

FRACTIONAL GRAPH HOMOMORPHISMS

Diploma Thesis for MFF UK

Petr Lukšan

Abstract

The graph colouring problem is classical to combinatorics. Recently many new concepts have been introduced relaxing this NP-complete problem or generalising it even further. The most inclusive of them is the question of existence of a homomorphism from one graph to another. Existence of a graph homomorphism is also an NP-complete problem and several relaxations, such as the concept of fractional graph homomorphism, try to introduce its polynomial approximations. This paper attempts to show the different concepts in relationship to one another and to give examples which emphasise their differences and similarities. Particular attention is drawn to the relatively new concepts of fractional colouring and fractional graph homomorphism.

Keywords. Colouring, chromatic number, graph homomorphism, fractional graph homomorphism, linear programming.

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Prohlašuji, že jsem diplomovou práci vykonal samostatně pouze s použitím uvedené literatury. Souhlasím se zapůjčováním práce.

V Praze, 15.9.1997

Petr Lukšan

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Chapter 1

Introduction

1.1 Aims

It is more than two years since I left MFF UK, and a little longer since I chose fractional graph homomorphisms as the subject for my diploma thesis. Several things have changed in my life since then. I graduated from the University of Cambridge and began to work as a bond trader in London. One thing, however, did not change and that is the fact that I have been thinking of myself more as a 'matfyzak' than anything else. There are a couple of reasons for this, one of which is that I have always enjoyed maths and liked to twiddle with computers, but perhaps even more importantly, 'matfyz' helped to shape my thinking into the structure in which I hope it still is and will remain for some time, and which I would describe as logical and rational, but with an essential degree of curiosity and willingness to question the existing and attempt to understand the new.

My work on this thesis suffered many forced breaks and I am glad that I reached this stage eventually. My primary aim is naturally to present a paper worthy of being accepted by the examination committee but I hope that there is going to be a little more to it than that. Graph Theory and Combinatorics are parts of mathematics I find immensely interesting and studying them can give one, such as me, enough satisfaction on its own. By writing this paper I hope to share some of this with the reader.

My particular focus is going to be on fractional graph colouring and fractional graph homomorphism, concepts of Graph Theory that are relatively new and that are usually excluded from mainstream publications. But what are they? It is the nature of Graph Theory that solutions of many of its puzzles lead to problems that are NP-complete. However, it is often desirable to approximate some of these problems in order to obtain the solutions faster. The fractional concepts provide such approximations for the more classical graph colouring and graph homomorphism. The original problem specifications are slightly altered into forms compatible with linear programming. Polynomial algorithms for finding the fractional graph colouring and homomorphism are then provided by Linear Programming Theory.

Interesting questions arise. To what extent are these approximations accurate and how do they relate to the original concepts? For what classes of instances do the original and the fractional concepts turn out to be equivalent? Does the existence of one imply the existence (or nonexistence) of the other? Do similar theorems hold for the fractional concepts as well as for the original ones? Are their proofs similar? What about alternative approximations? And many more.

1.2 Structure

The second chapter is devoted to graph colouring. Firstly, the paper introduces traditional concepts such as chromatic and clique numbers and presents their alternative definitions. These then provide the link between the original concepts and some of their relaxations (such as fractional chromatic and clique numbers) leading to problems with only polynomial complexity. Several generalisations of the chromatic number (such as the star chromatic number) are also introduced in this chapter together with connecting theorems and corollaries. The section on fractional chromatic and clique numbers presents alternative definitions of the two concepts which prove useful later in the paper. A subsection on perfect k, m -strings, with some original ideas in derivations of their chromatic and fractional chromatic numbers, is included among examples and it produces interesting limit results for fractional chromatic and clique numbers. The chapter then proceeds with generalisations of some of the concepts for weighted graphs. Finally it concludes with a section on products of graphs showing some relationships between the characteristics of graphs and their various products. Some of the proofs in that part are original to this paper.

Chapter three is devoted to homomorphisms. It introduces graph homomorphism together with a few elementary results and shows how it generalises the concept of colouring. A large part of the chapter is concerned with fractional graph homomorphism and related topics. After some basic results are proved for fractional graph homomorphism and a few related examples are presented the paper proceeds with a section on duality centred around a theorem of Bačík and Mahajan. This part also includes some results original to this paper (such as the Weak Density Theorem for \rightarrow_f). Then there follows an alternative polynomial relaxation of the graph homomorphism problem - the pseudo graph homomorphism - and a theorem which shows that this concept is both a relaxation of the original graph homomorphism and a generalisation of the fractional graph homomorphism. The final segment presents interesting results of Mycielski and Larsen, Propp and Ullman regarding Mycielski's Graphs and demonstrates the construction of fractional chromatic and clique numbers on two well-known examples - Petersen's and Grötzsch's graphs.

1.3 Notation Used

Unless otherwise indicated, graphs in this paper will be simple, loopless and finite. For a given graph G denote $V(G)$ its vertex set and $E(G)$ its edge set. For given $s, t \in V(G)$ we denote $s \sim t$ or $s \sim_G t$ whenever $(s, t) \in E(G)$ and $s \not\sim t$ or $s \not\sim_G t$ otherwise. Unless specified otherwise, let K_n , C_n and P_n denote a complete graph, a cycle and a path on n vertices. The shorter $u \in G$ will sometimes be used instead of $u \in V(G)$. The vector of all ones will be denoted simply by $\mathbf{1}$ in situations where there is no confusion.

Chapter 2

Colouring

2.1 Chromatic Number

Definition 1 (Colouring) Let k be a positive integer. A k -colouring of a graph G is a mapping $c : V(G) \rightarrow \{1, \dots, k\}$ such that, $\forall u, v \in V(G) : u \sim v \Rightarrow c(u) \neq c(v)$. Denote the set of all k -colourings by \mathcal{C}_k .

Definition 2 (Chromatic Number) The chromatic number $\chi(G)$ of a graph G is defined as:

$$\chi(G) = \min\{k : G \text{ has a } k\text{-colouring}\}. \quad (2.1)$$

Definition 3 (Clique Number) The clique number $\omega(G)$ of a graph G is defined as the maximum size of a subset of $V(G)$ such that the subset induces a complete graph (i.e. a graph where each two vertices are adjacent) in G (such a subset is called a clique).

Definition 4 (Independence Number) The independence number $\alpha(G)$ of a graph G is defined as the maximum number of vertices in an independent set of G .

Definition 5 (Independence Ratio) The independence ratio of G , $i(G)$, is defined as $i(G) = \alpha(G)/|V(G)|$.

Theorem 1 (Alternative Specifications of $\chi(G)$ and $\omega(G)$) .

$$\chi(G) = \min\{1^T \cdot y : y^T A \geq 1\}, \quad (2.2)$$

$$\omega(G) = \max\{1^T \cdot x : Ax \leq 1\}, \quad (2.3)$$

where x, y are 0-1 column vectors and A is a matrix with rows indexed by maximal independent sets of G and columns indexed by vertices of G such that $A_{\sigma,i} = 1$ if the vertex i belongs to the maximal independent set σ and otherwise is 0.

Proof. Vector y defines a partitioning of $V(G)$ into maximal independent sets such that each vertex belongs to at least one such set. This is exactly the definition of a colouring, where each independent set represents colouring by a different colour (for vertices which are in more than one independent set we can choose the colour arbitrarily). $1^T \cdot y$ is equivalent to the number of colours in such colouring. Hence the minimization problems (2.1) and (2.2) are equivalent.

The set $K = \{u : x_u = 1\}$ is a clique, as for any independent set I in G there is at most one vertex from K in I . On the other hand, for each clique K in G there is a vector x such that $K = \{u : x_u = 1\}$ and $Ax \leq 1$. The size of each such a clique is then $1^T \cdot x$. Therefore (2.3) defines the size of the maximal clique in G . \square

2.1.1 Examples

$$\omega(C_5) = 2, \chi(C_5) = 3$$

2.2 Star Chromatic Number

Vince [28] introduced a generalization of the chromatic number, the star chromatic number. The colours on adjacent vertices are not only required to be distinct but also, in certain sense, as far apart as possible. Because Vince's proofs of some basic facts about the star chromatic number rely on continuous methods, we introduce an alternative approach of Bondy and Hell [5] allowing purely combinatorial treatment.

Definition 6 (Circular Norm) Let k be a positive integer. For $x \in \{-(k-1), \dots, 0, \dots, k-1\}$ define $|x|_k = \min\{|x|, k-|x|\}$.

Definition 7 (k-chromatic number) Define the k -chromatic number $\chi_k(G)$ of a graph G as

$$\chi_k(G) = \frac{k}{\max_{c \in \mathcal{C}_k} \min_{u \sim_G v} |c(u) - c(v)|_k} \quad (2.4)$$

Definition 8 ((k,d)-colouring) Let k and d be positive integers, $k \geq d$. A (k,d) -colouring of a graph G is a mapping $c : V(G) \rightarrow \{1, \dots, k\}$ such that, $\forall u, v \in V(G) : u \sim v \Rightarrow |c(u) - c(v)|_k \geq d$.

Corollary 1 $(k,1)$ -colouring of a graph G is also a k -colouring of G .

Theorem 2 If G has a (k,d) -colouring and $\frac{k}{d} \leq \frac{k'}{d'}$ for some positive integers k', d' , then G has a (k', d') -colouring.

Proof. Let $c : V(G) \rightarrow \{1, \dots, k\}$ be a (k,d) -colouring of G . Define $c' : V(G) \rightarrow \{1, \dots, k'\}$ by

$$c'(u) = \left\lfloor \frac{d'}{d} c(u) \right\rfloor \quad \text{for all } u \in G. \quad (2.5)$$

Consider $u \sim v$ such that $c(u) > c(v)$. Because c defines a (k, d) -colouring of G , $d \leq c(u) - c(v) \leq k - d$. Hence

$$c'(v) + d' = \left\lfloor \frac{d'}{d}(c(v) + d) \right\rfloor \leq \left\lfloor \frac{d'}{d}c(u) \right\rfloor = c'(u) \leq \quad (2.6)$$

$$\leq \left\lfloor \frac{d'}{d}(c(v) + k - d) \right\rfloor \leq \left\lfloor \frac{d'}{d}c(v) + \frac{kd'}{d} - d' \right\rfloor \leq c'(v) + k' - d' \\ \text{and} \quad d' \leq c'(u) - c'(v) \leq k' - d'. \quad (2.7)$$

Thus c' is a (k', d') -colouring of G . \square

Corollary 2 *If G has a (k, d) -colouring, then it also has a (k', d') -colouring where $k'/d' = k/d$ and the greatest common denominator $\gcd(k', d')$ is 1.*

Lemma 1 *Let G be a graph on n vertices that has a (k, d) -colouring with $\gcd(k, d)=1$ and $k > n$. Then G has a (k', d') -colouring with $k' < k$ and $k'/d' < k/d$.*

Proof. The proof of this lemma is not included here, but it can be found in [5]. In the context of this paper Lemma 1 serves only the purpose of helping to derive the result of Lemma 2. \square

Definition 9 (Star Chromatic Number) *The star chromatic number of a graph G , $\chi^*(G)$, is defined as:*

$$\chi^*(G) = \inf\{k/d : G \text{ has a } (k, d)\text{-colouring}\}. \quad (2.8)$$

Definition 10 (t-circular colouring) *Let t be a positive real number and let C be a circle in the plane of length t . A t -circular colouring of a graph G is a mapping Δ which assigns an open arc of C of length 1 to each vertex of G in such a way that*

$$\text{if } u \sim_G v \text{ then } \Delta(u) \cap \Delta(v) = \emptyset \quad (2.9)$$

Definition 11 (Circular Chromatic Number) *The circular chromatic number, $\chi_c(G)$, of a graph G is defined as*

$$\chi_c(G) = \inf\{t : \text{there is a } t\text{-circular colouring of } G\}. \quad (2.10)$$

Lemma 2 *Let G be a graph on n vertices, then*

$$\chi^*(G) = \min\{k/d : G \text{ has a } (k, d)\text{-colouring and } k \leq n\}. \quad (2.11)$$

Proof. By Corollary 2 and Lemma 1, if G has a (k, d) -colouring then it has a (k', d') -colouring with $k' \leq n$ and $k'/d' \leq k/d$. Therefore

$$\chi^*(G) = \inf\{k/d : G \text{ has a } (k, d)\text{-colouring and } k \leq n\}. \quad (2.12)$$

Because the set $\{k/d : G \text{ has a } (k, d)\text{-colouring and } k \leq n\}$ is finite, the infimum can be replaced by minimum. \square

Theorem 3 (The Original Specification of $\chi^*(G)$ - Vince [28]) *If G is a connected graph with n vertices,*

$$\chi^*(G) = \min_{1 \leq k \leq n} \chi_k(G). \quad (2.13)$$

Proof. Let k and d be positive integers such that a (k,d) -colouring of G (call it c) exists. Then from the definition of (k,d) -colouring it follows that $\min_{u \sim_G v} |c(u) - c(v)|_k \leq d$ and therefore $\chi_k(G) \leq k/d$. From Lemma 1 it follows that $\chi^*(G) \geq \min_{1 \leq k \leq n} \chi_k(G)$.

On the other hand, let $\chi_k(G) = k/d$ for some $d = \max_{c \in \mathcal{C}_k} \min_{u \sim_G v} |c(u) - c(v)|_k$. Let c be the k -colouring for which the maximal value of d is attained. Then c is clearly a (k,d) -colouring of G . Therefore $\chi^*(G) \leq \min_{1 \leq k \leq n} \chi_k(G)$. This concludes the proof of $\chi^*(G) = \min_{1 \leq k \leq n} \chi_k(G)$. \square

Theorem 4 (Another Specification of $\chi^*(G)$ - Zhu [31]) *For any graph G*

$$\chi^*(G) = \chi_c(G) \quad (2.14)$$

Proof. Firstly, let c be a (k,d) -colouring of G , for some k and d . Put k points, p_1, \dots, p_k , evenly spaced on a circle of length k . For all $u \in V(G)$ let $\Delta(u)$ be the interval of length d centred at $p_{c(u)}$. Then Δ is a k -circular colouring of G and therefore $\chi_c(G) \leq \chi^*(G)$. \square

Theorem 5 *Let G be a graph and let $u \in G$ be a vertex such that u is adjacent to all but one vertex in G . Then*

$$\chi^*(G) = \chi(G) \quad (2.15)$$

Proof. Let Δ be a t -circular colouring of G for some t . Then $\Delta(u)$ is disjoint from $\Delta(v)$ for all $v \in G$ but one and therefore one of the endpoints of $\Delta(u)$ is not covered by any of the intervals $\Delta(v)$. Hence, we can construct a t -interval colouring of G by cutting the circle at that end point. \square

Corollary 3 *If G has a universal vertex then $\chi^*(G) = \chi(G)$.*

Corollary 4 *Let $\chi(G) = n$ for some n and let $u \in G$ be a vertex such that its neighbours in G induce a subgraph G_u with $\omega(G_u) = n - 1$. Then $\chi^*(G) = \chi(G)$.*

Theorem 6 *Let G be a graph. Then*

$$\chi(G) = \lceil \chi^*(G) \rceil \quad (2.16)$$

Proof. $\chi^*(G) > \chi(G) - 1$ since by Lemma 1, $\chi^*(G) \leq \chi(G) - 1$ implies that there is a (k,d) -colouring of G for any positive integers k, d such that $\frac{k}{d} \leq \chi - 1$. By Theorem 2.2 there exists also a $(\chi(G) - 1, 1)$ -colouring of G which is a contradiction. Corollary 1 implies $\chi^*(G) \leq \chi(G)$. \square

Theorem 7 *Let G be a graph and let H be its subgraph. Then*

$$\chi^*(H) \leq \chi^*(G) \quad (2.17)$$

Proof. Let c be a (k,d) -colouring of G for some integers k, d . Then $c'(u) = c(u)$ for all $u \in H$ is clearly a (k,d) -colouring of H . Therefore $\chi^*(H) \leq \chi^*(G)$. \square

2.3 Fractional Chromatic and Clique Numbers

Given the alternative specification of the chromatic number (2.2) the fractional chromatic number represents a natural relaxation of the colouring concept. The constraint on a colouring y to be strictly a vector of zeros and ones is replaced by a weaker constraint $y \geq 0$. This enables the minimization problem in (2.18) to be solvable in a polynomial time using linear programming. An equivalent treatment of the clique number leads to a dual problem, and from the theory of linear programming it follows that the two fractional concepts are equivalent as shown later in this section.

Definition 12 (Fractional Chromatic Number) *We define the fractional chromatic number of a given graph G , $\chi_f(G)$ to be equal to:*

$$\chi_f(G) = \inf\{1^T \cdot y : y^T A \geq 1^T, y \geq 0\}, \quad (2.18)$$

where y is a rational column vector and A is the matrix with rows indexed by maximal independent sets of G and columns indexed by vertices of G such that $A_{\sigma,i} = 1$ if the vertex i belongs to the maximal independent set σ and otherwise is 0.

Definition 13 (Fractional Clique Number) *We define the fractional clique number of a given graph G , $\omega_f(G)$ to be equal to:*

$$\omega_f(G) = \sup\{1^T \cdot x : Ax \leq 1, x \geq 0\}, \quad (2.19)$$

where x is a rational column vector and A is as above.

Call any vector y satisfying the feasibility conditions in (2.18) a fractional colouring of G and similarly call any vector x satisfying the feasibility conditions in (2.19) a fractional clique of G and call $1^T \cdot x$ its size. Also let us call an x for which the optimum is attained a maximum fractional clique of G .

Theorem 8 (Duality Theorem of Linear Programming) *Let A be a matrix and let b and c be vectors. Then*

$$\max\{c^T \cdot x : Ax \leq b, x \geq 0\} = \min\{b^T \cdot y : y^T A \geq c^T, y \geq 0\} \quad (2.20)$$

provided that both sets in (2.20) are nonempty.

Proof. This theorem is a well-known result of the theory of linear programming. Its proof can be found for example in [29]. \square

Theorem 9 (Complementary Slackness) *Consider the linear programming duality*

$$\max\{c^T \cdot x : Ax \leq b, x \geq 0\} = \min\{b^T \cdot y : y^T A \geq c^T, y \geq 0\}. \quad (2.21)$$

Assume that both optima are finite and let x_0 and y_0 be feasible solutions. Then the following are equivalent

$$(i) \quad x_0 \text{ and } y_0 \text{ are optimum solutions in (2.21)} \quad (2.22)$$

$$(ii) \quad c^T x_0 = y_0^T b \quad (2.23)$$

$$(iii) \quad (y_0^T A - c^T)x_0 = 0 \quad \text{and} \quad y_0^T(b - Ax_0) = 0 \quad (2.24)$$

Proof. The equivalence of (i) and (ii) follows directly from Theorem 8. The equivalence of (ii) and (iii) follows from

$$\{(y_0^T A - c^T)x_0 = 0 \quad \text{and} \quad y_0^T(b - Ax_0) = 0\} \Leftrightarrow \{y_0^T Ax_0 = c^T x_0 \quad \text{and} \quad y_0^T b = y_0^T Ax_0\} \quad (2.25)$$

and the fact that $c^T x_0 \leq y_0^T Ax_0 \leq y_0^T b$. \square

Theorem 10 *inf and sup in definitions of the fractional chromatic and clique numbers are attained, they are equal to rational numbers and $\chi_f(G) = \omega_f(G)$ for all graphs G .*

Proof. $x = 0$ and $y = 1$ are feasible solutions of (2.19) and (2.18). Therefore by Theorem 8 the inf and sup are attained and equal. Since A is an integer matrix, the optimum is a rational number. \square

Theorem 11 *Let G be a graph and let H be its subgraph. Then*

$$\chi_f(H) = \omega_f(H) \leq \omega_f(G) = \chi_f(G). \quad (2.26)$$

Proof. Let x^* be a fractional clique in H . Then we can construct the following fractional clique x in G

$$x_u = x_u^* \quad \forall u \in V(H) \quad \text{and} \quad x_v = 0 \quad \forall v \in V(G) \setminus V(H). \quad (2.27)$$

From the construction it follows that $\sum_{v \in G} x_v = \sum_{u \in H} x_u^*$ and therefore $\omega_f(H) \leq \omega_f(G)$. \square

The fractional chromatic number is also referred to as the set chromatic number or the multicolouring number or the ultimate chromatic number or the fuzzy chromatic number and it is useful to introduce its equivalent characterisations.

Definition 14 (a/b-colouring) *Let a and b be positive integers. An a/b -colouring of a graph G is a mapping from $V(G)$ to b -element subsets of $\{1, \dots, a\}$ such that all adjacent vertices are assigned disjoint subsets.*

Definition 15 (Kneser graph) Let a and b be positive integers. The Kneser graph $K(a, b)$ is the graph whose vertices are the b -element subsets of $\{1, \dots, a\}$. There is an edge between such two vertices iff they are disjoint sets.

Although graph homomorphism has not yet been defined (see Chapter 3), we include the definition of Kneser colouring and related theorems here because they naturally belong in this section.

Definition 16 (Kneser colouring) Let a and b be positive integers. A Kneser a/b -colouring of a graph G is a homomorphism from G to the Kneser graph $K(a, b)$.

Theorem 12 A graph G has an a/b -colouring iff it has a Kneser a/b -colouring.

Proof. A mapping from $V(G)$ to b -subsets of $\{1, \dots, a\}$ representing the a/b -colouring defines a homomorphism from G to the Kneser graph $K(a, b)$. On the other hand, a homomorphism $f : G \rightarrow K(a, b)$ defines a mapping from G to b -subsets of $\{1, \dots, a\}$ satisfying the properties of an a/b -colouring. \square

Theorem 13

$$\chi_f(G) = \inf\{a/b : G \text{ has an } a/b\text{-colouring}\} = \sup\{|H|/\alpha(H) : H \rightarrow G\}. \quad (2.28)$$

Proof. Firstly, we will show that $\chi_f(G) \leq \inf\{a/b : G \text{ has an } a/b\text{-colouring}\}$. Let G have an a/b -colouring for some a and b , let $K(a, b)$ be the Kneser graph, and let $f : G \rightarrow K(a, b)$ be a homomorphism. Construct a fractional colouring, y , of G as follows: $y_{A_j} = \frac{k_{A_j}}{b}$, where A_j is the j -th independent set and k_{A_j} is the number of indices $i \in \{1, \dots, a\}$ such that $A_j = \{u \in V(G) : i \in f(u)\}$. This shows that $\chi_f(G) \leq \inf\{a/b : G \text{ has an } a/b\text{-colouring}\}$.

Secondly, we will show that $\chi_f(G) \geq \sup\{|H|/\alpha(H) : H \rightarrow G\}$. Let $f : V(H) \rightarrow V(G)$ be a homomorphism. Define a fractional clique x in G such that $\forall u \in G : x_u = |\{s \in H : f(s) = u\}|/\alpha(H)$. It follows that

$$\sum_{u \in G} x_u = \sum_{u \in G} \frac{|\{s \in H : f(s) = u\}|}{\alpha(H)} = \frac{|H|}{\alpha(H)}. \quad (2.29)$$

Let I be an independent set in G . Then the set $\{s \in H : f(s) \in I\}$ must also be independent because $s \sim_H t \Rightarrow u \sim_G v$. Therefore for any independent set I in G

$$\sum_{u \in I} x_u = \frac{|\{s \in H : f(s) \in I\}|}{\alpha(H)} \leq \frac{\alpha(H)}{\alpha(H)} = 1. \quad (2.30)$$

Hence x is a fractional clique in G of size $|H|/\alpha(H)$. This concludes the proof that $\omega_f(G) = \chi_f(G) \geq \sup\{|H|/\alpha(H) : H \rightarrow G\}$. It remains to show that

$$\inf\{a/b : G \text{ has an } a/b\text{-colouring}\} \leq \sup\{|H|/\alpha(H) : H \rightarrow G\}. \quad (2.31)$$

The proof of this last inequality is not included here. See for example [14] for further references. \square

2.3.1 Examples

Perfect k,m-strings

Definition 17 (Almost Complete Graphs) *An almost complete graph on n vertices is K_n with one of its edges removed.*

Definition 18 (Perfect k,m-string) *Let $k \geq 2$ and m be positive integers. Define a perfect k,m -string as a graph $S_{k,m}$ such that*

- (i) $V(S_{k,m}) = \bigcup_{i=1}^m V(H_i)$, where H_i 's are almost complete graphs on k vertices;
- (ii) Let x_i, y_i be such that $x_i \not\sim_{H_i} y_i$. Then $x_{i+1} = y_i$ for $i = 1, \dots, m-1$ and no other vertices belong to more than one H_i ;
- (iii) $E(S_{k,m}) = \{(u, v) : \exists i; u \sim_{H_i} v\} \cup (x_1, y_m)$

Theorem 14 *Let k, m be positive integers. If $k \geq 3$ and $m \geq 2$ then*

$$\omega(S_{k,m}) = k - 1, \quad \chi_f(S_{k,m}) = k - 1 + \frac{1}{m} \quad \text{and} \quad \chi(S_{k,m}) = k \quad (2.32)$$

Proof. Let $S_{k,m}$ be a perfect k,m -string where $k \geq 3$ and $m \geq 2$ and let H_i, x_i and $y_i, i = 1, \dots, m$ be as in the definition above. Define $\bar{H}_i = H_i \setminus \{x_i, y_i\}$ and $\tilde{H} = S_{k,m} \setminus (\bigcup_{i=1}^m \bar{H}_i) = \{x_1, \dots, x_m, y_m\}$. Note that each \bar{H}_i is a clique. Clearly $2 \leq k-1 \leq \omega(H_1) \leq \omega(S_{k,m})$. Construct the following k -colouring of $S_{k,m}$. Use one colour for $\{x_1, \dots, x_m\}$, second colour for y_m and the same set of another $(k-2)$ colours for each one of $\bar{H}_i, i = 1, \dots, m$. This is a k -colouring of $S_{k,m}$ and hence $\chi(S_{k,m}) \leq k$. Since $\omega(S_{k,m}) \leq \omega_f(S_{k,m}) = \chi_f(S_{k,m}) \leq \chi(S_{k,m})$ and because the fractional and the clique numbers are integers it will be enough to show that $\chi_f(S_{k,m}) = k - 1 + \frac{1}{m}$ to conclude the proof.

Firstly, we will argue that each independent set in $S_{k,m}$ has at most m vertices. Clearly, for any independent set I at most one of x_1 and y_m belongs to I .

Consider an independent set I such that $y_m \notin I$ and let z_{i_1}, \dots, z_{i_p} be the p vertices from I that belong to \tilde{H} ($z_{i_l} = x_{i_l}$ or $z_{i_l} = y_{i_l}$ for each l). Then there are at most $m - p$ vertices from $\bigcup_{i=1}^m \bar{H}_i$ in I since all vertices from \bar{H}_j are adjacent to x_j for each $j = i_1, \dots, i_p$ and \bar{H}_j 's are complete graphs on $k-2$ vertices. Therefore there are at most $(m-p) + p = m$ in I .

Similarly, consider an independent set I such that $x_1 \notin I$. The same arguments apply, with the difference that all vertices from \bar{H}_j are now adjacent to y_j for certain j 's.

Secondly, construct a fractional clique w in $S_{k,m}$ such that

$$w_u = \frac{1}{m} \forall u \in S_{k,m} \quad (2.33)$$

Therefore $\sum_{u \in S_{k,m}} w_u = \frac{|V(S_{k,m})|}{m} = \frac{m(k-1)+1}{m} = (k-1) + \frac{1}{m}$ and $(k-1) + \frac{1}{m} \leq \omega_f(S_{k,m})$. It remains to show that $\omega_f(S_{k,m}) \leq (k-1) + \frac{1}{m}$. We will show that by constructing a $(m(k-1)+1)/m$ -colouring of $S_{k,m}$. Consider the mapping c from $V(S_{k,m})$ to m -element subsets of $\{0, \dots, m(k-1)\}$ which satisfies

$$c(x_1) = \{m(k-1) - m + 1, \dots, m(k-1)\} \quad (2.34)$$

$$c(x_2) = \{m(k-1) - m + 2, \dots, m(k-1)\} \cup \{0\} \quad (2.35)$$

$$c(x_3) = \{m(k-1) - m + 3, \dots, m(k-1)\} \cup \{0, 1\} \quad (2.36)$$

...

$$c(x_m) = \{m(k-1)\} \cup \{0, 1, \dots, m-2\} \quad (2.37)$$

$$c(y_m) = \{0, 1, \dots, m-1\}, \quad (2.38)$$

and where each \bar{H}_i is coloured by $(m(k-1)+1) - m - 1 = m(k-2)$ colours $\{0, \dots, m(k-1)\} \setminus (c(x_i) \cup c(y_i))$. Note that c was constructed in such a way that $c(x_1) \cap c(y_m) = \emptyset$. Since $|V(\bar{H}_i)| = (k-2)$ for each $i = 1, \dots, m$, c is a proper $(m(k-1)+1)/m$ -colouring of $S_{k,m}$ and therefore $\omega_f(S_{k,m}) \leq (k-1) + \frac{1}{m}$. By showing that $\omega_f(S_{k,m}) = (k-1) + \frac{1}{m}$ we have concluded the proof of

$$\omega(S_{k,m}) = k-1, \quad \chi_f(S_{k,m}) = k-1 + \frac{1}{m} \quad \text{and} \quad \chi(S_{k,m}) = k \quad (2.39)$$

□

The above theorem produces an interesting corollary. There is a sequence of graphs for which the fractional chromatic number converges to the clique number while the chromatic number does not.

Corollary 5

$$\lim_{m \rightarrow \infty} \frac{\omega_f(S_{k,m})}{\omega(S_{k,m})} = 1, \quad \forall k \quad (2.40)$$

and

$$\lim_{k \rightarrow \infty} \frac{\chi(S_{k,m})}{\chi_f(S_{k,m})} = 1, \quad \forall m \quad (2.41)$$

Moreover, also

$$\lim_{m \rightarrow \infty} \omega_f(S_{k,m}) - \omega(S_{k,m}) = 0, \quad \forall k \quad (2.42)$$

2.4 Weighted Graphs

Some of the concepts introduced in this paper can be extended naturally to graphs with defined assignment of nonnegative real weights to its vertices.

Definition 19 (Weighted Graph) *A weighted graph is a pair (G, w) where G is a graph and $w : V(G) \rightarrow [0, \infty)$ is a weight function.*

Definition 20 (Clique Number of (G, w)) For a weighted graph (G, w) its clique number $\omega(G, w)$ is defined as the maximum weight of a clique in G (weight of a clique is the sum of weights of vertices of that clique).

Definition 21 (t-interval colouring of (G, w)) Let I be an interval of length t . A t -interval colouring of a weighted graph (G, w) is a mapping Δ which assigns to each vertex of G an open sub-interval in I such that

$$(i) \quad \text{if } u \sim_G v \text{ then } \Delta(u) \cap \Delta(v) = \emptyset \quad (2.43)$$

$$(ii) \quad \forall u \in V(G) : \text{the length of } \Delta(u) \text{ is } w(u) \quad (2.44)$$

Definition 22 (Interval Chromatic Number of (G, w)) The interval chromatic number $\chi(G, w)$ of a weighted graph (G, w) is defined as

$$\chi(G, w) = \min\{t : \text{there is a } t\text{-interval colouring of } (G, w)\}. \quad (2.45)$$

Deuber and Zhu introduced a concept of circular colouring to weighted graphs.

Definition 23 (t-circular colouring of (G, w)) Let t be a positive real number and let C be a circle in the plane of length t . A t -circular colouring of a weighted graph (G, w) is a mapping Δ which assigns an open arc of C to each vertex of G in such a way that

$$(i) \quad \text{if } u \sim_G v \text{ then } \Delta(u) \cap \Delta(v) = \emptyset \quad (2.46)$$

$$(ii) \quad \forall u \in V(G) : \text{the length of arc } \Delta(u) \text{ is at least } w(u) \quad (2.47)$$

Definition 24 (Circular Chromatic Number of (G, w)) The circular chromatic number $\chi_c(G, w)$ of a weighted graph (G, w) is defined as

$$\chi_c(G, w) = \inf\{t : \text{there is a } t\text{-circular colouring of } (G, w)\}. \quad (2.48)$$

Definition 25 (Fractional Chromatic Number of (G, w)) The fractional chromatic number $\chi_f(G, w)$ of a weighted graph (G, w) is defined as

$$\chi_f(G, w) = \inf\{1^T \cdot y \mid \forall u \in V(G) : (y^T A)_u \geq w(u), y \geq 0\}, \quad (2.49)$$

where y is a column vector and A is a matrix with rows indexed by maximal independent sets of G and columns indexed by vertices of G such that $A_{\sigma, i} = 1$ if the vertex i belongs to the maximal independent set σ and otherwise is 0.

A vector y satisfying the feasibility conditions in (2.49) is called a fractional colouring of (G, w) . Again, a fractional clique is a vector x satisfying the feasibility conditions in a problem of linear programming dual to (2.49) and the fractional clique number is equal to the fractional chromatic number as for ordinary graphs.

Theorem 15 (Alternative Specification of $\chi_f(G, w)$) *Let (G, w) be a weighted graph. A fractional colouring of size t of (G, w) , where t is a positive number, exists iff there exists a mapping Δ of $V(G)$ to measurable subsets of I , where I is an interval of length t , such that $\Delta(u)$ has measure $w(u)$ for all $u \in G$ and adjacent vertices are mapped to disjoint subsets of I .*

Proof. A detailed proof is not included here. Some details are discussed in [32]. \square

Theorem 16

$$\omega(G, w) \leq \omega_f(G, w) = \chi_f(G, w) \leq \chi_c(G, w) \leq \chi(G, w) \quad (2.50)$$

Proof. A t -interval colouring of a weighted graph (G, w) can be interpreted as a t -circular colouring where the end points of an interval are joined together. Hence $\chi_c(G, w) \leq \chi(G, w)$.

Let Δ be a t -circular colouring of (G, w) on a circle of length t and let x be a point on that circle. Construct a mapping Δ' by cutting the circle at x and stretching it on an interval I of length t . Δ' clearly maps vertices of G to measurable subsets of I and by Theorem 15 there is a fractional colouring of (G, w) of size t . Therefore $\chi_f(G, w) \leq \chi_c(G, w)$.

The equality follows again from the duality of linear programming. Finally, $\omega(G, w) \leq \omega_f(G, w)$ follows from the fact that $\omega(G, w)$ can be expressed through an equivalent maximization problem as $\omega_f(G, w)$ only with an additional constraint (the control variable being a 0-1 vector). \square

Corollary 6

$$\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi_c(G) \leq \chi(G) \quad (2.51)$$

2.5 Products of Graphs

Theorem 17 (Characteristics of Disjoint Union of Graphs) *Let G and H be graphs and let F be a disjoint union of G and H . Then*

$$\omega(F) = \max(\omega(G), \omega(H)) \quad (2.52)$$

$$\chi^*(F) = \max(\chi^*(G), \chi^*(H)) \quad (2.53)$$

$$\chi_f(F) = \max(\chi_f(G), \chi_f(H)) \quad (2.54)$$

$$\chi(F) = \max(\chi(G), \chi(H)) \quad (2.55)$$

Proof. The first and fourth equalities are trivial. The second and third equalities clearly hold with \geq since both G and H are subgraphs of F .

To prove the other side of the second equality, consider circular colourings Δ^G, Δ^H of G and H with the lengths of the circles t^G and t^H and assume $t^G \geq t^H$. Then construct a mapping Δ from $V(F)$ to unit arcs of a circle with length $t = t^G$ such

that $\Delta(u) = \Delta^G(u) \quad \forall u \in G$, and the vertices of H are mapped onto unit arcs with centres at the same angles as $\Delta(v)$'s for $v \in H$.

To prove the other side of the third equality, let y^G, y^H be maximal fractional colourings in G and H and let $\chi_f(G) = \max\{\chi_f(G), \chi_f(H)\}$. Each maximal independent set in F is of the form $I_G \cup I_H$ where I_G, I_H are maximal independent sets in G and H . Construct the following fractional colouring y of F

$$y_{I_G, I_H} = y_{I_G}^G \cdot \frac{y_{I_H}^H}{\chi_f(H)}. \quad (2.56)$$

y is a proper fractional colouring since $\forall u \in G$:

$$\sum_{I \ni u} y_I = \sum_{I_G \ni u} y_{I_G}^G \sum_{I_H} \frac{y_{I_H}^H}{\chi_f(H)} = \sum_{I_G \ni u} y_{I_G}^G \geq 1 \quad (2.57)$$

and $\forall v \in H$:

$$\sum_{I \ni v} y_I = \sum_{I_G} y_{I_G}^G \sum_{I_H \ni v} \frac{y_{I_H}^H}{\chi_f(H)} = \chi_f(G) \sum_{I_H \ni v} \frac{y_{I_H}^H}{\chi_f(H)} \geq \frac{\chi_f(G)}{\chi_f(H)} \geq 1. \quad (2.58)$$

The size of y is equal to

$$\sum_{I \ni u} y_I = \sum_{I_G} y_{I_G}^G \sum_{I_H} \frac{y_{I_H}^H}{\chi_f(H)} = \sum_{I_G} y_{I_G}^G = \chi_f(G). \quad (2.59)$$

Therefore $\chi_f(f) \leq \max\{\chi_f(G), \chi_f(H)\}$. \square

Definition 26 (Sum of Graphs) Let G and H be graphs. Define the sum of G and H as follows

$$V(G + H) = \{s : s \in V(G)\} \cup \{u : u \in V(H)\} \quad \text{and} \quad (2.60)$$

$$E(G + H) = \{st : s \sim_G t\} \cup \{uv : u \sim_H v\} \cup \{su : s \in G, u \in H\} \quad (2.61)$$

Theorem 18 (Characteristics of the Sum of Graphs) Then

$$\omega(G + H) = \omega(G) + \omega(H) \quad (2.62)$$

$$\chi_f(G + H) = \chi_f(G) + \chi_f(H) \quad (2.63)$$

$$\chi^*(G + H) = \chi(G) + \chi(H) \quad (2.64)$$

$$\chi(G + H) = \chi(G) + \chi(H) \quad (2.65)$$

Proof. $G + H$ contains all edges of G and H plus all edges between vertices from G and H . Clearly, any two cliques in G and H form together a clique in $G + H$. Similarly, any clique in $G + H$ consists of two cliques from G and H and edges between them. Hence the first equality is trivial.

Each independent set in $G + H$ contains only vertices from G or H . Thus for x^G, x^H fractional cliques in G and H

$$x_u = x_u^G \quad \forall u \in G \text{ and } x_v = x_v^H \quad \forall v \in H \quad (2.66)$$

is a fractional clique in $G + H$ and for y^G, y^H fractional colourings of G and H

$$x_I = x_I^G \quad \forall I \subset G \text{ and } x_I = x_I^H \quad \forall I \subset H, \quad (2.67)$$

where I denotes a maximal independent set, is a fractional colouring of $G + H$. The second equality therefore holds as well.

G and H have to be coloured by two distinct sets of colours since each vertex from G is adjoint to each vertex from H . The last equality then follows.

Let Δ be a circular colouring of $G + H$ of size t . Because each vertex s from G is adjoint to each vertex u from H the arcs $\Delta(s)$ and $\Delta(u)$ must also be disjoint for any such s and u . We can therefore separate and cut out the parts of the circle of length t which are covered by arcs $\Delta(s), s \in G$. Because we were allowed to cut the circle into such pieces we can now glue them together in such an order as to produce a t -interval colouring of $G + H$ and moreover in such a way that all the pieces $\Delta(s), s \in G$ remain on the left of the pieces $\Delta(u), u \in H$. Thus we showed that $\chi^*(G + H) \geq \chi(G) + \chi(H)$. The other side follows from equality four. \square

Definition 27 (Graph Products) For graphs G and H we define the following graph products. For all of them the vertex set of a product is $V(G \text{ prod } H) = \{su : s \in G, u \in H\}$. The corresponding edge sets are as follows

$$\begin{aligned} \text{Wreath product: } E(G[H]) = \{ & su, tv : \text{ either } s \sim_G t \text{ or} \\ & s = t \text{ and } u \sim_H v \} \end{aligned} \quad (2.68)$$

$$\text{Categorical product: } E(G \times H) = \{su, tv : s \sim_G t \text{ and } u \sim_H v\} \quad (2.69)$$

$$\begin{aligned} \text{Cartesian product: } E(G \square H) = \{ & su, tv : \text{ either } s = t \text{ and } u \sim_H v \\ & \text{or } s \sim_G t \text{ and } u = v \} \end{aligned} \quad (2.70)$$

Theorem 19 (Characteristics of the Wreath Product) Let G and H be graphs. Then

$$\omega(G[H]) = \omega(G)\omega(H) \quad (2.71)$$

$$\chi_f(G[H]) \leq \chi_f(G)\chi_f(H) \quad (2.72)$$

$$\chi^*(G[H]) \leq \chi^*(G)\chi(H) \quad (2.73)$$

$$\chi(G[H]) \leq \chi(G)\chi(H) \quad (2.74)$$

Proof. Let $K_m \subset G$ and $K_n \subset H$ be cliques in G and H such that $m = \omega(G)$ and $n = \omega(H)$. Clearly $\omega(G[H]) \geq \omega(G)\omega(H)$ since $K_{mn} = \{su : s \in K_m, u \in K_n\}$ is a clique in $G[H]$. On the other hand all maximal independent sets in $G[H]$ are of the form $\bigcup_{i=1}^p \{s_i u : s_i \in K_p, u \in K_{q_i}\}$ where $p \leq m$ and $q_i \leq n$ for all $i = 1, \dots, p$.

Therefore also $\omega(G[H]) \leq \omega(G)\omega(H)$ and the first equality holds. Let y^G and y^H be fractional colourings of G and H . Let I_G and I_H be independent sets in G and H . Then clearly $I_{I_G, I_H} = \{su : s \in I_G, u \in I_H\}$ is an independent set in $G[H]$. Construct a fractional colouring y of $G[H]$ as follows

$$y_{I_{I_G, I_H}} = y_{I_G}^G \cdot y_{I_H}^H \quad (2.75)$$

and zero otherwise. Then for each $su \in G[H]$

$$\sum_{I \ni su} y_I = \sum_{I_G \ni s} y_{I_G}^G \sum_{I_H \ni u} y_{I_H}^H \geq \sum_{I_G \ni s} y_{I_G}^G \geq 1 \quad (2.76)$$

hence y is a proper fractional colouring. Clearly, the size of y is equal to

$$\sum_I y_I = \sum_{I_G} y_{I_G}^G \sum_{I_H} y_{I_H}^H = \chi_f(H) \sum_{I_G} y_{I_G}^G = \chi_f(H) \chi_f(G). \quad (2.77)$$

Therefore $\chi_f(G[H]) \leq \chi_f(G) \chi_f(H)$ and the second inequality holds.

Suppose $\chi(H) = n$ and $\chi^*(G) = k/d$ for some positive integers n, k, d . Let c' be an n -colouring of H and c^* be a (k, d) -colouring of G . We will show that the following mapping $c : V(G[H]) \rightarrow \{1, \dots, n\}$ is a (kn, d) -colouring of $G[H]$

$$c(gh) = c^*(g) + c'(h) \cdot k. \quad (2.78)$$

For $g_1, g_2 \in G$, $g_1 \neq g_2$

$$|c(g_1 h_1) - c(g_2 h_2)|_{kn} = |c^*(g_1) - c^*(g_2) + pk|_{kn} \geq \quad (2.79)$$

$$\geq |c^*(g_1) - c^*(g_2) + pk|_k = |c^*(g_1) - c^*(g_2)|_k \geq d. \quad (2.80)$$

On the other hand For $g_1 = g_2 = g \in G$ and $h_1, h_2 \in H$, $h_1 \neq h_2$

$$|c(gh_1) - c(gh_2)|_{kn} = |pk|_{kn} \geq k \geq d. \quad (2.81)$$

Therefore $\chi^*(G[H]) \leq n \frac{k}{d} = \chi^*(G) \chi(H)$.

Similar arguments can be used to prove the last inequality. Take c^* to be a $(k, 1)$ -colouring where $k = \chi(G)$ and the same proof gives the result. \square

Theorem 20 (Characteristics of the Categorical Product) *Let G and H be graphs. Then*

$$\omega(G \times H) = \min(\omega(G), \omega(H)) \quad (2.82)$$

$$\chi_f(G \times H) \leq \min(\chi_f(G), \chi_f(H)) \quad (2.83)$$

$$\chi^*(G \times H) \leq \min(\chi^*(G), \chi^*(H)) \quad (2.84)$$

$$\chi(G \times H) \leq \min(\chi(G), \chi(H)) \quad (2.85)$$

Proof. Let $\omega(G) = m \leq n = \omega(H)$ for some integers m, n . Take a clique of size m in each G and H . Then a subset of $G \times H$ consisting of pairs of vertices representing a one-to-one correspondence between these two sets is a clique in $G \times H$. On the other hand each clique in $G \times H$ must represent a one to one correspondence between vertices in two cliques from G and H . Therefore it follows that $\omega(G \times H) = \min(\omega(G), \omega(H))$ and the first equality holds. Suppose $\chi_f(G) = p \leq q = \chi_f(H)$ and let y^G be a fractional colouring of G of size p . Let I_G be a maximal independent set in G . Then the set

$$I_{I_G, H} = \{su : s \in I_G, u \in H\} \quad (2.86)$$

is independent in $G \times H$. Construct the following fractional colouring y of $G \times H$

$$y_{I_{I_G, H}} = y_{I_G}^G \quad (2.87)$$

for all I_G and zero otherwise. For each $su \in G \times H$

$$\sum_{I \ni su} y_I = \sum_{I_G \ni s} y_{I_{I_G, H}} = \sum_{I_G \ni s} y_{I_G}^G \geq 1 \quad (2.88)$$

hence y is a proper fractional colouring. Clearly, the size of y is equal to the size of y^G and therefore $\chi_f(G \times H) \leq \min(\chi_f(G), \chi_f(H))$ and the second inequality holds. Let c' be a (k, d) -colouring of G . Then $c(gh) = c'(g)$ is clearly a (k, d) -colouring of $G \times H$ since $g_1 h_1 \sim_{G \times H} g_2 h_2 \Rightarrow g_1 \sim_G g_2$. Therefore the third inequality holds. Similarly, let c' be a k -colouring of G . Then $c(gh) = c'(g)$ is k -colouring of $G \times H$ since $g_1 h_1 \sim_{G \times H} g_2 h_2 \Rightarrow g_1 \sim_G g_2$ and the last inequality also holds. \square

Theorem 21 (Characteristics of the Cartesian Product) *Let G and H be graphs. Then*

$$\omega(G \square H) = \max(\omega(G), \omega(H)) \quad (2.89)$$

$$\chi_f(G \square H) = \max(\chi_f(G), \chi_f(H)) \quad (2.90)$$

$$\chi^*(G \square H) = \max(\chi^*(G), \chi^*(H)) \quad (2.91)$$

$$\chi(G \square H) = \max(\chi(G), \chi(H)) \quad (2.92)$$

Proof. G and H are subgraphs of $G \square H$. Therefore

$$\omega(G \square H) \geq \max(\omega(G), \omega(H)) \quad (2.93)$$

$$\chi_f(G \square H) \geq \max(\chi_f(G), \chi_f(H)) \quad (2.94)$$

$$\chi^*(G \square H) \geq \max(\chi^*(G), \chi^*(H)) \quad (2.95)$$

$$\chi(G \square H) \geq \max(\chi(G), \chi(H)). \quad (2.96)$$

Each clique in $G \square H$ must form a bunch $\{su : u \in K_n\}$, $n \leq \omega(H)$ or $\{su : s \in K_m\}$, $m \leq \omega(G)$. A size of a clique in $G \square H$ is therefore at most $\max(\omega(G), \omega(H))$ and the first equality holds.

Let G have an \bar{a}_G/\bar{b}_G -colouring and H has an \bar{a}_H/\bar{b}_H -colouring such that $\bar{a}_G/\bar{b}_G = \chi_f(G)$ and $\bar{a}_H/\bar{b}_H = \chi_f(H)$. Assume $\chi_f(G) \leq \chi_f(H)$. Construct an a_G/b -colouring c^G of G and an a_H/b -colouring c^H of H such that $b = \bar{b}_G\bar{b}_H$, $a_G = \bar{a}_G\bar{b}_H$ and $a_H = \bar{a}_H\bar{b}_G$. Let g be a one to one mapping of b -element subsets of $\{1, \dots, a_H\}$ to $\{1, \dots, \binom{a_H}{b}\}$. Construct the following a_H/b -colouring c of $G \square H$

$$c(su) = g^{-1} \left(g(c^G(s)) + g(c^H(u)) \bmod \binom{a_H}{b} \right) \quad (2.97)$$

It is a proper a_H/b -colouring of $G \square H$ where $a_H/b = \max\{\chi_f(G), \chi_f(H)\}$. Therefore the second equality holds.

It remains to prove the last two equalities. Let c^G and c^H be (k, d) -colourings of G and H for some positive integers k, d . Then

$$c(gh) = c^G(g) + c^H(h) \pmod{k} \quad (2.98)$$

is a (k, d) colouring of $G \square H$. This is so, because $|c(g_1h_1) - c(g_2h_2)|_k = |c^G(g_1) - c^G(g_2) + c^H(h_1) - c^H(h_2)|_k$ which is equal to $|c^G(g_1) - c^G(g_2)|_k \geq d$ when $h_1 = h_2$ and $g_1 \sim_G g_2$ and to $|c^H(h_1) - c^H(h_2)|_k \geq d$ when $g_1 = g_2$ and $h_1 \sim_H h_2$. Therefore $\chi^*(G \square H) \leq \max(\chi^*(G), \chi^*(H))$.

Because each k -colouring is also a $(k, 1)$ -colouring the above arguments hold also for $\chi(G \square H) \leq \max(\chi(G), \chi(H))$. \square

Chapter 3

Homomorphisms

3.1 Graph Homomorphism

Definition 28 (Homomorphism) For given graphs G, H we say that G is homomorphic to H or that there is a homomorphism from G to H (we write $G \rightarrow H$) if there is a function $f : V(G) \rightarrow V(H)$ satisfying $\forall s, u \in V(G) : s \sim_G u \Rightarrow f(s) \sim_H f(u)$.

Theorem 22 (Alternative Characterisation of Colouring) Let n be a positive integer. A graph G has a n -colouring iff $G \rightarrow K_n$, where K_n is the complete graph on n vertices.

Proof. Let $f : G \rightarrow K_n$ be a homomorphism and let $c : V(G) \rightarrow \{1, \dots, n\}$ be a colouring such that for all $u \in V(G)$, $c(u) = i \Leftrightarrow f(u) = v_i$, where v_i is the i -th vertex of K_n in some canonical ordering. It is trivial to check that c is indeed a colouring iff f is a homomorphism. \square

Theorem 23 (Transitivity of \rightarrow) $G \rightarrow H \ \& \ H \rightarrow H' \Rightarrow G \rightarrow H'$

Proof. Let $f : V(G) \rightarrow V(H)$ and $g : V(H) \rightarrow V(H')$ be the functions defining $G \rightarrow H$ and $H \rightarrow H'$. Construct a function $f' : V(G) \rightarrow V(H')$ as $g \circ f$. Then $\forall s, u \in V(G) : s \sim_G u \Rightarrow f(s) \sim_H f(u)$ and $\forall f(s), f(u) \in V(H) : f(s) \sim_H f(u) \Rightarrow g(f(s)) \sim_{H'} g(f(u))$ thus $\forall s, u \in V(G) : s \sim_G u \Rightarrow f'(s) \sim_{H'} f'(u)$. \square

Lemma 3 Let G be a graph with a (k, d) -colouring. Then $G \rightarrow K(k, d)$, where $K(k, d)$ is the Kneser graph.

Proof. Let $c : V(G) \rightarrow \{1, \dots, k\}$ be a (k, d) colouring of G . Define a mapping $f : V(G) \rightarrow K(k, d)$ such that $f(u) = \{c(u), c(u) + 1, \dots, c(u) + d \bmod k\}$ for $u \in G$. Clearly $f(u) \cap f(v) = \emptyset$ for $u \sim_G v$ because $|c(u) - c(v)|_k \geq d$. \square

Theorem 24 *Let G and H be graphs such that $G \rightarrow H$. Then*

$$\omega(G) \leq \omega(H), \quad (3.1)$$

$$\chi_f(G) \leq \chi_f(H), \quad (3.2)$$

$$\chi^*(G) \leq \chi^*(H) \quad \text{and} \quad (3.3)$$

$$\chi(G) \leq \chi(H). \quad (3.4)$$

Proof. Assume $H \rightarrow K_n$. Then $G \rightarrow H$ and the transitivity of \rightarrow imply $G \rightarrow K_n$. Similarly, assume $K_n \rightarrow G$. Then $G \rightarrow H$ and the transitivity of \rightarrow imply $K_n \rightarrow H$. Thus the first and the last inequalities hold.

Suppose $c : V(H) \rightarrow \{1, \dots, k\}$ is a (k, d) -colouring of H for some positive integers k, d . Then by the proof of Lemma 3 there is a homomorphism $g : V(H) \rightarrow K(k, d)$ such that $g(v) = \{c(v), c(v) + 1, \dots, c(v) + d \bmod k\}$ for $v \in H$. Construct $f' : V(G) \rightarrow K(k, d)$ such that $f'(u) = g(f(u))$ for $u \in G$ where $f : V(G) \rightarrow V(H)$ is a homomorphism from G to H . Then construct a mapping $c' : V(G) \rightarrow \{1, \dots, k\}$ such that $\forall u \in G : c'(u) = i$ where $g(f(u)) = \{i, i + 1, \dots, i + d \bmod k\}$. Such i clearly exists and it is unique. c' is a (k, d) -colouring because c was assumed to be a (k, d) -colouring. Thus the third inequality holds.

Let $\chi_f(H) = a/b$ for some positive integers a, b . Then by Theorem 12 $H \rightarrow K(a, b)$ where $K(a, b)$ is the Kneser graph. Then $G \rightarrow H$ and the transitivity of \rightarrow imply $G \rightarrow K(a, b)$. By Theorem 12 G has an a/b -colouring and by Theorem 13 $\chi_f(G) \leq \chi_f(H)$. Therefore the second inequality also holds. \square

Definition 29 *For given graphs G, H we denote by $G \circ H$ the graph with vertices $V(G \circ H) = \{su : s \in G \text{ and } u \in H\}$ and edges $E(G \circ H) = \{su \sim tw : s \sim_G t \Rightarrow u \sim_H w \text{ and } (ii) s \neq t\}$*

Lemma 4 $\omega(G \circ H) \leq |V(G)|$

Proof. Assume $\omega(G \circ H) > |V(G)|$. Choose a clique, C , in $G \circ H$ of size greater than $|V(G)|$. Then there must exist $su, sw \in C$ such that $u \neq w$. However, $u \neq w$ contradicts $su \sim_{G \circ H} sw$. \square

Theorem 25 (Alternative Specification of \rightarrow) $G \rightarrow H$ iff $\omega(G \circ H) = |V(G)|$

Proof. Let $f : G \rightarrow H$ be a homomorphism of G to H . Construct a set $C = \{su : u = f(s)\}$. $|C| = |V(G)|$ and $\forall su, tw \in C : su \sim_{G \circ H} tw$ from the definition of f . Therefore C is a clique in $G \circ H$ of size $|V(G)|$. From Lemma 4 it follows that $\omega(G \circ H) = |V(G)|$.

On the other hand, let C be a maximum clique in $G \circ H$ of size $\omega(G \circ H)$. Because $su \not\sim_{G \circ H} sw$ for $u \neq w$ and $\omega(G \circ H) = |V(G)|$, C defines a homomorphism $f : G \rightarrow H$ in the following way: $f(s) = u$ iff $su \in C$. \square

Theorem 26 (Graph Homomorphism Density - Welzl) *Let G and H be graphs such that*

$$G \rightarrow H \quad \text{and} \quad G \not\rightarrow H. \quad (3.5)$$

Then there always exists a graph F such that

$$G \rightarrow F \rightarrow H \quad \text{and} \quad G \not\rightarrow F \not\rightarrow H. \quad (3.6)$$

Proof. This theorem is given without a proof. The original proof was published in [30]. A shorter proof can be found in [14]. \square

3.2 Fractional Graph Homomorphism

3.2.1 Definitions and Basic Results

Definition 30 (Fractional Homomorphism) *For given graphs G, H we say that G is fractionally homomorphic to H or that there is a fractional homomorphism from G to H (we write $G \rightarrow_f H$) if $\omega_f(G \circ H) = |V(G)|$.*

Lemma 5 *Let $G \rightarrow_f H$ and let x be a maximal fractional clique in $G \circ H$. Then $\forall s \in V(G) : \sum_{u \in V(H)} x_{su} = 1$.*

Proof. From the definitions we have $\omega_f(G \circ H) = \sum_{su} x_{su} = \sum_s \sum_u x_{su} = |V(G)|$. It is therefore enough to show that $\forall s \in G : \sum_{u \in V(H)} x_{su} \leq 1$. This and $\sum_s \sum_u x_{su} = |V(G)|$ then implies $\forall s \in V(G) : \sum_{u \in V(H)} x_{su} = 1$. Using the notation from Definition 3, it follows from the definitions that $\sum_{su} A_{\sigma, su} \cdot x_{su} \leq 1$ and $A_{\sigma, su}, x_{su} \geq 0$. Together with the fact that each vertex belongs to at least one maximal independent set this implies that $\forall s \in G : \sum_{u \in V(H)} x_{su} \leq 1$. \square

Theorem 27 (Transitivity of \rightarrow_f) $G \rightarrow_f F \text{ \& } F \rightarrow_f H \Rightarrow G \rightarrow_f H$

Proof. Let x and x' be maximal fractional cliques in graphs $G \circ F$ and $F \circ H$ respectively. Construct a fractional clique z in the graph $G \circ H$ as follows: $z_{gh} = \sum_{f \in V(F)} x_{gf} \cdot x'_{fh}$. We want to show that $\omega_f(G \circ H) = |V(G)|$, i.e. that $\sum_{gh \in V(G \circ H)} z_{gh} = |V(G)|$. But $\sum_{gh \in V(G \circ H)} z_{gh} = \sum_{g \in V(G), h \in V(H)} \sum_{f \in V(F)} x_{gf} \cdot x'_{fh} = \sum_{g \in V(G), f \in V(F)} x_{gf} \sum_{h \in V(H)} x'_{fh}$. Now, applying the previous lemma on $F \rightarrow_f H$ we know that the last sum is equal to 1 and $\sum_{g \in V(G), f \in V(F)} x_{gf} = \sum_{gf \in V(G \circ F)} x_{gf} = |V(G)|$ due to the assumption that $G \rightarrow_f F$. \square

Lemma 6 $G \rightarrow_f H \Rightarrow G^* \rightarrow_f H$, where G^* is a subgraph of G .

Proof. Let x be a maximal fractional clique in $G \circ H$ of the size $|V(G)|$. Construct a fractional clique, x^* in $G^* \circ H$ such that $\forall u \in G^*$ and $\forall v \in H : x_{uv}^* = x_{uv}$. Using Lemma 5, we know that $\forall u \in G^* : \sum_{v \in H} x_{uv}^* = 1$ and therefore $\sum_{uv \in G^* \circ H} x_{uv}^* = |V(G^*)|$ which means that $G^* \rightarrow_f H$. \square

Lemma 7 $G \rightarrow_f K_{\omega(G)}$

Proof. Construct a fractional clique, x , in $G \circ K_{\omega(G)}$ such that $\forall uv \in G \circ K_{\omega(G)} : x_{uv} = \frac{1}{\omega(G)}$. It is easy to verify that x is indeed a fractional clique and that it has the size $|V(G)|$. \square

Lemma 8 $G \rightarrow_f H \Rightarrow K_{\omega(H)+1} \not\rightarrow_f G$

Proof. Assume $K_{\omega(H)+1} \rightarrow G$ and that $G \rightarrow_f H$. We will show that such assumptions lead to a contradiction and therefore the original statement must hold.

Denote $\omega(H)+1$ by n , i.e. we have $K_n \rightarrow G$ which implies $\omega(G) \geq n$. Take a clique, K_n^G , of size n in G .

We will show that $H \rightarrow_f K_{n-1}$. Construct a fractional clique x^* in $H \circ K_{n-1}$ in the following way

$$\forall u \in H, \forall v \in K_{n-1} : x_{uv}^* = \frac{x_v^{K_{n-1}}}{\omega_f(K_{n-1})} = \frac{1}{n-1}, \quad (3.7)$$

where $x^{K_{n-1}}$ is the maximal fractional clique in K_{n-1} (it assigns weight 1 to each vertex). Now recall that $\omega(H) = n-1$ and therefore any clique K^H in H has size at most $n-1$. The only maximal independent sets in $H \circ K_{n-1}$ are of the following two types

$$(i) \quad I_{K^H, v} = \{uv : u \in K_{n-1}^H\}, \quad K^H \text{ is a clique in } H, \quad v \in K_{n-1} \quad (3.8)$$

$$(ii) \quad I_u = \{uv : v \in K_{n-1}\}, \quad u \in G \quad (3.9)$$

Clearly, $|I| \leq n-1$ for any such independent set I and therefore $\sum_{uv \in I} x_{uv}^* v = (n-1) * \frac{1}{n-1} \leq 1$ and x^* is indeed a fractional clique. Moreover, also $\sum_{u \in H} \sum_{v \in K_{n-1}} x_{uv}^* v = |V(H)| * (n-1) * \frac{1}{n-1} = |V(H)|$ which means that $H \rightarrow_f K_{n-1}$.

Now using Theorem 27 we know that

$$G \rightarrow_f H \quad \& \quad H \rightarrow_f K_{n-1} \quad \Rightarrow \quad G \rightarrow_f K_{n-1}. \quad (3.10)$$

We will show that $G \rightarrow_f K_{n-1}$ leads to a contradiction. Lemma 6 implies that $K_n^G \rightarrow_f K_{n-1}$. Let x be a maximal fractional clique in $K_n^G \circ K_{n-1}$. All maximal independent sets in $K_n^G \circ K_{n-1}$ are of the following form

$$I_v = \{uv : u \in K_n^G\}, \quad v \in K_{n-1} \quad (3.11)$$

It must hold that $\sum_{uv \in I_v} x_{uv} \leq 1$ for all $v \in K_{n-1}$. Clearly, $\sum_{v \in K_{n-1}} \sum_{u \in K_n^G} x_{uv} \leq (n-1) < |V(K_n^G)| = n$ which contradicts that x defines a fractional homomorphism $K_n^G \rightarrow_f K_{n-1}$. \square

3.2.2 Examples

Triangulated Graphs

Definition 31 (n-Triangulated Graph) (i) K_n is an n -triangulated graph. (ii) Let $H = K_{n-1}$ be a subgraph of an n -triangulated graph G and let v be a new vertex. Then a graph $(V(G) \cup \{v\}, E(G) \cup \{uv : u \in H\})$ is also an n -triangulated graph.

For example, 1-triangulated graphs are discrete graphs and 2-triangulated graphs are trees.

Lemma 9 Let G be an n -triangulated graph. Then $G \rightarrow_f K_n$ and $G \not\rightarrow_f K_{n-1}$.

Proof. Firstly, we will show that $G \not\rightarrow_f K_{n-1}$. By definition, any n -triangulated graph contains K_n or in other words $K_n \rightarrow G$. Put $H = K_{n-1}$, $\omega(H) = n - 1$ and by Lemma 8 $G \not\rightarrow_f H$ or $G \not\rightarrow_f K_{n-1}$.

We will now construct a fractional clique in $G \circ K_n$ in the following way:

$$\forall u \in V(G), \forall v \in V(K_n) : x_{uv}^* = \frac{x_u^{K_n}}{\omega_f(K_n)} = \frac{1}{n}, \quad (3.12)$$

where x^{K_n} is the maximal fractional clique in K_n (it assigns weight 1 to each vertex). The only maximal independent sets in $G \circ K_n$ are of the following two types

$$(i) \quad I_{H,v} = \{uv : u \in V(H)\}, \quad H \text{ is a } K_n \text{ subgraph in } G, \quad v \in V(K_n) \quad (3.13)$$

$$(ii) \quad I_u = \{uv : v \in V(K_n)\}, \quad u \in V(G) \quad (3.14)$$

Clearly, $|I| = n$ for any such independent set I and therefore $\sum_{uv \in I} x_{uv}^* = n * \frac{1}{n} = 1$ and x^* is indeed a fractional clique. Moreover, also $\sum_{u \in V(G)} \sum_{v \in K_n} x_{uv}^* = |V(G)| * n * \frac{1}{n} = |V(G)|$ which means that $G \rightarrow_f K_n$. \square

Fractional Homomorphisms to C_5

Lemma 10 $G \rightarrow_f C_5$ iff G does not contain K_3 as its subgraph.

Proof. Firstly, if G does not contain K_3 we construct a fractional clique x in $G \circ C_5$ as follows:

$$\forall u \in V(G), \forall v \in V(C_5) : x_{uv} = \frac{x_u^{C_5}}{\omega_f(C_5)} = \frac{1/2}{5/2} = \frac{1}{5}, \quad (3.15)$$

where x^{C_5} is the maximal fractional clique in C_5 . Again, let us describe all possible maximal independent sets in $G \circ C_5$. They are of the following three types

$$(i) \quad I_{u_1, u_2, v_1, v_2} = \{u_i v_j : i, j = 1, 2\}, \quad (3.16)$$

where $u_1 \sim_G u_2$ and $v_1 \not\sim_{C_5} v_2$

$$(ii) \quad I_{u_1, u_2, v_1, v_2, v_3} = \{u_i v_3 : i = 1, 2\} \cup \{u_1 v_j : j = 1, 2\}, \quad (3.17)$$

where $u_1 \sim_G u_2$, $v_1 \not\sim_{C_5} v_3$ and $v_2 \not\sim_{C_5} v_3$

$$(iii) \quad I_u = \{uv : v \in V(C_5)\}, \quad \text{where } u \in V(G) \quad (3.18)$$

Because $|I| \leq 5$ for any such independent set I , $\sum_{uv \in I} x_u v \leq 1$ and x is indeed a fractional clique. Moreover, $\sum_{u \in V(G), v \in C_5} x_u v = 5 * \frac{1}{5} * |V(G)| = |V(G)|$ which means that $G \rightarrow_f C_5$.

Note that the fractional clique specified in 3.15 is not unique. $x'_u v = 1/4$ for $v \neq v_1$ and $x'_u v_1 = 0$ is also a fractional clique of size $|V(G)|$.

Secondly, assume that G contains a triangle T , $V(T) = a, b, c$. Then we can construct the following independent sets in $G \circ C_5$

$$I_{T,v} = \{uv : u \in V(T)\}, \quad v \in V(C_5) \quad (3.19)$$

and a graph H such that $V(H) = \{I_{T,v} : v \in V(C_5)\}$, $E(H) = \{(I_{T,v_1}, I_{T,v_2}) : v_1, v_2 \in V(C_5) \text{ and } v_1 \not\sim_{C_5} v_2\}$. Graph H is isomorphic with C_5 and therefore $\omega_f(H) = 5/2$. Also, two vertices I_{T,v_1}, I_{T,v_2} constitute an independent set in H iff $V(I_{T,v_1}) \cup V(I_{T,v_2})$ induces an independent set in $G \circ C_5$.

Let x be a fractional clique in $G \circ C_5$. Then we can construct a fractional clique in H , x^* , such that

$$x_{I_{T,v}}^* = \sum_{u \in V(T)} x_{uv} \quad \text{and} \quad (3.20)$$

$$\sum_{u \in T, v \in C_5} x_{uv} = \sum_{I_{T,v} \in V(H)} x_{I_{T,v}}^*. \quad (3.21)$$

This implies that $\omega_f(T \circ C_5) \leq \omega_f(H) = 5/2 < 3 = |V(T \circ C_5)|$ and therefore $T \not\rightarrow_f C_5$. Because T is a subgraph of $G \circ C_5$ it also implies that $G \not\rightarrow_f C_5$. \square

3.2.3 Duality

Lemma 11 *Let G and H be graphs. All independent sets in $G \circ H$ are of the form*

$$I \cup \left(\bigcup_{i=1}^n I_i \right), \quad \text{where} \quad (3.22)$$

$$I \subseteq \{su : s \in X \subseteq V(G), u \in Y \subseteq V(H)\} \quad (3.23)$$

$$I_i \subseteq \{s_i u : u \in Y_i \subseteq V(H)\}, \quad (3.24)$$

X induces a clique in G of size n with vertices s_1, \dots, s_n , Y induces an independent set in H and Y_i 's are such that there are no edges in H between any two Y_i, Y_j where $i \neq j$ and for all i there are no edges between Y_i and Y . Any two of the Y -sets have no vertices in common.

Moreover, all sets of the form (3.22) are independent and for maximal independent sets in addition

$$I = \{su : s \in X \subseteq V(G), u \in Y \subseteq V(H)\} \quad (3.25)$$

Proof. Firstly, we will show that the specified sets are independent.

(i) $I = \{su : s \in X \subseteq V(G), u \in Y \subseteq V(H)\}$ is independent:

Let $su \neq tv$ be from I and $su \sim_{G \circ H} tv$. Then $s \neq t$ and $s \not\sim_G t$ or $u \sim_H v$ by the definition of $G \circ H$. This, however contradicts $su, tv \in I$ and therefore I must be independent.

(ii) $I_i = \{s_i u : u \in Y_i \subseteq V(H)\}$ is independent for all i :

Let $s_i u \neq s_i v$ be from I_i and $s_i u \sim_{G \circ H} s_i v$. Then because $u \neq v$ clearly $s_i u \sim s_i v$ clearly contradicts the definition of $G \circ H$.

(iii) There are no edges between I and I_i for all i :

Let $su \in I, t_i v \in I_i$ and $su \sim_{G \circ H} t_i v$. If $s = t_i$ then the fact that $u \neq v$ ($I \cap I_i = \emptyset$) clearly contradicts $su \sim t_i v$. If $s \neq t_i$ then $s \sim_G t_i$ by the definition of X . Also, $u \not\sim_H v$ by the definitions of Y and Y_i . Now $s \sim_G t_i$ and $u \not\sim_H v$ contradict $su \sim_{G \circ H} t_i v$.

(iv) There are no edges between I_i and I_j for $i \neq j$:

Let $s_i u \in I_i, s_j v \in I_j$ and $s_i u \sim_{G \circ H} s_j v$. $s_i \neq s_j$ by assumption and $u \not\sim_H v$ by the definitions of Y_i and Y_j . Moreover, $s_i \sim_G s_j$ by the definition of Y . Again, $s_i \sim_G s_j$ and $u \not\sim_H v$ contradict $su \sim_{G \circ H} t_i v$.

Secondly, we will show that all maximal independent sets are in the specified form. Let I be an independent set in $G \circ H$ such that

$$\begin{aligned} A = \quad & \{su : s \in G, u \in H\}, \quad \text{where} \quad \forall su, tv \in A : \\ & s = t \quad \text{and} \quad u \neq v \quad \text{or} \\ & s \sim_G t \quad \text{and} \quad u \not\sim_H v \end{aligned} \quad (3.26)$$

and define $X = \{s : su \in A\}$ and $\bar{Y} = \{u : su \in A\}$.

(i) X induces a clique in G :

Let $su, tv \in A$ such that $s \neq t$. Then from the definition of A it follows that $s \sim_G t$.

(ii) $su, tv \in A, s \neq t \Rightarrow u \not\sim_H v$:

From (i) follows that $s \sim_G t$ and from the definition of A then follows that $u \not\sim_H v$.

Let $|X| = n$. Define a partition of \bar{Y} into $n + 1$ disjoint sets Y, Y_1, \dots, Y_n such that

$$Y = \{u : s_1 u, s_2 u \in A \text{ for some } s_1 \neq s_2\} \quad (3.27)$$

$$Y_i = \{u : s_i u \in A \text{ and } s_i \neq t \Rightarrow tu \notin A\} \text{ where } s_i \in X \quad (3.28)$$

(iii) There are no edges between Y_i and Y_j for $i \neq j$:

Assume $u \sim_H v$ for some $u \in Y_i, v \in Y_j$. Then Y_i, Y_j are nonempty and there exist $s_i, s_j \in X$ such that $s_i u, s_j v \in A$. However, $i \neq j$ implies that $s_i \neq s_j$ and (ii) concludes the contradiction. Therefore $u \not\sim_H v$ for all $u \in Y_i, v \in Y_j$.

(iv) There are no edges between Y and Y_i for all i :

Assume $u \sim_H v$ for some $u \in Y, v \in Y_i$. Then Y, Y_i are nonempty and there exist $s_i, s_j \in X$ such that $s_i u, s_j v \in A$ and $s_i \neq s_j$ ($s_i = s_1$ in the definition of Y if $s_j \neq s_1$ and $s_i = s_2$ otherwise). (ii) then concludes the contradiction. Therefore $u \not\sim_H v$ for all $u \in Y, v \in Y_i$.

(v) Y induces an independent set in H :

Assume $u \sim_H v$ for some $u, v \in Y$. By the definition of Y there exist $s_1 \neq s_2$ such that $s_1 u, s_2 v \in A$. (ii) immediately concludes the contradiction. Therefore Y induces an independent set in H .

Finally, let A be a maximal independent set in $G \circ H$. We have proved above that A has to satisfy (3.23) and (3.24). Assume that (3.25), i.e. I is a primitive subset. Let $su \in \{su : s \in X, u \in Y\} \setminus I$. It is a contradiction with A being a maximal independent set, because $\{su\} \cup A$ is also an independent set. \square

Theorem 28 (Bačík, Mahajan [3]) $G \rightarrow_f H$ iff $K_{\lfloor \omega_f(H)+1 \rfloor} \not\rightarrow G$.

Proof. Let $K_{\lfloor \omega_f(H)+1 \rfloor} \rightarrow G$ and let C be a maximal clique in G such that $|C| \geq \lfloor \omega_f(H) + 1 \rfloor$. Let y^H be a minimum fractional colouring of H of the size $\chi_f(H) = \omega_f(H)$. Construct the following fractional colouring of $G \circ H$

$$(i) \quad \forall \text{ maximal independent set } A \subseteq H : y_{C \times A} = y_A^H \quad (3.29)$$

$$(ii) \quad \forall u \in V(G) \setminus C : y_{u \times V(H)} = 1 \quad (3.30)$$

$$(iii) \quad y_I = 0 \text{ otherwise.} \quad (3.31)$$

All of type (i) and (ii) are clearly independent sets by Lemma 11. To show that y is a proper colouring take any $su \in G \circ H$. If $s \in V(G) \setminus C$ then $y_{s \times V(H)} = 1$. If $s \in C$ then $\sum_{I \ni su} y_I = \sum_{\text{indep. } A \ni u} y_{C \times A} = \sum_{\text{indep. } A \ni u} y_A^H \geq 1$ because y^H was taken such to be a proper fractional colouring of H . The size of y is at most $\omega_f(H) + (|V(G)| - \lfloor \omega_f(H) + 1 \rfloor) < |V(G)|$. It implies that $\omega_f(G \circ H) < |V(G)|$ and $G \not\rightarrow H$. To prove the other direction assume $K_{\lfloor \omega_f(H)+1 \rfloor} \not\rightarrow G$. Again, take x^H and y^H fractional clique and colouring in H dual to each other, $\omega_f(H) = \chi_f(H)$. From the complementary slackness (Theorem 9) follows that

$$x_u^H = x_u^H \sum_{B \ni u} y_B^H, \quad (3.32)$$

where B runs through all maximal independent sets in H . Construct a fractional clique in $G \circ H$ such that

$$\forall s \in G, u \in H : x_{su} = \frac{x_u^H}{\omega_f(H)}. \quad (3.33)$$

To show that x is a proper clique we need $\sum_{su \in A} x_{su} \leq 1$, where $A \subseteq G \circ H$ is an independent set. By Lemma 11 $A = I \cup (\bigcup_{i=1}^n I_i)$ where $I \subseteq K_n \times Y, I_i = s_i \times Y_i$ and $n \leq \omega_f(H)$ by the assumption $K_{\lfloor \omega_f(H)+1 \rfloor} \not\rightarrow G$. Therefore

$$\sum_{su \in A} x_{su} = \sum_{su \in I} x_{su} + \sum_{i=1}^n \sum_{s_i u \in I_i} x_{s_i u} \leq \quad (3.34)$$

$$\leq \frac{n}{\omega_f(H)} \sum_{u \in Y} x_u^H + \frac{1}{\omega_f(H)} \sum_{i=1}^n \sum_{u \in Y_i} x_u^H \leq \quad (3.35)$$

$$\leq \frac{1}{\omega_f(H)} \left(\sum_B y_B^H \right) \sum_{u \in Y} x_u^H + \frac{1}{\omega_f(H)} \sum_{i=1}^n \sum_{u \in Y_i} \left(x_u^H \sum_{B \ni u} y_B^H \right) = \quad (3.36)$$

$$= \frac{1}{\omega_f(H)} \sum_B y_B^H \left(\sum_{u \in Y} x_u^H \right) + \frac{1}{\omega_f(H)} \sum_{i=1}^n \sum_B y_B^H \left(\sum_{u \in B \cap Y_i} x_u^H \right) = \quad (3.37)$$

$$= \frac{1}{\omega_f(H)} \sum_B y_B^H \left(\sum_{u \in Y} x_u^H + \sum_{i=1}^n \sum_{u \in B \cap Y_i} x_u^H \right), \quad (3.38)$$

where B runs through all maximal independent sets in H . $Y \cup (\bigcup_{i=1}^n (B \cap Y_i))$ is an independent set in H because Y and B are independent sets and there are no edges between Y and Y_i for all i . Therefore $(\sum_{u \in Y} x_u^H + \sum_{i=1}^n \sum_{u \in B \cap Y_i} x_u^H) \leq 1$. Hence

$$\sum_{su \in A} x_{su} \leq \frac{1}{\omega_f(H)} \sum_B y_B^H = 1. \quad (3.39)$$

The size of x is $|V(G)|$ because

$$\sum_{su \in G \circ H} x_{su} = \sum_{s \in G, u \in H} \frac{x_u^H}{\omega_f(H)} = |V(G)| \sum_{u \in H} \frac{x_u^H}{\omega_f(H)} = |V(G)|. \quad (3.40)$$

□

Theorem 29 (Transitivity of $\not\rightarrow_f$) $G \not\rightarrow_f F \ \& \ F \not\rightarrow_f H \Rightarrow G \not\rightarrow_f H$

Proof. Suppose $G \not\rightarrow_f F$ & $F \not\rightarrow_f H$. By Theorem 28

$$\lfloor \omega_f(H) + 1 \rfloor \leq \omega(F) \quad (3.41)$$

and

$$\lfloor \omega_f(F) + 1 \rfloor \leq \omega(G). \quad (3.42)$$

Since $\omega(F) < \lfloor \omega_f(F) + 1 \rfloor$ it follows that $\lfloor \omega_f(H) + 1 \rfloor < \omega(G)$. By Theorem 28 $G \not\rightarrow_f H$. □

Theorem 30 $G \not\rightarrow_f H \Rightarrow H \rightarrow_f G$

Proof. Let $G \not\rightarrow_f H$ By Theorem 28

$$\lfloor \omega_f(H) + 1 \rfloor \leq \omega(G). \quad (3.43)$$

Since $\omega(H) < \lfloor \omega_f(H) + 1 \rfloor$ and $\omega(G) < \lfloor \omega_f(G) + 1 \rfloor$ it follows that

$$\omega(H) < \lfloor \omega_f(G) + 1 \rfloor \quad (3.44)$$

and hence by Theorem 28 $H \rightarrow_f G$. □

Lemma 12 *Let G and H be graphs. Then*

$$(\exists F \text{ such that } G \not\rightarrow_f F \not\rightarrow_f H) \Leftrightarrow (\lfloor \omega_f(G) + 1 \rfloor < \omega(H)). \quad (3.45)$$

Proof. Firstly suppose $H \not\rightarrow_f F \not\rightarrow_f G$ for some graph F . By Theorem 28 $K_{\lfloor \omega_f(G)+1 \rfloor} \rightarrow F$ and $K_{\lfloor \omega_f(F)+1 \rfloor} \rightarrow H$ or in other words

$$\lfloor \omega_f(G) + 1 \rfloor \leq \omega(F) \quad (3.46)$$

and

$$\lfloor \omega_f(F) + 1 \rfloor \leq \omega(H) \quad . \quad (3.47)$$

Since $\omega(F) < \lfloor \omega_f(F) + 1 \rfloor$ it follows that

$$\lfloor \omega_f(G) + 1 \rfloor < \omega(H). \quad (3.48)$$

On the other hand suppose $\lfloor \omega_f(G) + 1 \rfloor < \omega(H)$. Take $F = K_{\omega(H)-1}$, clearly $\omega_f(F) = \omega(F) = \omega(H) - 1$ and therefore

$$\omega(H) = \omega(F) + 1 = \lfloor \omega_f(F) + 1 \rfloor. \quad (3.49)$$

Since $\lfloor \omega_f(G) + 1 \rfloor$ and $\omega(H)$ are integers we also have $\lfloor \omega_f(G) + 1 \rfloor \leq \omega(H) - 1$ and therefore

$$\omega(F) = \omega(H) - 1 \geq \lfloor \omega_f(G) + 1 \rfloor. \quad (3.50)$$

Hence $H \not\rightarrow_f F \not\rightarrow_f G$. This concludes the proof. \square

The following theorem is a corollary of Lemma 12.

Theorem 31 (Weak Density Theorem for \rightarrow_f) *Let G and H be graphs such that*

$$G \rightarrow_f H \quad \text{and} \quad G \not\leftarrow_f H. \quad (3.51)$$

Then a graph F such that

$$G \rightarrow_f F \rightarrow_f H \quad \text{and} \quad G \not\leftarrow_f F \not\leftarrow_f H. \quad (3.52)$$

exists if and only if

$$\lfloor \omega_f(G) + 1 \rfloor < \omega(H). \quad (3.53)$$

Proof. By Lemma 12 a graph F such that $H \not\rightarrow_f F \not\rightarrow_f G$ exists iff $\lfloor \omega_f(G) + 1 \rfloor < \omega(H)$. By Theorem 30 $H \not\rightarrow_f F \not\rightarrow_f G$ implies $G \rightarrow_f F \rightarrow_f H$ \square

Corollary 7 *Let G and H be graphs such that*

$$G \rightarrow_f H \quad \text{and} \quad G \not\leftarrow_f H \quad (3.54)$$

and let $\omega(G) = \omega(H) - 1$ and $\chi(G) < \omega(H)$ (e.g. $G = K_3$ and $H = K_4$). Then there is no graph F such that

$$G \rightarrow_f F \rightarrow_f H \quad \text{and} \quad G \not\leftarrow_f F \not\leftarrow_f H. \quad (3.55)$$

Corollary 8 *Let $M_n, n \geq 2$ be Mycielsky's graphs defined in Section 3.4. Then since M_n are triangle-free $M_n \rightarrow_f K_{\lfloor \omega_f(M_n)+1 \rfloor}$. Also by Theorem 28 $K_{\lfloor \omega_f(M_n)+1 \rfloor} \not\rightarrow_f M_n$. But despite*

$$\lim_{n \rightarrow \infty} \omega(K_{\lfloor \omega_f(M_n)+1 \rfloor}) - \omega(M_n) = \infty, \quad (3.56)$$

there is no n such that

$$M_n \rightarrow_f F_n \rightarrow_f K_{\lfloor \omega_f(M_n)+1 \rfloor} \quad \text{and} \quad M_n \not\rightarrow_f F_n \not\rightarrow_f K_{\lfloor \omega_f(M_n)+1 \rfloor} \quad (3.57)$$

for some graph F_n . This is because $\lfloor \omega_f(M_n) + 1 \rfloor = \omega(K_{\lfloor \omega_f(M_n)+1 \rfloor})$ for all $n \geq 2$.

Lemma 13

$$G \rightarrow_f H \not\Rightarrow \omega(G) \leq \omega(H) \quad (3.58)$$

$$G \rightarrow_f H \not\Rightarrow \chi_f(G) \leq \chi_f(H) \quad (3.59)$$

$$G \rightarrow_f H \not\Rightarrow \chi(G) \leq \chi(H) \quad (3.60)$$

Proof. To show that the first implication does not hold take $G = K_3$ and $H = M_5$ where M_5 is the Mycielsky's graph defined in Section 3.4. Then

$$G \rightarrow_f H, \quad (3.61)$$

and

$$\omega(G) = 3 > 2 = \chi_f(H). \quad (3.62)$$

Clearly $\omega(K_3) = 3$ and $\omega(M_5) = 2$. By Theorem 36 (Section 3.4)

$$\chi_f(M_5) = \frac{29}{10} + \frac{10}{29} = 941/290 \doteq 3.24 \quad (3.63)$$

Therefore $\lfloor \omega_f(H) + 1 \rfloor = 4$ and $K_{\lfloor \omega_f(H)+1 \rfloor} \not\rightarrow G$ hence $G \rightarrow_f H$ by Theorem 28.

To show that the second and the third implications do not hold take $G = C_5$ and $H = K_2$. Then

$$G \rightarrow_f H, \quad (3.64)$$

$$\chi_f(G) = 5/2 > 2 = \chi_f(H), \quad (3.65)$$

and

$$\chi(G) = 3 > 2 = \chi(H). \quad (3.66)$$

Clearly $\omega(G) = 2$ and $\omega(H) = \omega_f(H) = 2$. Since $\lfloor \omega_f(H) + 1 \rfloor = 3$ it follows that $K_{\lfloor \omega_f(H)+1 \rfloor} \not\rightarrow G$ and hence by Theorem 28 $G \rightarrow_f H$. $C_5 = S_{3,2}$ and therefore $\chi_f(C_5) = 5/2$. \square

3.3 Pseudo Graph Homomorphism

Feige and Lovász in [10] introduced a general technique for a polynomial approximation of any problem in NP . Given an NP -set L they construct a polynomial set L' (the hoax set for L) with the property $L \cap X = L' \cap X$ for a certain family of instances X . The technique can be applied to the following Graph Homomorphism Problem

Instance: (G, H) where G and H are graphs (3.67)

Question: Is G homomorphic to H ? (3.68)

Consider the 0-1 matrix V with rows and columns indexed by su where $s \in G, u \in H$ with the property that

$$V_{su,tv} = 0 \Leftrightarrow (i)s = t \& u \neq v \text{ or } (ii)s \sim_G t \& u \not\sim_H v \quad (3.69)$$

(note that the matrix V is the adjacency matrix of $G \circ H$ with a self-loop added to each vertex). Construct matrices $C = \frac{1}{|V(G)|^2} V$ and $Q_{su,tv} = p_{su}p_{tv}$ where p_{su} have the following properties (they are usually interpreted as probabilities over choices su). $\forall s \in G : \sum_{u \in H} p_{su} = 1$ and $\forall s \in G, \forall u \in H : p_{su} \geq 0$. The following theorem gives an alternative specification of the Graph Homomorphism Problem in terms of the above matrices

Theorem 32 *Let G and H be given graphs. Then $G \rightarrow H$ iff the optimum of the following maximization problem is 1*

$$\max_Q \sum_{s,u,t,v} C_{su,tv} Q_{su,tv} \quad (3.70)$$

s.t.

$$Q \text{ is a rank 1 matrix} \quad (3.71)$$

$$Q \text{ is symmetric} \quad (3.72)$$

$$\forall s, t : \sum_{u,v} Q_{su,tv} = 1 \quad (3.73)$$

$$\forall s, t, u, v : Q_{su,tv} \geq 0 \quad (3.74)$$

Proof. Suppose $G \rightarrow H$ and let $f : V(G) \rightarrow V(H)$ be a homomorphism from G to H . Construct a 0-1 vector p such that $p_{su} = 1$ iff $f(s) = u$, $s \in G, u \in H$. Then $Q = pp^T$ is a symmetric rank 1 matrix and $Q_{su,tv} \geq 0$ for all $s, t \in G, u, v \in H$. Clearly, $\forall s, t \in G : Q_{su,tv} = 1 \Leftrightarrow (u = f(s) \& v = f(t))$, and $Q_{su,tv} = 0$ otherwise. Therefore $\forall s, t \in G : \sum_{u,v \in H} Q_{su,tv} = 1$ and all conditions (3.71) - (3.74) are met (Q is a feasible solution).

As stated above, $Q_{su,tv} = 1$ iff $(f(s) = u \text{ and } f(t) = v)$, where f is a homomorphism. Thus $Q_{su,tv} = 1 \Rightarrow V_{su,tv} = 1$, where V is defined as in (3.69). Therefore

$$\sum_{s,u,t,v} C_{su,tv} Q_{su,tv} = \sum_{s,u,t,v} \frac{1}{|V(G)|^2} V_{su,tv} Q_{su,tv} = \sum_{s,u,t,v} \frac{1}{|V(G)|^2} Q_{su,tv} =$$

$$= \frac{1}{|V(G)|^2} \sum_{su} p_{su} \sum_{tv} p_{tv} = \frac{1}{|V(G)|^2} |V(G)|^2 = 1, \quad (3.75)$$

where C is defined as above. The value of the objective function of the maximization problem (3.70) is equal to 1 for Q and therefore Q is an optimal solution (note that 1 is an upper bound for (3.70)).

On the other hand, suppose Q is an optimal solution of the above maximization problem and $\sum_{s,u,t,v} C_{su,tv} Q_{su,tv} = 1$. By (3.73) $\sum_{s,t,u,v} Q_{su,tv} = \sum_{s,t} \sum_{u,v} Q_{su,tv} = \sum_{s,t} 1 = |V(G)|^2$. Since V is a 0-1 matrix and $Q_{su,tv} \geq 0$ the following must hold

$$\sum_{s,u,t,v} C_{su,tv} Q_{su,tv} = 1 \Rightarrow (Q_{su,tv} > 0 \Rightarrow V_{su,tv} = 1). \quad (3.76)$$

By (3.73) $\forall s \in G, \forall t \in G : \sum_{u,v \in H} Q_{su,tv} = 1 > 0$. By (3.74) all $Q_{su,tv}$ are nonnegative and therefore

$$\forall s \in G : \exists u \in H \text{ such that } Q_{su,tv} > 0 \text{ for some } t \in G, v \in H. \quad (3.77)$$

Since Q is a nonnegative rank 1 matrix, there is also a nonnegative vector p such that $Q_{su,tv} = p_{su} p_{tv}$ for all s, t, u, v . Because $Q_{su,tv} > 0$ implies that $p_{su} > 0$ and $p_{tv} > 0$, by (3.77) we have $\forall s \in G : \exists u \in H$ such that $p_{su} > 0$. Define a mapping $f : V(G) \rightarrow V(H)$ such that $f(s) = u \Rightarrow p_{su} > 0$ (note that for each s there can be more than one u for which $p_{su} > 0$).

It remains to verify that f is a homomorphism. Let $f(s) = u$ and $f(t) = v$. Then $p_{su} > 0$ and $p_{tv} > 0$ and hence also $Q_{su,tv} = p_{su} p_{tv} > 0$. Moreover, by (3.76) also $V_{su,tv} = 1$ and therefore $s \sim_G t \Rightarrow u \sim_H v$ by the definition of V . In other words, f satisfies the properties of a homomorphism and $G \rightarrow H$. \square

The polynomial approximation of Feige and Lovász replaces the rank 1 constraint (3.71) with the requirement that Q is a positive semidefinite matrix. The ellipsoid algorithm can be used to solve the new problem in a polynomial time (c.f. [10]). The optimal solution of the modified problem with objective value 1 is called hoax and the original and the new problems will be referred to as (*) and (**) respectively (c.f. [10]).

Definition 32 (Pseudo Graph Homomorphism) *Let G and H be graphs. We say that G is pseudo homomorphic to H , write $G \rightarrow_h H$, if the problem (**)*

$$\max_Q \sum_{s,u,t,v} C_{su,tv} Q_{su,tv} \quad (3.78)$$

s.t.

$$Q \text{ is a positive semidefinite matrix} \quad (3.79)$$

$$Q \text{ is symmetric} \quad (3.80)$$

$$\forall s, t : \sum_{u,v} Q_{su,tv} = 1 \quad (3.81)$$

$$\forall s, t, u, v : Q_{su,tv} \geq 0 \quad (3.82)$$

has a hoax for (G, H) .

Corollary 9 Clearly, $G \rightarrow H$ implies $G \rightarrow_h H$.

Lemma 14 Let $(**)$ has a hoax Q with instance (G, H) . Then $Q = MM^T$ for some matrix M . Denote each column su of M^T by m_{su} . Then for any independent set $I = \{s_1u_1, \dots, s_ku_k\}$ in $G \circ H$ the set of vectors $\{m_{s_1u_1}, \dots, m_{s_ku_k}\}$ is orthogonal.

Proof. By assumption, Q is a positive semidefinite matrix. Therefore $Q = MM^T$ for some M . Let $I = \{s_1u_1, \dots, s_ku_k\}$ be an independent set in $G \circ H$ where k is a positive integer. Then $V_{s_iu_i, s_ju_j} = 0$ for all $i \neq j$. By (3.76) also $0 = Q_{s_iu_i, s_ju_j} = m_{s_iu_i}^T m_{s_ju_j}$ and hence $m_{s_iu_i}$ and $m_{s_ju_j}$ are orthogonal. \square

Bačík and Mahajan [3] proved the following implication.

Theorem 33 $G \rightarrow_h H$ implies $G \rightarrow_f H$.

Proof. Suppose Q is a hoax of the problem $(**)$ and write Q as MM^T for some matrix M (Q is positive semidefinite). Denote each column of M^T with index su by m_{su} . Then $Q_{su, tv} = m_{su}^T m_{tv}$ and by (3.82)

$$\forall s, t \in G, \forall u, v \in H : m_{su}^T m_{tv} \geq 0 \quad (3.83)$$

For each $s \in G$ define vector $\tilde{m}_s = \sum_{u \in H} m_{su}$. The condition (3.81) of $(**)$ then becomes

$$\forall s, t \in G : 1 = \sum_{u, v \in H} Q_{su, tv} = \sum_{u, v \in H} m_{su}^T m_{tv} = \sum_{u \in H} m_{su}^T \sum_{v \in H} m_{tv} = \tilde{m}_s^T \tilde{m}_t. \quad (3.84)$$

Choosing $s=t$ in (3.84) gives $\tilde{m}_s^T \tilde{m}_s = 1$ which means that \tilde{m}_s are all vectors with unit length. Given this fact, (3.84) then also implies that the angle between any two \tilde{m}_s and \tilde{m}_t is zero and hence they are all equal. Denote

$$\tilde{m} = \sum_{u \in H} m_{su}. \quad (3.85)$$

Then

$$|\tilde{m}| = 1. \quad (3.86)$$

From the definition of V , $V_{su, sv} = 0$ for $u \neq v$. Therefore by (3.76) $m_{su}^T m_{sv} = 0$ for $u \neq v$ and

$$\tilde{m}^T m_{su} = \sum_v m_{sv} m_{su} = m_{su} m_{su} = |m_{su}|^2. \quad (3.87)$$

Construct a fractional clique x in $G \in H$ such that

$$x_{su} = |m_{su}|^2. \quad (3.88)$$

By Lemma 14 for any independent set $I = \{s_1 u_1, \dots, s_k u_k\}$ in $G \circ H$ vectors $\{m_{s_1 u_1}, \dots, m_{s_k u_k}\}$ are all mutually orthogonal, i.e. $m_{s_i u_i}^T m_{s_j u_j} = 0$ for $i \neq j$. Denote $\tilde{m}_I = \sum_{su \in I} m_{su}$. Then

$$|\tilde{m}_I|^2 = \sum_{su \in I} m_{su}^T \sum_{tv \in I} m_{tv} = \sum_{su \in I} |m_{su}|^2 \quad (3.89)$$

By (3.87) $|m_{su}|^2 = \tilde{m}^T m_{su}$ and thus

$$|\tilde{m}_I|^2 = \sum_{su \in I} |m_{su}|^2 = \tilde{m}^T \sum_{su \in I} m_{su} = \tilde{m}^T \tilde{m}_I \leq |\tilde{m}| \cdot |\tilde{m}_I| \quad (3.90)$$

By (3.86) $|\tilde{m}| = 1$, hence

$$|\tilde{m}_I|^2 = \sum_{su \in I} |m_{su}|^2 \leq |\tilde{m}_I|. \quad (3.91)$$

The fact that $|\tilde{m}_I|^2 \leq |\tilde{m}_I|$ means that $|\tilde{m}_I| \leq 1$. Together with (3.91) this implies that

$$\sum_{su \in I} x_{su} = \sum_{su \in I} |m_{su}|^2 \leq |\tilde{m}_I| \leq 1 \quad (3.92)$$

and therefore x is a proper fractional clique (note that $x_{su} \geq 0$ follows trivially from its definition). Moreover,

$$\sum_{su \in G \circ H} x_{su} = \sum_{su} |m_{su}|^2 = \sum_{su} \tilde{m}^T m_{su} = \sum_s \tilde{m}^T \sum_u m_{su} = \sum_s \tilde{m}^T \tilde{m} = \sum_{s \in G} 1 = |V(G)|. \quad (3.93)$$

Therefore $G \rightarrow_f H$. \square

Thus we have $G \rightarrow H \Rightarrow G \rightarrow_h H \Rightarrow G \rightarrow_f H$. The converse, however, is not true.

Theorem 34 $G \rightarrow_f H \not\Rightarrow G \rightarrow_h H \not\Rightarrow G \rightarrow H$.

Proof. Firstly, we will show that for each perfect n, m -string $S_{n+1, m}$

$$S_{n+1, m} \rightarrow_f K_n \quad \text{and} \quad S_{n+1, m} \not\rightarrow_h K_n. \quad (3.94)$$

Clearly $\lfloor \omega_f(K_n) + 1 \rfloor = n + 1$. By Theorem 14, $\omega(S_{n+1, m}) = n$ and therefore $K_{n+1} \not\rightarrow S_{n+1, m}$. Theorem 28 then implies that $S_{n+1, m} \rightarrow_f K_n$.

Now assume $S_{n+1, m} \rightarrow_h K_n$ with a hoax $Q = MM^T$ and denote each column of M^T with index su by m_{su} . From the proof of Theorem 33 it follows that

$$\sum_{u \in V(K_n)} m_{su} = \tilde{m}. \quad (3.95)$$

Take a partitioning of $S_{n+1, m}$ similar to that in Definition 18. Then $H_i \setminus \{x_i\} = K_n$ and $H_i \setminus \{y_i\} = K_n$, $i = 1, \dots, m$. For each $u \in K_n$, the sets $I_{x_i} = \{su : s \in V(H_i) \setminus \{x_i\}\}$, $i = 1, \dots, m$ and $I_{y_i} = \{su : s \in V(H_i) \setminus \{y_i\}\}$, $i = 1, \dots, m$ are

independent in $S_{n+1,m} \circ K_n$ and therefore by Lemma 14, vectors m_{su} , $su \in I_{x_i}$ are orthogonal as well as vectors m_{su} , $su \in I_{y_i}$. From the proof of Theorem 33 it follows that

$$\left| \sum_{su \in I_{x_i}} m_{su} \right| = |\tilde{m}_{I_{x_i}}| \leq 1 \quad (3.96)$$

$$\left| \sum_{su \in I_{y_i}} m_{su} \right| = |\tilde{m}_{I_{y_i}}| \leq 1 \quad (3.97)$$

The fact that $H_i \setminus \{x_i\}$ and $H_i \setminus \{y_i\}$ are cliques of size n then gives us

$$n \geq \sum_{u \in V(K_n)} \left| \sum_{s \in H_i \setminus \{x_i\}} m_{su} \right| \geq \left| \sum_{s \in H_i \setminus \{x_i\}} \sum_{u \in V(K_n)} m_{su} \right| = |n\tilde{m}| = n \quad (3.98)$$

$$n \geq \sum_{u \in V(K_n)} \left| \sum_{s \in H_i \setminus \{y_i\}} m_{su} \right| \geq \left| \sum_{s \in H_i \setminus \{y_i\}} \sum_{u \in V(K_n)} m_{su} \right| = |n\tilde{m}| = n. \quad (3.99)$$

The above two inequalities are therefore in fact satisfied as equalities and by dividing both sides by n we obtain

$$\left| \sum_{s \in H_i \setminus \{x_i\}} m_{su} \right| = |\tilde{m}| = 1 \quad (3.100)$$

$$\left| \sum_{s \in H_i \setminus \{y_i\}} m_{su} \right| = |\tilde{m}| = 1. \quad (3.101)$$

Also,

$$\left| \sum_{su \in I} m_{su} \right|^2 = |\tilde{m}_I|^2 = \sum_{su \in I} |m_{su}|^2 = \tilde{m}^T \sum_{su \in I} m_{su} = \tilde{m}^T \tilde{m}_I \leq |\tilde{m}| \cdot |\tilde{m}_I| \quad (3.102)$$

is true for any independent set I . In our case, thanks to (3.100) and (3.101), the inequality in (3.102) holds as equality and we can conclude that

$$\sum_{s \in H_i \setminus \{x_i\}} m_{su} = \tilde{m} \quad (3.103)$$

$$\sum_{s \in H_i \setminus \{y_i\}} m_{su} = \tilde{m} \quad (3.104)$$

(c.f. [3]). Therefore, when rearranged, we obtain

$$\sum_{s \in H_i \setminus \{x_i\}} m_{su} = \tilde{m} = \sum_{s \in H_j \setminus \{y_j\}} m_{su} \quad (3.105)$$

$$\tilde{m} - m_{x_i,u} = \tilde{m} - m_{y_j,u} \quad (3.106)$$

which implies $m_{x_i,u} = m_{y_j,u}$, $i, j = 1, \dots, m$ and thus also $m_{x_1,u} = m_{y_m,u}$. But vertices x_1 and y_m are adjacent and hence cannot be mapped to the same vertex u . Therefore

$m_{x_1,u}$ and $m_{y_m,u}$ have to be at the same time equal and orthogonal, which can only be satisfied if they are both zero vectors. That, however, contradicts that

$$\left| \sum_{u \in V(K_n)} m_{x_1,u} \right| = \left| \sum_{u \in V(K_n)} m_{y_m,u} \right| = |\tilde{m}| = 1. \quad (3.107)$$

Therefore $S_{n+1,m} \not\rightarrow_h K_n$.

Secondly, for the Grötzsch's graph \mathcal{G}

$$\mathcal{G} \rightarrow_h K_3 \quad \text{and} \quad \mathcal{G} \not\rightarrow_h K_3. \quad (3.108)$$

In the next section it is shown that $\chi(\mathcal{G}) = 4$, thus $\mathcal{G} \not\rightarrow_h K_3$. Conversely, the following matrix is a hoax in $(**)$ for the instance (\mathcal{G}, K_3)

$$Q_{s,s} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \quad (3.109)$$

$$Q_{s,t} = \begin{pmatrix} 0 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 0 \end{pmatrix} \text{ for } s \sim t, \text{ and} \quad (3.110)$$

$$Q_{s,t} = \begin{pmatrix} 1/6 & 1/12 & 1/12 \\ 1/12 & 1/6 & 1/12 \\ 1/12 & 1/12 & 1/6 \end{pmatrix} \text{ for } s \not\sim t. \quad (3.111)$$

Where $Q_{s,t}$ denotes the s, t block of Q . Therefore $\mathcal{G} \rightarrow_h K_3$. \square

Theorem 35 (Transitivity of \rightarrow_h) $G \rightarrow_h F \not\rightarrow_h F \rightarrow_h H \Rightarrow G \rightarrow_h H$

Proof. Let Q^{GF} be a hoax in $(**)$ for the instance (G, F) and let Q^{FH} be a hoax in $(**)$ for the instance (F, H) . Construct a hoax Q for (G, H) as follows

$$Q_{g_1 h_1, g_2 h_2} = \sum_{f_1, f_2 \in F} Q_{g_1 f_1, g_2 f_2}^{GF} Q_{f_1 h_1, f_2 h_2}^{GF} \quad (3.112)$$

The value of the objective function in $(**)$ for Q is equal to

$$\sum_{g_1, g_2 \in G, h_1, h_2 \in H} C_{g_1 h_1, g_2 h_2} Q_{g_1 h_1, g_2 h_2} = \sum_{g_1, g_2 \in G, h_1, h_2 \in H} C_{g_1 h_1, g_2 h_2} \sum_{f_1, f_2 \in F} Q_{g_1 f_1, g_2 f_2}^{GF} Q_{f_1 h_1, f_2 h_2}^{GF} = 1 \quad (3.113)$$

Clearly Q is symmetric and $Q_{g_1 h_1, g_2 h_2} \geq 0$ for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$ since both Q^{GF} and Q^{FH} have those properties. For all $g_1, g_2 \in G$

$$\begin{aligned} \sum_{h_1, h_2 \in H} Q_{g_1 h_1, g_2 h_2} &= \sum_{h_1, h_2 \in H} \sum_{f_1, f_2 \in F} Q_{g_1 f_1, g_2 f_2}^{GF} Q_{f_1 h_1, f_2 h_2}^{GF} = \\ &= \sum_{f_1, f_2 \in F} Q_{g_1 f_1, g_2 f_2}^{GF} \left(\sum_{h_1, h_2 \in H} Q_{f_1 h_1, f_2 h_2}^{GF} \right) = \\ &= \sum_{f_1, f_2 \in F} Q_{g_1 f_1, g_2 f_2}^{GF} = \\ &= 1 \quad (\text{c.f. [3]}). \end{aligned} \quad (3.114)$$

Q is also positive semidefinite (c.f. [3]). Hence Q is a hoax in $(^{**})$ for the instance (G, H) . \square

3.4 Examples

3.4.1 Mycielski's Graphs

It is interesting to show that there can be graphs whose clique number is much smaller than their chromatic number. The sequence of Mycielski's Graphs has the property that all its members are triangle-free, however the sequence of their chromatic numbers increases without a bound hence offering an arbitrarily wide gap between ω and χ .

Definition 33 (Mycielski's Graph Transformation) *Given a graph G such that $V(G) = v_1, \dots, v_k$, the Mycielski's Transformation of $G, \mu(G)$, is defined as follows*

$$V(\mu(G)) = \{x_1, \dots, x_k, y_1, \dots, y_k, z\}, \quad (3.115)$$

$$x_i \sim_{\mu(G)} x_j \Leftrightarrow v_i \sim_G v_j, \quad (3.116)$$

$$x_i \sim_{\mu(G)} y_j \Leftrightarrow v_i \sim_G v_j, \quad (3.117)$$

$$y_i \sim_{\mu(G)} z \quad \forall i = 1, \dots, k, \quad (3.118)$$

$$\text{and there are no other edges.} \quad (3.119)$$

Theorem 36 (Mycielski [24], Larsen, Propp and Ullman [21]) *Let G be a graph with at least one edge. Then*

$$(a) \quad \omega(\mu(G)) = \omega(G), \quad (3.120)$$

$$(b) \quad \chi(\mu(G)) = \chi(G) + 1 \quad \text{and} \quad (3.121)$$

$$(c) \quad \chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}. \quad (3.122)$$

Proof. The proof is not included here. It can be found in [21]. \square

When $M_2 = K_2$ graphs recursively defined by $M_{n+1} = \mu(M_n)$ for $n \geq 2$ are called Mycielski's graphs. First few examples are $M_3 = C_5$ and M_4 which is Grötzsch's graph. By the previous theorem all M_n 's are triangle free (or $\omega(M_n) = 2$) and $\chi(M_n) = n$. Also, the fractional chromatic numbers form an infinite sequence where $\chi_f(M_2) = 2$ and $\chi_f(M_{n+1}) = \chi_f(M_n) + (\chi_f(M_n))^{-1}$. This sequence grows like $\sqrt{2n}$ in the sense that $\frac{\chi_f(M_n)}{\sqrt{2n}} \rightarrow_{n \rightarrow \infty} 1$.

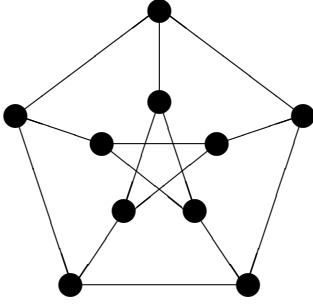
Therefore, the sequence of Mycielski's graphs has the property that $(\chi(M_n) - \chi_f(M_n)) \rightarrow_{n \rightarrow \infty} \infty$ and $(\omega_f(M_n) - \omega(M_n)) \rightarrow_{n \rightarrow \infty} \infty$. Moreover, Fisher [11]

showed that the fractional chromatic numbers of Mycielski's graphs have denominators on the order of e^{cn} , where c is a constant.

Despite the colouring complexity of the Mycielski's graphs they are, as triangle-free graphs, trivially fractionally homomorphic to K_2 . On the other hand $\forall k \exists n$ such that $K_k \rightarrow_f M_n$.

3.4.2 Petersen's Graph

Petersen's graph \mathcal{P} is defined as $V(\mathcal{P}) = \{u_1, \dots, u_5, v_1, \dots, v_5\}$, where $\{u_1, \dots, u_5\}$ induce a 5-cycle, $\{v_1, \dots, v_5\}$ induce a star (i.e. v_1, v_3, v_5, v_2, v_4 form a 5-cycle in that order) and for each $i = 1, \dots, 5$ $u_i \sim v_i$. Formally $E(\mathcal{P}) = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_1), (v_1, v_3), (v_3, v_5), (v_5, v_2), (v_2, v_4), (v_4, v_1), (u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4), (u_5, v_5)\}$.



Petersen's graph is triangle free. Therefore it is fractionally homomorphic to K_2 (take a fractional clique x^* of size $|V(\mathcal{P})|$ in $\mathcal{P} \circ K_2$ such that $x_{st_1}^* = x_{st_2}^* = 1/2$ for all $s \in \mathcal{P}$ and where t_1 and t_2 are the two vertices of K_2).

We will also show the following

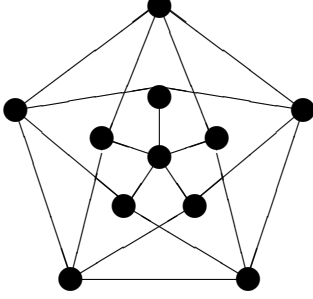
$$\omega(\mathcal{P}) = 2 \quad \omega_f(\mathcal{P}) = \frac{5}{2} = \chi_f(\mathcal{P}) \quad \chi(\mathcal{P}) = 3 \quad (3.123)$$

One possible 3-colouring of \mathcal{P} is the partition $A_1 = \{u_1, u_3, v_2\}$, $A_2 = \{u_2, u_4, v_1, v_5\}$, $A_3 = \{u_5, v_3, v_4\}$. One fractional clique of size $5/2$ is x such that $x_{v_i} = 1/2$ for all $i = 1, \dots, 5$ and zero otherwise. A fractional colouring of size $5/2$ is y such that $y_{\{u_1, u_4, v_2, v_3\}} = y_{\{u_2, u_5, v_3, v_4\}} = y_{\{u_3, u_1, v_4, v_5\}} = y_{\{u_4, u_2, v_5, v_1\}} = y_{\{u_5, u_3, v_1, v_2\}} = 1/2$ and zero otherwise.

3.4.3 Grötzsch's Graph

Grötzsch's graph \mathcal{G} is isomorphic to M_4 . More formally, it is defined as $V(\mathcal{G}) = \{u_1, \dots, u_5, v_1, \dots, v_5, w\}$ and $E(\mathcal{G}) = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_1),$

$(u_1, v_2), (u_2, v_3), (u_3, v_4), (u_4, v_5), (u_5, v_1), (v_1, u_2), (v_2, u_3), (v_3, u_4), (v_4, u_5), (v_5, u_1),$
 $(v_1, w), (v_2, w), (v_3, w), (v_4, w), (v_5, w)\}$



Grötzsch's graph is also triangle-free and therefore it is fractionally homomorphic to K_2 (similarly as for Petersen's graph). The following is also true

$$\omega(\mathcal{G}) = 2 \quad \omega_f(\mathcal{G}) = \frac{29}{10} = \chi_f(\mathcal{G}) \quad \chi(\mathcal{G}) = 4 \quad (3.124)$$

A 4-colouring of \mathcal{G} is the following partition $A_1 = \{u_1, u_3, v_1, v_3\}$, $A_2 = \{u_2, u_4, v_2, v_4\}$, $A_3 = \{u_5, v_5\}$, $A_4 = \{w\}$. A fractional clique x in \mathcal{G} whose size is $29/10$ is defined as follows: $x_{u_i} = 3/10$, $x_{v_i} = 2/10$ for all $i = 1, \dots, 5$ and $x_w = 4/10$. A dual fractional colouring of size $29/10$ is Y such that $y_{\{u_1, u_4, v_1, v_4\}} = y_{\{u_2, u_5, v_2, v_5\}} = y_{\{u_3, u_1, v_3, v_1\}} = y_{\{u_4, u_2, v_4, v_2\}} = y_{\{u_5, u_3, v_5, v_3\}} = 3/10$, $y_{\{u_1, u_4, w\}} = y_{\{u_2, u_5, w\}} = y_{\{u_3, u_1, w\}} = y_{\{u_4, u_2, w\}} = y_{\{u_5, u_3, w\}} = 2/10$ and $y_{\{v_1, v_2, v_3, v_4, v_5\}} = 4/10$ (there are no other maximal independent sets in \mathcal{G}).

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