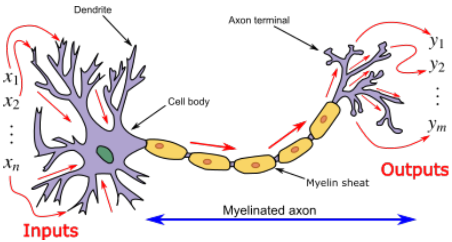
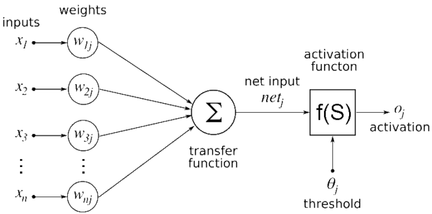
Basic concepts of artificial neural networks

Neural network inspiration



(a) Neuron in biological neural network



(b) Neuron in artificial neural network

Neural network inspiration

Biological Neural Network (BNN)	Artificial Neural Network (ANN)
Soma (Neuron body)	Node
Dendrites	Input
Synapse	Weights or Interconnections
Axon	Output

Neural network as a graph

Neurons

Let $u, v \in \mathcal{V}$ are neurons represented as vertices of a graph.

Connection links

The tuples (u, v) or (v, u) are connection links represented as oriented edges. $\mathcal{C} \subset \mathcal{V} \times \mathcal{V}$ is the set of all edges.

Neural net

$(\mathcal{V}, \mathcal{C})$ is a graph representing a neural net.

Communication with environment

- In our definition, the neurons can communicate with each other.
- We need to communicate with an environment ϖ.

Input and output connection links

$\varepsilon \subset \{\varpi\} \times \mathcal{V} \cup \mathcal{V} \times \{\varpi\}$

Input node

If $((\varpi, u) \in \varepsilon$, then the node u receives signals from the environment.

Output node

If $((v, \varpi) \in \varepsilon$, then the node v transfers signals to the environment.

Topology

- The triplet (V, C, ε) is called topology.

Input and output sets

Let us define:

- input set of neuron v:

$$i(v) : \{u : u \in \mathcal{V} \& (u, v) \in \mathcal{C}\}$$

- output set of neuron v:

$$o(v) : \{u : u \in \mathcal{V} \& (v, u) \in \mathcal{C}\}$$

Neuron types with respect to connections

- Input nodes:

$$\mathcal{I} = \{v : v \in \mathcal{V} \& i(v) = \emptyset\}$$

- Output nodes:

$$\mathcal{O} = \{v : v \in \mathcal{V} \& o(v) = \emptyset\}$$

- Hidden nodes

$$\mathcal{H} = \mathcal{V} \setminus (\mathcal{I} \cup \mathcal{O})$$

Important neural net conditions

- The graph (V, C) is non-redundant.

$$(\forall v \in \mathcal{V})(\exists u \in \mathcal{V})\{(u, v), (v, u)\} \cap \mathcal{C} \neq \emptyset$$

- A neuron can transfer a signal to other neurons only if it received a signal from one or more neurons or from the environment.
- A neuron that received a signal has to transfer a signal to other neurons or to the environment.

Neuron types with respect to connections

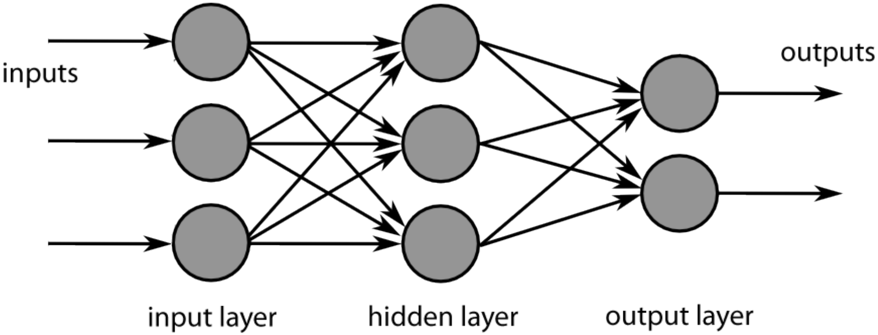


Figure: Feed forward neural network organized in layers

Neuron activity

- Time: $\mathcal{T} \subset \mathcal{R}$
- $\mathcal{T}_t^- = \mathcal{T} \cap (-\infty, t\rangle$
- We can define the *activity of a neuron* v:

$$z_v : \mathcal{T} \to \mathcal{R}$$

- The activity can have range restrictions:
    - $z_v : \mathcal{T} \to \langle 0, 1\rangle$ - normalized activity
    - $z_v : \mathcal{T} \to \langle -1, 1\rangle$
- Network state: $z(t) = (z_v(t))_{v \in \mathcal{V}}$

Global active dynamics

- At each time t the network performs a mapping F_t of input neuron activities to output neuron activities.
- We define the set of all feasible mappings $\mathcal{F}_t$
- The system $(F_t)_{t\in\mathcal{T}}$ is called *active dynamics of the network*
- Requirements:
  - The same domain for all elements
  - A finite number of parameters

$$(\exists k \in \mathcal{N})(\forall t \in \mathcal{T})(\exists D_t \subset \{\mathcal{T}_t^- \to \mathcal{R}^{|\mathcal{I}|}\})(\exists \pi_t : \mathcal{R}^k \to \{D_t \to \mathcal{R}^{|\mathcal{O}|}\})$$
$$\mathcal{F}_t = \pi_t(\mathcal{R}^k)$$

  - Restrictions on possible parameter values

$$(\exists k \in \mathcal{N})(\forall t \in \mathcal{T})(\exists W_t \subset \mathcal{R}^k)(\exists D_t \subset \{\mathcal{T}_t^- \to \mathcal{R}^{|\mathcal{I}|}\})$$
$$(\exists \pi_t : \mathcal{W}_t \to \{D_t \to \mathcal{R}^{|\mathcal{O}|}\})\mathcal{F}_t = \pi_t(W_t)$$

Local active dynamics

- System of functions $(\psi_t^v)_{t \in \mathcal{T}, v \in \mathcal{V} \setminus \mathcal{I}}$ with the following properties:
  1. For each $t \in \mathcal{T}$, each F_t can be expressed as a composition of mappings ψ_t^v that transform the activities of the input neurons $i(v), v \in \mathcal{V} \setminus \mathcal{I}$ into the activity of the neuron v at the time t.
  2. For each time t and each $v \in \mathcal{V} \setminus \mathcal{I}$, the function ψ_t^v is taken from a set Ψ_t^v of possible functions.
  3. For each time t and each $v \in \mathcal{V} \setminus \mathcal{I}$, all elements of Ψ_t^v have the same domain.

$$(\forall v \in \mathcal{V} \setminus \mathcal{I})(\exists k_v \in \mathcal{N})(\forall t \in \mathcal{T})(\exists W_t^v \subset \mathcal{R}^{k_v})$$
$$(\exists D_t^v \subset \{T_t^- \to \mathcal{R}^{|i(v)|}\})(\exists \pi_t^v : W_t^v \to \{D_t^v \to \mathcal{R}\})\Psi_t^v = \pi_t^v(W_t^v)$$

Local active dynamics II.

- We can assign each parameter to a neuron $v \in \mathcal{V} \setminus \mathcal{I}$ or to a connection $(u, v) \in \mathcal{C}$.
- An example neuron parameter: threshold θ_v.
- A usual connection parameter: connection weight $w_{(u,v)}$.
- The activity z_v of a neuron $v \in \mathcal{V} \setminus \mathcal{I}$ is often defined as:

  $$z_v(t) = f(\sum_{u \in i(v)} w_{(u,v)}(t) z_u(t) + \theta(t)),$$

  where f is a function called activation function.
- For output neurons, an identity activation function is often used.

Time-independent global active dynamics

- Time independent version is very common in practical applications
- Global active dynamics:

$$(\exists D \subset \mathcal{R}^{|\mathcal{I}|})\mathcal{F} = \{F : D \to \mathcal{R}^{|\mathcal{O}|}, \}$$

or with a parametrization:

$$(\exists k \in \mathcal{N})(\exists W \subset \mathcal{R}^{k})(\exists D \subset \mathcal{R}^{|\mathcal{I}|})(\exists \pi : W \to \{D \to \mathcal{R}^{|O|}\})$$
$$\mathcal{F} = \pi(W) \quad (1)$$

Time-independent local active dynamics

- Local active dynamics:

$$(\forall v \in \mathcal{V} \setminus \mathcal{I})(\exists k_v \in \mathcal{N})(\exists W_v \subset \mathcal{R}^{k_v})(\exists D_v \subset \mathcal{R}^{|i(v)|})$$
$$(\exists \pi_v : W_v \to \{D_v \to \mathcal{R}\})\Psi_v = \pi_v(W_v)$$

- Time-independent neuron activity:

$$z_v = f(\sum_{u \in i(v)} w_{(u,v)} z_u + \theta_v)$$

Adaptive dynamics

- We have shown that the neuron activity can be time-dependent.
- Global and local dynamics can be time-dependent as well:

$$(F_\tau)_{\tau\in\mathcal{T}} \text{ , or } (\psi_\tau^v)_{\tau\in\mathcal{T}}$$

- They depend on the following factors:
  - Previous evolution of F_t, $(F_\tau)_{\tau\in\mathcal{T}_t^-\setminus\{t\}}$
  - Previous evolution and current value of neuron activities $(z_v|\mathcal{T}_t^-)_{v\in\mathcal{V}}$
  - Information from a supervisor:
    - correct (required) value that the network should output,
    - a non-negative value expressing dissimilarity of output and correct value (loss function),
    - a non-negative value expressing supervisor's satisfaction.

Loss function

- Mapping $\gamma : \mathcal{R}^{|\mathcal{O}|} \times \mathcal{R}^{|\mathcal{O}|} \to \mathcal{R}_0^+$.
- Function $\gamma(d, a)$ is called *error function* or *loss function*, where d is the correct value and a is output of the network.
- Common loss functions:
  - Sum of least squares: $\gamma(a, d) = \sum_{i=1}^{|\mathcal{O}|} |a_i - d_i|^2$
  - Cross entropy: $\gamma(a, d) = -\sum_{i=1}^{|\mathcal{O}|} \left(d_i \log a_i + (1 - d_i)\log(1 - a_i)\right)$
  - Logistic loss: $\gamma(a, d) = -da + \log(e^a + e^{-a}) = \log\frac{e^a + e^{-a}}{e^{da}}$