Hidden Conflict of Belief Functions
(working version of extented text)*

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Abstract. Hidden conflict of belief functions in case of the sum of all multiples of conflicting belief masses being equal to zero was observed. Degrees of hidden conflict and of non-conflictness are defined and analysed including full non-conflictness. Hidden conflict between two belief functions is distinguished from internal hidden conflict(s) of the individual belief function(s). Finally, computational issues of hidden conflict and non-conflictness are presented.

Keywords: belief functions · Dempster-Shafer theory· uncertainty· conflicting belief masses· internal conflict· conflict between belief functions· hidden conflict · full non-conflictness.

1 Introduction

When combining belief functions (BFs) by the conjunctive rules of combination, some conflicts often appear (they are assigned either to $\emptyset$ by non-normalised conjunctive rule $\odot$ or distributed among other belief masses by normalisation in Dempster's rule of combination $\oplus$). Combination of conflicting BFs and interpretation of their conflicts are often questionable in real applications. Thus a series of papers related to conflicts of BFs was published, e.g. [2, 7, 10, 8, 11, 13–15, 19, 22]. A new interpretation of conflicts of belief functions was introduced in [5]: important distinction of internal conflicts of individual BFs (due to their inconsistency) from conflicts between BFs (due to conflict/contradiction of evidences represented by the BFs) was introduced there. Note that zero sum of all multiples of conflicting belief masses (denoted by $m_{\odot}(\emptyset)$) is usually considered as non-conflictness of the belief functions in all these approaches.

When analysing the conflict between belief functions based on their non-conflicting parts[3] [8] a positive value of conflict was observed even in a situation when sum of all multiples of conflicting belief masses equals to zero. This arose

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[3] Conflicting and non-conflicting parts of belief functions originally come from [6].

a series of new questions: how to interpret the sum of conflicting masses, is the conflict based on non-conflicting parts of belief functions correct? Some of the answers are provided in this text. They are positive in favour of the conflict based on non-conflicting parts. It led to a definition of a hidden conflict of BFs (Section 3).

Going further, different levels / degrees of hidden conflicts are defined and a maximal degree of hidden conflict is investigated. Analogously to the degrees of hidden conflict, there also exist different degrees of non-conflictness. Full non-conflictness and conditions, under which belief functions are fully non-conflicting, are defined and presented in Section 4.

In accordance with approach from [5], there are observed and presented not only hidden conflicts between two belief functions, but also internal hidden conflicts of individual BFs (Section 5). Finally, computational aspects of hidden conflict are presented in Section 6, ideas and goals for a future research in Section 7.

2 Preliminaries

We assume classic definitions of basic notions from theory of *belief functions* [20] on finite exhaustive frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. See also [4].

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm).* $m(\emptyset) = 0$ is usually assumed. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is *power-set* of Ω. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. Because there is a unique correspondence among m and corresponding Bel and Pl thus we often speak about m as of belief function.

A *focal element* is a subset of the frame of discernment $X \subseteq \Omega$, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF); in fact, it is a probability distribution on Ω. If there are only focal elements such that $|X| = 1$ or $|X| = n$ we speak about *quasi-Bayesian BF* (qBBF). In the case of $m(\Omega) = 1$ we speak about *vacuous BF* (VBF) and otherwise about *a non-vacuous BF*; in the case of the only focal element $\emptyset \neq X \subset \Omega$, i.e., if $m(X) = 1$, we speak about *a categorical BF*. If all focal elements have a non-empty intersection, we speak about *a consistent BF*; and if all of them are nested, about *a consonant BF*.

Dempster's (normalized conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [20]. Putting $K = 1$ and $(m_1 \circledcirc m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, see e. g. [21].

Smets's *pignistic probability* is given by $BetP(\omega_i) = \sum_{\omega_i \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1-m(\emptyset)}$, see e.g. [21]. *Normalized plausibility of singletons*[4] of Bel is a probability distribution Pl_P such that $Pl\_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [3, 4].

A conflict of BFs Bel', Bel'' based on their non-conflicting parts is defined by the expression $Conf(Bel', Bel'') = (m'_0 \circledcirc m''_0)(\emptyset)$, where non-conflicting part Bel_0 (of a BF Bel) is unique consonant BF such that $Pl\_P_0 = Pl\_P$ (normalised plausibility of singletons corresponding to Bel_0 is the same as that corresponding to Bel). For an algorithm to compute Bel_0 see [8].

3 Observation of Hidden Conflict

3.1 An Introductory Example

Let us suppose two simple consistent belief functions Bel' and Bel'' on a three-element frame of discernment $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ given by the bbas $m'(\{\omega_1, \omega_2\}) = 0.6$, $m'(\{\omega_1, \omega_3\}) = 0.4$, and $m''(\{\omega_2, \omega_3\}) = 1.0$. Then $(m' \circledcirc m'')(\emptyset) = 0$ what seems — and it is usually considered — to be a non-conflictness of m' and m'', but there is positive conflict based on non-conflicting parts $Conf(Bel', Bel'') = (m'_0 \circledcirc m''_0)(\emptyset) = 0.4 > 0$. (This holds true despite of Theorem 4 from [8] which should be revised in future).

We can easily verify this situation: the only focal element of m'' has non-empty intersection with both focal elements of m', thus $(m' \circledcirc m'')(\emptyset) = \sum_{X \cap Y = \emptyset} m'(X)m''(Y) =$ (empty sum) $= 0$; Bel'' is already consonant itself, thus $Bel''_0 = Bel''$, $m''_0 = m''$, $Pl'(\{\omega_1\}) = 1$, $Pl'(\{\omega_2\}) = 0.6$, $Pl'(\{\omega_3\}) = 0.4$, thus $m'_0(\{\omega_1\}) = 0.4$, $m'_0(\{\omega_1, \omega_2\}) = 0.2$, $m'_0(\{\omega_1, \omega_2, \omega_3\}) = 0.4$, hence $Conf(Bel', Bel'') = (m'_0 \circledcirc m''_0)(\emptyset) = m'_0(\{\omega_1\})m''_0(\{\omega_2, \omega_3\}) = 0.4 \cdot 1 = 0.4$.

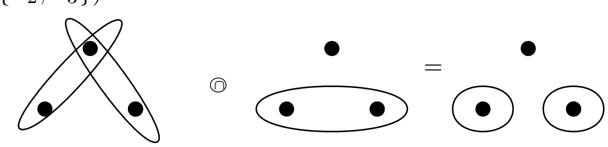

Fig. 1. Introductory Example: focal elements of m', m'', and of $m' \circledcirc m''$.

3.2 Interpretation of the Example

The following questions arise: Does $(m' \circledcirc m'')(\emptyset) = 0$ really represent non-conflictness of respective BFs? Is the definition of conflict based on non-conflicting parts correct? Are m' and m'' conflicting or non-conflicting? What does $(m' \circledcirc m'')(\emptyset) = 0$ mean?

[4] Plausibility of singletons is called *contour function* by Shafer in [20], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

Suppose that Bel' and Bel'' are non-conflicting now. Thus their combination should be also non-conflicting with both of them. Does this hold for BFs from our example? This holds true when we combine $m'\textcircled{\cap}m''$ with m'' one more time. It follows from the idempotency of categorical m'': $m'\textcircled{\cap}m''\textcircled{\cap}m'' = m'\textcircled{\cap}m''$ and therefore $(m'\textcircled{\cap}m''\textcircled{\cap}m'')(\emptyset) = 0$ again. On the other hand, we obtain positive $(m'\textcircled{\cap}m''\textcircled{\cap}m')(\emptyset) = (m'\textcircled{\cap}m'\textcircled{\cap}m'')(\emptyset) = 0.48$. See Table 1 and Figure 2. When m'' and m' are combined once, then we observe $m_{\textcircled{\cap}}(\emptyset) = 0$. When combining m'' with m' twice then $m_{\textcircled{\cap}}(\emptyset) = 0.48$. We observe some kind of *a hidden conflict*. Moreover, both individual BFs are consistent. I.e. there are no internal conflicts. Thus the hidden conflict is *hidden conflict between the BFs* and we have an argument for correctness of positive value of $Conf(Bel', Bel'')$.

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3	$\emptyset$
$m'(X)$:	0.0	0.0	0.0	0.60	0.40	0.00	0.00	—
$m''(X)$:	0.0	0.0	0.0	0.00	0.00	1.00	0.00	—
$(m'\textcircled{\cap}m'')(X)$:	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m'\textcircled{\cap}m''\textcircled{\cap}m'')(X)$:	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m'\textcircled{\cap}m''\textcircled{\cap}m')(X)$:	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48
$(m'\textcircled{\cap}m''\textcircled{\cap}m'\textcircled{\cap}m'')(X)$:	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48

Table 1. Hidden conflict in the Introductory Example

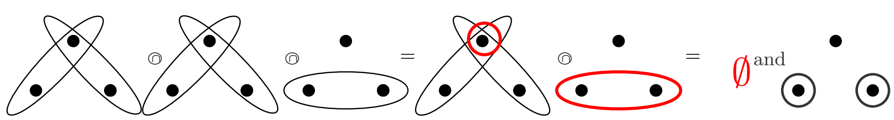

Fig. 2. Arising of a hidden conflict between BFs in the Introductory Example: focal elements of m', m', m'' — $m' \textcircled{\cap} m', m''$ and of $(m' \textcircled{\cap} m') \textcircled{\cap} m''$.

What is a decisional interpretation of our BFs? Contours, i.e. plausibilities of singletons are $Pl' = (1.0, 0.6, 0.4)$ and $Pl'' = (0.0, 1.0, 1.0)$, we obtain $Pl\_P' = (0.5, 0.3, 0.2)$ and $Pl\_P'' = (0.0, 0.5, 0.5)$ by normalization; thus at Bel', ω_1 is significantly preferred, whereas at Bel'', one of ω_2, ω_3; this is also an argument for mutual conflictness of the BFs. Considering Smets' pignistic probability we obtain $BetP' = (0.5, 0.3, 0.2)$ and $BetP'' = (0.0, 0.5, 0.5)$, just the same values as in the case when normalized plausibility of singletons (normalized contour) is used for decision. Thus the argument for mutual conflictness of the BFs is strengthened and we obtain a same pair of incompatible decisions based on the BFs in both frequent decisional approaches: using either normalized contour (which is compatible with conjunctive combination of BFs) or pignistic probability (designed for betting).

Hence $(m' \textcircled{\cap} m'')(\emptyset)$ does not mean non-conflictness of the BFs. It means a simple or partial compatibility of their focal elements only.

3.3 Definition of Hidden Conflict

Definition 1. *Let us suppose two BFs Bel', Bel'' defined by bbas m', m'', such that $(m' \circledcirc m'')(\emptyset) = 0$. If there further holds $(m' \circledcirc m'' \circledcirc m')(\emptyset) > 0$ or $(m' \circledcirc m'' \circledcirc m'')(\emptyset) > 0$ we say that there is* a hidden conflict *of the BFs.*

Observation 1 *A condition $(m' \circledcirc m'' \circledcirc m')(\emptyset) > 0$ or $(m' \circledcirc m'' \circledcirc m'')(\emptyset) > 0$ from Definition 1 is equivalent to the following condition $(m' \circledcirc m'' \circledcirc m' \circledcirc m'')(\emptyset) > 0$.*

We have to note that a hidden conflict is quite a new phenomenon, qualitatively different from the ideas of all previous Daniel's works on conflict of belief functions and also different from the other referred approaches. Till now, it was supposed that $m_{\circledcirc}(\emptyset)$ includes both conflict between BFs and also internal conflicts of individual BFs. Thus conflict between BFs was supposed to be less or equal to $m_{\circledcirc}(\emptyset)$. Here, we deal with a situation of a positive conflict between BFs while $m_{\circledcirc}(\emptyset) = 0$.

We have already observed that $m_{\circledcirc}(\emptyset) = 0$ does not mean full non-conflictness of BFs and that the condition $(m' \circledcirc m'' \circledcirc m' \circledcirc m'')(\emptyset) > 0$ together with $(m' \circledcirc m'')(\emptyset) = 0$ defines hidden conflict. What about the condition $(m' \circledcirc m'' \circledcirc m' \circledcirc m'')(\emptyset) = 0$? Is this condition sufficient for full non-conflictness of BFs Bel' and Bel''? May some conflict be still hidden there?

The zero version of the condition seems to imply non-conflictness on Ω_3, the frame of discernment of the Introductory Example. To solve the question in general, we have to consider a larger frame of discernment.

3.4 Little Angel Example

For Ω_5 one can find the following Little Angel Example, see Table 2. Similarly to the Introductory Example, we have two consistent BFs Bel^i and Bel^{ii}, which have disjoint sets of max-plausibility elements and where zero condition $(m^i \circledcirc m^{ii})(\emptyset) = 0$ holds true.

In addition to the Introductory Example, $(m^i \circledcirc m^{ii} \circledcirc m^i \circledcirc m^{ii})(\emptyset) = 0$ (see Table 2) while $Conf(Bel^i, Bel^{ii}) = 0.1$ is positive again. Positiveness of the $Conf$ value can be easily seen from the fact that sets of max-plausibility elements are disjoint for Pl^i and Pl^{ii}. Numerically, we have again $Bel_0^{ii} = Bel^{ii}$, and $Pl\_P^i = (\frac{10}{39}, \frac{4}{39}, \frac{9}{39}, \frac{9}{39}, \frac{7}{39})$. We obtain $m_0^i(\{\omega_1\}) = 0.1, m_0^i(\{\omega_1, \omega_3, \omega_4\}) = 0.2, m_0^i(\{\omega_1, \omega_3, \omega_4, \omega_5\}) = 0.3, m_0^i(\{\Omega_5\}) = 0.4$, and $Conf(Bel^i, Bel^{ii}) = m_0^i(\{\omega_1\}) m^{ii}(X) = 0.1$. Analogous arguments hold true for the positive $Conf$ and hidden conflict again (hidden in the 2nd degree this time). $BetP^i = (0.2583, 0.1083, 0.2250, 0.2250, 0.1833)$ which is not numerically the same as Pl_P^i, but both prefer ω_1, whereas $BetP^{ii} = Pl\_P^{ii} = (0.00, 0.25, 0.25, 0.25, 0.25)$.

For an existence of a hidden conflict, it is the structure of focal elements that is important — not their belief masses. Belief masses are important for the size of a conflict. In general, we can take $m^i(A) = a$, $m^i(B) = b$, $m^i(C) = c$, for A, B, C defined in Table 2 and for any $a, b, c > 0$, such that $a + b + c = 1$ and

X :	$A=\{\omega_1,\omega_2,\omega_5\}$	$B=\{\omega_1,\omega_2,\omega_3,\omega_4\}$	$C=\{\omega_1,\omega_3,\omega_4,\omega_5\}$	$X=\{\omega_2,\omega_3,\omega_4,\omega_5\}$	$\emptyset$
$m^i(X)$:	0.1	0.30	0.60	0.00	—
$m^{ii}(X)$:	0.0	0.00	0.00	1.00	—

X :	$A\cap X$	$B\cap X$	$C\cap X$	$A\cap B\cap X$	$A\cap C\cap X$	$B\cap C\cap X$	$\emptyset$
$(m^i \circledcirc m^{ii})(X)$:	0.1	0.3	0.6	0.0	0.0	0.0	0.000
$(m^i \circledcirc m^{ii} \circledcirc m^{ii})(X)$:	0.10	0.30	0.60	0.00	0.00	0.00	0.00
$(m^i \circledcirc m^i \circledcirc m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i \circledcirc m^i \circledcirc m^{ii} \circledcirc m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i \circledcirc m^{ii} \circledcirc m^{ii} \circledcirc m^{ii})(X)$:	0.010	0.090	0.360	0.060	0.120	0.360	0.000
$(m^i \circledcirc m^i \circledcirc m^i \circledcirc m^{ii})(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108
$m(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108

where $m = m^i \circledcirc m^i \circledcirc m^i \circledcirc m^{ii} \circledcirc m^{ii} \circledcirc m^{ii}$.

Table 2. Hidden Conflict in the Little Angel Example

we obtain $m(\emptyset) = 6abc$ as a hidden conflict of the 2nd degree (a conflict hidden in the 2nd degree). In our numeric case there is $6abc = 6 \cdot 0.1 \cdot 0.3 \cdot 0.6 = 0.108$. For graphical presentation of the Little Angel Example see Figure 3.

Degrees of hidden conflict, its maximal value, and the issue of full non-conflictness will be analyzed in the following section.

4 Degrees of Hidden Conflict and Full Non-conflictness

When analyzing examples from the previous section, we have observed different degrees of hidden conflict. We can formalize it in the next definition.

Definition 2. *Assume two BFs Bel^i, Bel^{ii} defined by bbas m^i, m^{ii}, such that for some $k > 0$ $(\bigcirc_{j=1}^{k}(m^i \circledcirc m^{ii}))(\emptyset) = 0$. If there further holds $(\bigcirc_{j=1}^{k+1}(m^i \circledcirc m^{ii}))(\emptyset) > 0$ we say that there is* a conflict of BFs Bel^i *and* Bel^{ii} hidden in the k-th degree*.*

Analogously to particular degrees of hidden conflict, there are degrees of non-conflictness. Particular degrees of non-conflictness are not very important. However, there is an important question whether there is some hidden conflict or not, i.e. whether or not the BFs in question are fully non-conflicting.

Definition 3. *We say that BFs Bel^i and Bel^{ii} are* fully non-conflicting *if there is no hidden conflict of any degree. I.e. if $(\bigcirc_{j=1}^{k}(m^i \circledcirc m^{ii}))(\emptyset) = 0$ for any $k > 0$.*

Thus there is a question how many times we have to combine $(m^i \circledcirc m^{ii})$, i.e., for which k value of $(\bigcirc_{j=1}^{k}(m^i \circledcirc m^{ii}))(\emptyset)$ shows whether there is some hidden conflict of the BFs Bel^i and Bel^{ii} or not. For answers to this question see corollaries of the following two theorems.

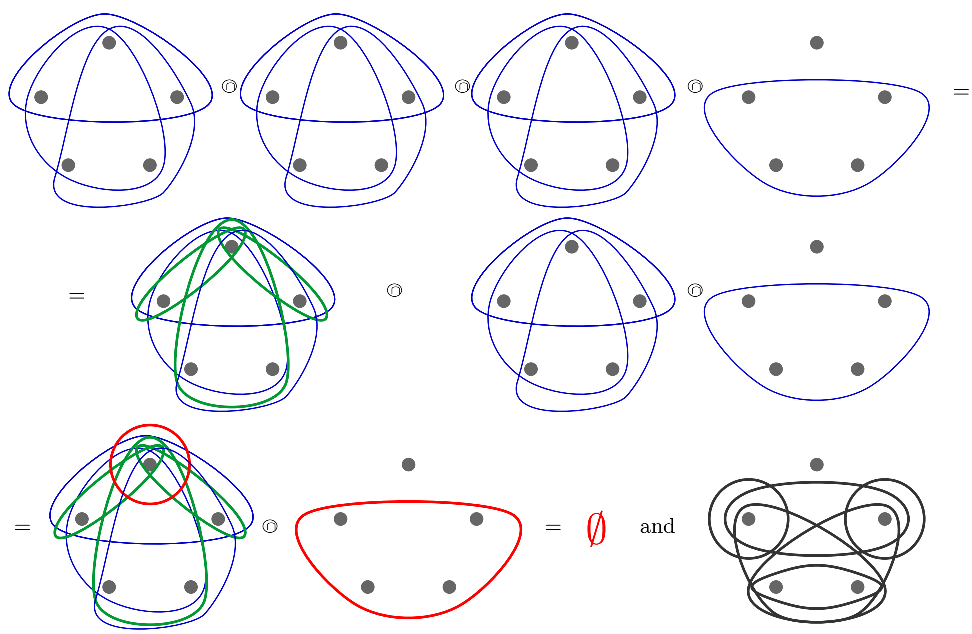

Fig. 3. Arrising of a hidden conflict between BFs in the Little Angel Example. Focal elements of m^i, m^{ii}, $m^i \circledcap m^i$, $m^i \circledcap m^i \circledcap m^i$ and of $(m^i \circledcap m^i \circledcap m^i) \circledcap m^{ii}$. Red-colored focal elements are those responsible for creation of the empty-set in the last step.

Theorem 1 (maximal degree of hidden conflict). *For any non-vacuous BFs Bel^i and Bel^{ii} defined by m^i and m^{ii} on any frame Ω_n it holds that*

$$(\textcircled{\cap}_{j=1}^{n-1}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0 \quad \textit{iff} \quad (\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$$

for any $k > n-2$.

Corollary 1.
(i) Hidden conflict of any non-vacuous BFs on any Ω_n has always degree less or equal to $n-2$.
(ii) Having any two BFs Bel^i, Bel^{ii} defined by m^i and m^{ii} on any frame of discernment Ω_n, zero value of the expression $(\textcircled{\cap}_{j=1}^{n-1}(m^i \textcircled{\cap} m^{ii}))(\emptyset)$, i.e., the condition

$$(\textcircled{\cap}_{j=1}^{n-1}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$$

always means full non-conflictness of the BFs.

Lemma 1.
(i) The degree of a hidden conflict is finite.
(ii) The degree of a hidden conflict is $\leq n$.

Lemma 2 (A series of statements for proving Lemma 1 and Theorem on maximal degree.).

(i) $Bel^i \textcircled{\cap} Bel^{ii}$: focal elements of Bel^i, Bel^{ii} are kept or decreased when the BFs are combined.
(ii) $Bel \textcircled{\cap} Bel$: focal elements of any Bel are kept + possibly decreased when the BF is combined with itself.
(iii) $(Bel^i \textcircled{\cap} Bel^{ii}) \textcircled{\cap} Bel^{ii}$: focal elements of $(Bel^i \textcircled{\cap} Bel^{ii})$ are kept + possibly decreased when $(Bel^i \textcircled{\cap} Bel^{ii})$ is combined with Bel^i or Bel^{ii} again.
(iv) If any focal element F of $(Bel^i \textcircled{\cap} Bel^{ii})$ is kept when $(Bel^i \textcircled{\cap} Bel^{ii})$ is combined with Bel^{ii}, then it is also kept when $(Bel^i \textcircled{\cap} Bel^{ii})$ is combined with $(Bel^{ii} \textcircled{\cap} Bel^{ii})$.
(v) If any focal element F of $(Bel^i \textcircled{\cap} Bel^{ii})$ is kept when $(Bel^i \textcircled{\cap} Bel^{ii})$ is combined with $(Bel^i \textcircled{\cap} Bel^{ii})$, then it is also kept when $(Bel^i \textcircled{\cap} Bel^{ii})$ is combined with $\textcircled{\cap}_{j=1}^{2}(Bel^i \textcircled{\cap} Bel^{ii})$.
(vi) If any focal element F of $\textcircled{\cap}_{j=1}^{k}(Bel^i \textcircled{\cap} Bel^{ii})$ is kept when $\textcircled{\cap}_{j=1}^{k}(Bel^i \textcircled{\cap} Bel^{ii})$ is combined with $(Bel^i \textcircled{\cap} Bel^{ii})$, then it is also kept when combined with $\textcircled{\cap}_{j=1}^{m}(Bel^i \textcircled{\cap} Bel^i)$ for any $m > 1$.
(vii) For any BFs Bel^i and Bel^{ii}: $\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset) > 0$ implies $\textcircled{\cap}_{j=1}^{k+1}(m^i \textcircled{\cap} m^{ii}))(\emptyset) > 0$.

Proof (of Theorem on maximal degree).
(i) Focal element (f.e.) Ω does not causes any conflict. F.e. of cardinality $\leq n-1$ — after maximally $n-2$ combinations $\textcircled{\cap}$ — possibly produces a singleton. It can be possibly conflicting with a f.e. of the other BF (using the above lemmata) and $\textcircled{\cap}_{j=1}^{n-1}(m^i \textcircled{\cap} m^{ii}))(\emptyset) > 0$.
(ii) Max degree at least $n-2$: this immediately follows Example 1. If this is not the case then, after $n-2$ combinations $\textcircled{\cap}$, no conflicting f.e. can appear.

Theorem 2. *(i) Any non-vacuous BFs Bel^i, Bel^{ii} have a conflict hidden at most in $(c-1)$-th degree where $c = min(c^i, c^{ii}) + sgn(|c^i - c^{ii}|)$. where c^i, c^{ii} are maximal cardinalities of focal elements of Bel^i, Bel^{ii} different from Ω. In the other words*

$$\textcircled{\cap}_{j=1}^{c}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0 \quad \mathit{iff} \quad \textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$$

for any $k \geq c = min(c^i, c^{ii}) + sgn(|c^i - c^{ii}|)$.
(ii) There are no hidden conflicts of any non-vacuous BFs on any two-element frame Ω_2.
(iii) There are no hidden conflicts of any non-vacuous quasi-Bayesian BFs on any frame Ω_n.
(iv) For a BF Bel^i and a qBBF Bel^{ii} there is a hidden conflict of (at most) the first degree; if it appears then it is an internal conflict of Bel^{ii} in fact.

Proof. All the statements follow from the statements of Lemma 2, analogously to the proof of Theorem on maximal degree. Ad (iv): it always holds $Conf(Bel^i, Bel^{ii}) = 0$ for a BF Bel^i and a qBBF Bel^{ii} such that $(m^i \textcircled{\cap} m^{ii})(\emptyset) = 0$.

Corollary 2. *(i) Assume two non-vacuous BFs Bel^i, Bel^{ii} on Ω_n. The zero value of the expression $(\textcircled{\cap}_{j=1}^{c}(m^i \textcircled{\cap} m^{ii}))(\emptyset)$, i.e., the condition*

$$(\textcircled{\cap}_{j=1}^{c}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$$

means full non-conflictness of the BFs for $c = min(c^i, c^{ii}) + sgn(|c^i - c^{ii}|)$, where c^i, c^{ii} are maximal cardinalities of focal elements of Bel^i, Bel^{ii} different from Ω_n.
(ii) For any two non-vacuous quasi Bayesian BFs Bel^i, Bel^{ii} on any frame of discernment Ω_n the condition $(m^i \textcircled{\cap} m^{ii})(\emptyset) = 0$ always means full non-conflictness of the BFs.
(iii) For any BF Bel^i and any quasi-Bayesian BF Bel^{ii} the condition $(\textcircled{\cap}_{j=1}^{2}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$ always means full non-conflictness of the BFs.

Example 1. Example of hidden conflict of the $(n-2)$-th degree: Let us suppose n-element frame of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. Bel^i and Bel^{ii} are given by $m^i(\{\omega_1, \omega_2, ..., \omega_{n-1}\}) = \frac{1}{n-1}$, $m^i(\{\omega_1, \omega_2, ..., \omega_{n-2}, \omega_n\}) = \frac{1}{n-1}$, $m^i(\{\omega_1, \omega_2, ..., \omega_{n-3}, \omega_{n-1}, \omega_n\}) = \frac{1}{n-1}$, . . . $m^i(\{\omega_1, \omega_3, \omega_4, ..., \omega_n\}) = \frac{1}{n-1}$, $m^{ii}(\{\omega_2, \omega_3, ..., \omega_n\}) = \frac{1}{1}$. At $m^i \textcircled{\cap} m^i$ $(n-2)$-element focal elements appear, at $m^i \textcircled{\cap} m^i \textcircled{\cap} m^i$ $(n-3)$-element focal elements appear, at $\textcircled{\cap}_{j=1}^{k} m^i$ $(n-k)$-element focal elements appear, at $\textcircled{\cap}_{j=1}^{n-2} m^i$ 2-element focal elements appear, all these focal elements have non-empty intersections with the only focal element of m^{ii}, and finally at $\textcircled{\cap}_{j=1}^{n-1} m^i$ singleton focal element $\{\omega_1\}$ appears which has empty intersection with the only focal element of m^{ii} $\{\omega_2, \omega_3, ..., \omega_n\}$.

What does m^i express? It gives a big support to all elements of the frame, to the entire frame Ω_n and even greater support to ω_1 which is included in all focal elements; ω_1 is preferred and, moreover, it has plausibility 1. We can modify m^i

and express this more easily: $\overline{m}^i(\Omega_n) = \frac{n-1}{n}$, $\overline{m}^i(\{\omega_1\}) = \frac{1}{n}$, or more generally, $\widetilde{m}^i(\Omega_n) = 1 - a$, $\widetilde{m}^i(\{\omega_1\}) = a$ for some $0 < a < 1$. We can easily see evident conflict corresponding to positive $\overline{m}(\emptyset) = (\overline{m}^i \circledcirc m^{ii})(\emptyset) = \frac{1}{n}$, $\widetilde{m}(\emptyset) = a$ for these modifications of m^i. Hence either hidden conflict of the $(n-2)$-th degree of m^i and m^{ii} or positive $Conf(m^i, m^{ii}) = Conf(\overline{m}^i, m^{ii}) = \frac{1}{n}$ are not surprising.

We have to note that the Introductory Example is a special instance of Example 1 for $n = 3$.

Structure of BFs on Ω_n which have the hidden conflict of $(n-2)$-th degree is spedified by the following lemma:

Lemma 3. *Non-vacuous BFs on Ω_n with hidden conflict of degree $(n-2)$ are just the BFs with focal elements of cardinality $\geq n-1$, such that one has at least $(n-1)$ focal elements of cardinalities $(n-1)$ and the other one has just one focal element of cardinality $(n-1)$. Moreover, every $(n-1)$-element subset of Ω_n must be a focal element of either one or both BFs.*

Proof. (Read: f.e. $|F| = n$ as a focal element F of cardinality n; h.c. as hidden conflict.) F.e. $|F| = n$ does not decrease (by intersection) neither cardinality of any other f.e. nor the degree of h.c. Note that BFs from Example 1 and all their extensions with f.e.(s) $|F| = n$ have h.c. of the $(n-2)$-th degree. Removing of a f.e. $|F| = n-1$ from the BFs remove h.c.

Without loss of generality we assume Bel^i with at least $(n-1)$ f.e. $|F| = n-1$ and Bel^{ii} with just one f.e. $|F| = n-1$. Addition a f.e. $|F| = n-1$ to Bel^{ii} excludes other element of Ω_n by Bel^{ii}; thus the degree of h.c. decreases by 1. Addition of the missing f.e. $|F| = n-1$ to Bel^i from Example 1 keeps the degree $n-2$ of h.c. There is — of course — a decrease of degree of h.c. by adding a f.e. $|F| < n-1$ either to Bel^i or Bel^{ii}. The degree of h.c. is analogously decreased by decreasing of cardinality of any f.e. $|F| = n-1$. Moving a f.e. $|F| = n-1$ from Bel^i to Bel^{ii}: F decreases its cardinality in $\circledcirc m^{ii}$ thus there not necessary $n-2$ combinations with Bel^i, hence there is a decrease in degree of h.c. by 1 again.

5 Internal Hidden Conflict

We can observe internal hidden conflict when at least one of BFs in hidden conflict is not consistent:

Example 2 (Little Angel Modified). Let us consider the following modification of the Little Angel Example on Ω_5. Let us take m^{iii} instead of m^i, such that $m^{iii}(A) = m^i(A)$, $m^{iii}(C) = m^i(C)$, and $m^{iii}(D) = m^{iii}(\{\omega_2, \omega_3, \omega_4\}) = 0.30$ instead of $m^i(B)$. There is $(m^{ii} \circledcirc m^{ii} \circledcirc m^{iii} \circledcirc m^{iii})(\emptyset) = 0$, but $(m^{ii} \circledcirc m^{iii} \circledcirc m^{iii} \circledcirc m^{iii})(\emptyset) > 0$, even $(m^{iii} \circledcirc m^{iii} \circledcirc m^{iii})(\emptyset) > 0$, i.e. $(\circledcirc_1^3 m^{iii})(\emptyset) > 0$, see Table 3.

We observe a conflict of the belief functions hidden in the 2-nd degree again. Nevertheless, the situation of focal elements is different now: the only focal element X of $m^{ii} = \circledcirc_1^3 m^{ii}$ has non-empty intersection with any focal element

X :	$A=\{\omega_1,\omega_2,\omega_5\}$	$D=\{\omega_2,\omega_3,\omega_4\}$	$C=\{\omega_1,\omega_3,\omega_4,\omega_5\}$	$X=\{\omega_2,\omega_3,\omega_4,\omega_5\}$	$\emptyset$
$m^{iii}(X)$:	0.1	0.30	0.60	0.00	—
$m^{ii}(X)$:	0.0	0.00	0.00	1.00	—

X :	$A\cap X$	$D\cap X$	$C\cap X$	$A\cap D\cap X$	$A\cap C\cap X$	$D\cap C\cap X$	$\emptyset$
$(m^{iii}\circledcirc m^{ii})(X)$:	0.1	0.3	0.6	0.0	0.0	0.0	0.000
$(m^{iii}\circledcirc m^{ii}\circledcirc m^{ii})(X)$:	0.10	0.30	0.60	0.00	0.00	0.00	0.00
$(m^{iii}\circledcirc m^{iii}\circledcirc m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^{iii}\circledcirc m^{iii}\circledcirc m^{ii}\circledcirc m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^{iii}\circledcirc m^{ii}\circledcirc m^{ii}\circledcirc m^{ii})(X)$:	0.010	0.090	0.360	0.060	0.120	0.360	0.000
$(m^{iii}\circledcirc m^{iii}\circledcirc m^{iii}\circledcirc m^{ii})(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108
$m(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108

where $m = m^{iii}\circledcirc m^{iii}\circledcirc m^{iii}\circledcirc m^{ii}\circledcirc m^{ii}\circledcirc m^{ii}$.

Table 3. Hidden Conflict in the Little Angel Modified Example

of $\circledcirc_1^3 m^{iii}$, but $(\circledcirc_1^3 m^{iii})(\emptyset) > 0$ now. Thus this is not a hidden conflict between m^{ii} and m^{iii}, but *an internal hidden conflict* of m^{iii} comming from its non-consistency.

We can see, that the values in Table 3 are numerically same to those of Table 2 not only in rows defining the input bbms m^{ii} and m^{iii}, but also in all other rows, thus the tables are identical up to desription of row and collumns, corresponding to different focal elements. For the focal elements see also Figure 4.

In relation to conflict based on non-conflicting parts of belief functions we can observe $Conf(m^{ii}, m^{iii}) = 0$, as $m_0^{iii}(\omega_3, \omega_4\}) = 0.2\overline{2}$, $m_0^{iii}(\omega_1, \omega_3, \omega_4, \omega_5\}) = 0.3\overline{3}$, and $m_0^{iii}(\Omega_5) = 0.4\overline{4}$. Hence the minimal focal elements of $m_0^{ii} = m^{ii}$ and of m_0^{iii} have non-empty intersection.

(. . . figure in preparation . . .)

Fig. 4. Arrising of an internal hidden conflict in the Little Angel Modified Example. Focal elements of m^{iii}, $m^{iii}\circledcirc m^{iii}$,iii $m^{iii}\circledcirc m^{iii}\circledcirc m^{iii}$ and of m^{ii} and $(m^{iii}\circledcirc m^{iii}\circledcirc m^{iii})\circledcirc m^{ii}$. Red-colored focal elements are those responsible for creation of the empty-set in the next step.

Example 3. Let us also consider the following modification of Example 1 of a conflict hidden in maximal degree on Ω_n. Instead of m^i we take m^{iii} having all focal elements of cardinality $n-1$, such that $m^{iii}(\Omega_n \setminus \{\omega\}) = \frac{1}{n}$ for any $\omega \in \Omega_n$; m^{ii} same as in Example 1. m^{iii} is not consistent; $Pl^{iii}(\{\omega\}) = \frac{n-1}{n}$ for any $\omega \in \Omega_n$. We observe hidden conflict of the (n–2)-th degree again. Because of same plausibilities of all singletons ${m'''}_0(\Omega_n) = 1$ and $Conf(m^{iii}, m^{ii}) = 0$ now.

There is a positive hidden conflict of BFs Bel^{iii} and Bel^{ii}, but zero conflict between them. We say that there is *an internal hidden conflict.* This corresponds to non-consistency of BF Bel^{iii}; Bel^{ii} is consistent thus there is an internal hidden conflict of BF Bel^{iii} in this case.

A numeric example was computed on Ω_{16}, see Table 4 for a comparison of focal elements and $m_{\textcircled{\cap}}(\emptyset)$ values of Examples 1 and 3. For simplicity, same bbms $m^i(X) = \frac{1}{15}$ and $m^{iii}(X) = \frac{1}{16}$ were used there.

Ω_{16}		$m' = m^i$	$m'' = m^{ii}$		$m' = m^{iii}$	$m'' = m^{ii}$	
Degree	$m_{\textcircled{\cap}}$	no of f.e.	Card. of f.e.	$m_{\textcircled{\cap}}(\emptyset)$	no of f.e.	Card. of f.e.	$m_{\textcircled{\cap}}(\emptyset)$
–	m'	15	15	–	16	15	–
–	m''	1	15	–	1	15	–
0	$m' \textcircled{\cap} m''$:	15	14	0	16	14–15	0
1	$m' \textcircled{\cap} m'' \textcircled{\cap} m' \textcircled{\cap} m''$	120	13–14	0	121	13–15	0
2	$\textcircled{\cap}_{j=1}^{3}(m' \textcircled{\cap} m'')$	575	12–14	0	576	12–15	0
	. . .			0			0
k	$\textcircled{\cap}_{j=1}^{k+1}(m' \textcircled{\cap} m'')$	. . .	(14-k)–14	0	. . .	(14-k)–15	0
	. . .			0			0
13	$\textcircled{\cap}_{j=1}^{14}(m' \textcircled{\cap} m'')$	32766	1–14	0	32767	1–15	0
14	$\textcircled{\cap}_{j=1}^{15}(m' \textcircled{\cap} m'')$	32766	1–14	2.98–06	32767	1–15	1.13–06

Table 4. Hidden conflict between BFs Bel^i and Bel^{ii} from Example 1 and internal hidden conflict of Bel^{ii} and Bel^{iii} from Example 2, both on Ω_{16}.

Theorem 3. *For the BFs from Lemma 3 the following holds:*
(i) If one of the BFs has just $n-1$ focal elements of cardinality $n-1$ then there is hidden conflict between the BFs.
(ii) If one of the BFs has all n focal elements of cardinality $n-1$ then there is an internal hidden conflict.

Proof. (i) Boh the BFs are consistent, thus there is no internal conflict there.
(ii) Bel with n or $n+1$ focal element is not consistent, There is $Conf(Bel^i, Bel^{ii}) = 0$ in the case of conflict based on non-conflicting parts and also all conflicts which are $\leq (m^i \textcircled{\cap} m^{ii})(\emptyset)$, thus there is no conflict between. Rest follows from Lemma 3.

In general, there may be a mixture of internal hidden conflict and hidden conflict between BFs for general BFs defined under different assumptions.

Remark Accepting vacuous BF and internal conflict we can obtain two degenerated cases of hidden conflict of degree $(n-1)$ as modification of Examples 1 and 3. (i): m^{iii} from Example 3 and VBF instead of m^{ii}; (ii): m^{iv} which have all focal elements of m^{iii} plus additional focal element Ω_n and VBF again. These hidden conflicts are hidden internal conflicts of m^{iii} and m^{iv}. Using Lemma 2 we can show that there are no other hidden conflicts of $(n-1)$-th degree of any other BFs on Ω_n.

Thus $(\textcircled{\cap}_{j=1}^{n}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$ iff $(\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$ holds true for any pair of BFs on Ω_n in full generality.

Remark Computation of $\textcircled{\cap}_1^k(m' \textcircled{\cap}\, m'')()$ and internal hidden conflict have a relation to Martin's auto-conflict [16, **?**], thus we can speak about hidden auto-conflict here. For more detail see [9].

6 Computational Complexity and Computations of Examples

Based on Definition 2 and Theorem 1, the complexity of computation of the degree of hidden conflict of two BFs Bel^i and Bel^{ii} is — on a general Ω_n — $O(n)$ of $\textcircled{\cap}$ operations. In the case of checking existence of a hidden conflict of the BFs we obtain the complexity $O(log_2(n))$ of $\textcircled{\cap}$ operations utilizing a simplification of computation based on $\textcircled{\cap}_{j=1}^{2k}(m^i \textcircled{\cap} m^{ii}) = \textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}) \textcircled{\cap}\ \textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii})$. Note that the complexity of $\textcircled{\cap}$ operation depends on the number and structure of focal elements.

During our analysis of hidden conflicts a series of example computations was performed on frames of discernment of cardinality from 5 to 16. A number of focal elements rapidly grows up to $|\mathcal{P}(\Omega)| = 2^{|\Omega|}-1$ when conjunctive combination $\textcircled{\cap}$ is repeated, see e.g. 32766 and 32767 focal elements in the presented Examples 1 and 3 at Table 4. Because the degree of the hidden conflict and existence of the hidden conflict depends on the number and the structure of focal elements not on their bbms, we have used same bbms for all focal elements of a BF in our computations on frames of cardinality greater than 10.

All our experiments were performed in R [17] using R Studio [18]. We are currently developing an R package for dealing with belief functions on various frames of discernment. It is based on a relational database approach - nicely implemented in R in package called data.table [12].

7 Several Important Remarks

We have to underline that hidden conflict of belief functions is not a new measure of conflict. This notion serves for deeper understanding of conflictness / non-conflictness, it enables to point out the conflict also in situations where conflicts had not been expected, in situations where $m_{\textcircled{\cap}}(\emptyset) = 0$; hence to point out and to help to understand the conflicts which are hidden by $m_{\textcircled{\cap}}(\emptyset) = 0$.

Particular numeric values of hidden conflict have not yet any enough reasonable interpretation. We are only interesting whether the value is zero (thus no hidden conflict is there) or whether it is positive (thus a hidden conflict appears there). Degrees of hidden conflict do not present any size or a strength of the conflict. They present the level / degree how the conflict is hidden, i.e. a conflict of degree k is hidden in k-th degree. Thus the degrees are rather degrees of hiddeness of the conflicts. The higher degree, the higher hiddeness, thus less conflict and less strength of the same value of conflict in fact.

Repeating application of conjunctive combination $\oplus$ of a BF with itself is used here to simulate situation where different independent believers have numerically the same bbm. Thus this has nothing to do with idempotent belief combination (where of course no conflict between two BFs is possible).

There is brand new idea of hidden conflicts in [9] and in this contribution. The assumption of non-conflictness when $m_{\odot}(\emptyset) = 0$ was relaxed, due to observation of conflict even in the cases where $m_{\odot}(\emptyset) = 0$. Both these studies want to point out the existence of hidden (auto-)conflicts in situations where no conflict was expected till now. Thus the definitions of hidden conflict and hidden auto-conflict are not anything against the previous Daniel's research and results on conflict of belief functions e.g. [5, 7, 8]. Of course, some parts of the previous approaches should be updated to be fully consistent with the new presented results on hidden conflicts and auto-conflicts.

8 Ideas for a Future Research

Presenting new ideas and results concerning conflicts of belief functions make a challenge for a future investigation in the following directions.

System of degrees of hidden conflict resembles Martin's auto-conflict [16, 15], especially in the case of internal hidden conflict. Moreover, repeating conjunctive combination of input BFs is included in computations. Thus a relationship of hidden conflict to Martin's auto-conflict was the first idea for a future research. This topic was already investigated and the first resupts were presented at CJS 2017, see [9].

Because the first observation of hidden conflict was made in a context of dealing with conflict of BFs based on their non-conflicting parts, consequences for the conflict based on non-conflicting parts parts should be also investigated. Theorem 4 from [8] should be reformulated - results on hidden conflict should be applied.

Relation and consequences of the presented qualitatively new ideas and results to the axiomatic approaches to conflicts of belief functions [11, 15] and [1] should be analysed and updated. The new versions of these approaches (or a new axiomatic approach considering hidden conflict of belief functions) should be elaborated.

9 Summary and Conclusion

Hidden conflicts of belief functions in situations where mutual intersections of any focal element of one BF with all focal element of the other BF are non-empty has been presented and analysed. There may be a positive conflict in situations, where sums of conflicting belief masses are empty, i.e. in situations which have been usually considered to be non-conflicting till now.

Several levels — degrees of hidden conflict were observed, maximal degree of hidden conflicts dependent on size of corresponding frame of discernment was found. A variety of hidden conflicts of degrees 1 – $(n-2)$ was described for an n-element frame of discernment. A necessary and satisfiable condition for full non-conflictness of BFs in dependence on maximal cardinality of their focal elements has been specified and computational aspects presented. Analogously to the evident conflicts, internal hidden conflicts are distinguished from the internal conflicts between BFs.

This qualitatively new phenomenon of conflicts of BFs moves us to better understanding of nature of conflicts of belief functions in general and brings a challenge to elaborate and update existing approaches to conflicts of BFs.

This may consequently serve as a basis for better combination of conflicting belief functions and better interpretation of the results of belief combination whenever conflicting belief functions appear in real applications.

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