligned} \frac{c_3 + lc_1}{c_2 l + 1} &= 2\zeta\omega_n + a, \\ \frac{c_2}{c_2 l + 1} &= 2\zeta\omega_n a + \omega_n^2, \\ -9.8 \frac{c_1}{c_2 l + 1} &= \omega_n^2 \end{aligned}$$

We find a candidate (Step 1) by replacing the universal quantifier by a conjunction over the points 7.39 and 17.51 (by using the endpoints of an interval enclosing the uncertainty interval $[8, 16]$ we use a stronger condition that increases the likelihood of success for Step 2). After renaming the existentially quantified variables for each conjunction we arrive at an underdetermined systems of equations. Now one can use the constraint satisfaction software RealPaver (Granvilliers 2002) to prune the possible space for such a $\mathbf{k_0}$ (i.e., remove elements for which the quantified constraint provably does not hold), arriving at:

```
OUTER BOX: HULL of 1024 boxes
  c1 = -0.01890930621287335 + [-9.597e-10,+9.597e-10]
  c2 = -0.05416974204003023 + [-1.059e-10,+1.059e-10]
  c3 = 0.8970984281170524 + [-1.607e-08,+1.607e-08]
```

After choosing the values

$$\begin{aligned} c1 &= -0.0189093062128733 \\ c2 &= -0.541697420400302 \\ c3 &= 0.89709842811705 \\ l &= 12 \end{aligned}$$

within their corresponding intervals, the computer algebra system MuPAD (*MuPAD* 2002) could symbolically prove the existence of corresponding ζ, ω, a.

For all of the eight border checks needed by Step 2 we use RealPaver to prove the non-existence of according solutions.

3.2 Unstable plant

This problem is taken from the PhD thesis by Bondia (2002). The objective is to design a controller for an unstable plant corresponding to the identified model for an scale helicopter. The plant model is given by:

$$G(s,q) = K/s^2$$

where $K = [4.9, 6.125]$.

We design a PID controller such that $\zeta = [1, 1.01]$ and $\omega = [4.6, 5]$, for the dominant poles of the closed loop characteristic polynomial. The third pole can be located between -5 and -20.

In other words, given uncertainty domain $K = [4.9, 6.125]$, and the desired pole locations $\zeta \in [1, 1.01]$, $\omega \in [4.6, 5]$ and $a = [5, 20]$ the robust pole placement consists of finding K_p, K_i and K_d such that

$$\begin{aligned} &\forall K \in [4.9, 6.125] \\ &\quad \exists \zeta \in [1, 1.01]\ \exists \omega \in [4.6, 5]\ \exists a \in [5, 20] \\ &\qquad KK_d = a + 2\zeta\omega, \\ &\qquad KK_p = 2\zeta\omega + \omega^2, \\ &\qquad KK_i = a\omega^2 \end{aligned}$$

For $\mathbf{Q} = [4.9, 6.125]$, $\zeta \in [0.9, 1.18]$, $\omega \in [4.6, 5]$, $a \in [5, 20]$ we find a candidate (Step 1) by replacing the universal quantifier by a conjunction over both endpoints 4.9 and 6.125. After renaming the existentially quantified variables for each conjunction we arrive at an underdetermined systems of equations. Now one can use RealPaver (Granvilliers 2002) to prune the possible space for such a $\mathbf{k_0}$ (i.e., remove elements for which the quantified constraint provably does not hold), arriving at:

```
kd = 3.231037414966007 + [-1.004e-13,+1.004e-13]
kp = 6.008163265306122 + [-8.882e-16,+1.776e-15]
ki = 32.61267006802763 + [-2.188e-12,+2.196e-12]
```

Using the MuPad we can then prove symbolically the existence of a solution for the midpoint of the above intervals.

However, in Step 2 the checks for the lower borders corresponding to ω and both ζ checks fail! As Theorem 1 ensures, this can only happen if the solutions for ζ, ω, a touch the borders of their D-regions. So we enlarge the D-regions: after changing the lower bound of ω to 4.59 and changing the ζ-interval to $[0.899, 1.19]$ all checks succeeded, proving the original quantified constraint with slightly enlarged D-regions.

3.3 Second Order SISO Problem

Given a generic second order SISO system with uncertain parameter vector $\mathbf{q}$:

$$\dot{x} = A(\boldsymbol{q})x + B(\boldsymbol{q})u$$

with

$$A = \begin{bmatrix} a_{11}(\boldsymbol{q}) & a_{12}(\boldsymbol{q}) \\ a_{21}(\boldsymbol{q}) & a_{22}(\boldsymbol{q}) \end{bmatrix}; \qquad B = \begin{bmatrix} b_1(\boldsymbol{q}) \\ b_2(\boldsymbol{q}) \end{bmatrix};$$

the characteristic polynomial $p(s, \boldsymbol{q})$ is given by:

$$\begin{aligned} &s^2 + (b_2k_2 - a_{11} - a_{22} + b_1k_1)s + \\ &+ a_{11}a_{22} - a_{11}b_2k_2 + b_1k_2a_{21} - b_1k_1a_{22} - a_{12}a_{21} + a_{12}b_2k_1 \end{aligned}$$

and the design polynomial $p_D(s)$ is $s^2 + 2\zeta\omega s + \omega^2$.

The pole placement problem is to find k_1 and k_2 such that

$$\forall \boldsymbol{q} \in \boldsymbol{Q}\ \exists \zeta \in [\underline{\zeta}, \overline{\zeta}]\ \exists \omega \in [\underline{\omega}, \overline{\omega}] \quad p(s, \boldsymbol{q}) = p_D(s)$$

In the following example a state controller is found for a systems with three uncertain parameters.

Given

$$A = \begin{bmatrix} q_1 + q_2 + q_3 & q_2q_3 \\ q_1^2q_2 & q_1 + q_3 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

The characteristic polynomial $p(s, \mathbf{q})$ is $s^2 + c_1(\mathbf{q})s + c_0(\mathbf{q})$ where

$$\begin{aligned} c_1(\mathbf{q}) &= (k_2 - 2q_1 - q_2 - 2q_3), \\ c_0(\mathbf{q}) &= (q_1 + q_2 + q_3)(q_1 + q_3) - \\ &\quad -(q_1 + q_2 + q_3)k_2 - q_2^2q_3q_1^2 + q_2q_3k_1. \end{aligned} \tag{7}$$

Given uncertainty domain $\boldsymbol{Q} = [0.4, 0.6] \times [0.4, 0.6] \times [0.4, 0.6]$, and the desired pole locations $\zeta \in [0.5, 1]$, and $\omega \in [0.5, 2]$ the robust pole placement consist of finding k_1 and k_2 such that

$$\begin{aligned} &\forall q_1 \in [0.4, 0.6]\ \forall q_2 \in [0.4, 0.6]\ \forall q_3 \in [0.4, 0.6] \\ &\quad \exists \zeta \in [0.5, 1]\ \exists \omega \in [0.5, 2] \\ &\qquad c_1(\mathbf{q}) = 2\zeta\omega, c_0(\mathbf{q}) = \omega^2 \end{aligned}$$

with $c_1(\mathbf{q})$ and $c_0(\mathbf{q})$ as above.

For finding a candidate (Step 1), we drop the universal quantifier after substituting the midpoint of the uncertainty intervals $[0.4, 0.6]$ for q_1, q_2, q_3, respectively. We solve the resulting undetermined system of equations by substituting the midpoint of $[0.5, 1]$ and $[0.5, 2]$ for ζ and ω, respectively. This results in a simple system of equations which has the symbolic solutions $k_1 = 213/8$ and $k_2 = 35/8$.

Since we have computed a symbolic solution in Step 1, Step 2a is already done. For Step 2b we use a interval based solver for quantified constraints (Ratschan 2002*b*), which fails for the original uncertainty intervals $[0.4, 0.6]$ but succeeds for the smaller uncertainty intervals $q_1 \in [0.4, 0.55]$, $q_2 \in [0.49, 0.55]$, $q_3 \in [0.46, 0.55]$.

4. CONCLUSION

In this paper we have introduced a new interval-based methodology for solving the pole clustering problem for parametric uncertain systems, and we have demonstrated the use of the methodology on three examples. In future work we try to design an algorithm that fully automatizes the methodology, and we will combine the verification part (Step 2) of our methodology with other traditional methods for finding good controllers.

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