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Katedra aplikované matematiky

DIPLOMOVÁ PRÁCE

Jan Hladký

Szemerédi Regularity Lemma a jeho aplikace v kombinatorice – (Szemerédi Regularity Lemma and its Applications in Combinatorics)

Vedoucí: RNDr. Daniel Král', Ph.D., Institut teoretické informatiky
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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

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Jan Hladký

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Autor: Jan Hladký
Katedra: Katedra aplikované matematiky
Vedoucí diplomové práce: RNDr. Daniel Král', Ph.D., Institut teoretické informatiky
E-mail vedoucího: kral@kam.mff.cuni.cz

Abstrakt: V práci podáme důkaz domněnky Loebla, Komlóse a Sósové (1995) pro husté grafy. Dokážeme následujíci tvrzení. Pro libovolné $q > 0$ existuje číslo n_0 takové, že pokud má libovolný graf G řádu $n > n_0$ alespoň polovinu vrcholů se stupněm alespoň $k > qn$, pak G obsahuje každý strom na $k+1$ vrcholech jako podgraf. Tím vylepšujeme předchozí výsledky autorů Zhao (2002) a Piguet a Stein (2007). Ukážeme, že v jistých případech lze předpoklady věty oslabit. Je diskutována dolní mez k problému.

Jako důsledek hlavní věty dostaneme těsný odhad Ramseyova čísla dvou stromů.

Důkaz hlavní věty kombinuje vnořovací techniku založenou na Regularity Lemmatu s Metodou stability.

Výsledku bylo dosaženo ve společné práci s Dianou Piguet.

Klíčová slova: domněnka Loebla, Komlóse a Sósové; extremální teorie grafů; Szemerédi Regularity Lemma; Ramseyovo číslo stromů

Title: Szemerédi Regularity Lemma and its Applications in Combinatorics
Author: Jan Hladký
Department: Department of Applied Mathematics
Supervisor: RNDr. Daniel Král', Ph.D., Institute for Theoretical Computer Science
Supervisor's e-mail address: kral@kam.mff.cuni.cz

Abstract: In the thesis we provide a solution of the Loebl-Komlós-Sós Conjecture (1995) for dense graphs. We prove that for any $q > 0$ there exists a number $n_0 \in \mathbb{N}$ such that for any $n > n_0$ and $k > qn$ the following holds. Let G be a graph of order n with at least $n/2$ vertices of degree at least k. Then any tree of order $k+1$ is a subgraph of G. This improves previous results by Zhao (2002), and Piguet and Stein (2007). A strengthened version of the above theorem together with a lower bound for the problem is discussed. As a corollary a tight bound on the Ramsey number of two trees is stated.

The proof of the main theorem combines a Regularity-Lemma based embedding technique with the Stability Method of Simonovits.

Results presented here are based on joint work with Diana Piguet.

Keywords: Loebl-Komlós-Sós Conjecture; Extremal Graph Theory; Szemerédi Regularity Lemma; Ramsey number of trees

Contents

1 Introduction

Embedding problems play central role in Graph Theory. A variety of graph embeddings (subgraphs, minors, subdivisions, immersions, etc) have been studied extensively. A graph (finite, undirected, loopless, simple; here as well as in the rest of the thesis) H *embeds* in a graph G if there exists an injective mapping $\phi : V(H) \to V(G)$ which preserves edges of H, i. e., $\phi(x)\phi(y) \in E(G)$ for every edge $xy \in E(H)$. As a synonym we say that G *contains* H (*as a subgraph*) and write $H \subseteq G$. Let $\mathscr{H}$ be a family of graphs. The graph G is $\mathscr{H}$*-universal* if it contains every graph from $\mathscr{H}$. This fact is denoted by $\mathscr{H} \subseteq G$.

In this thesis we investigate embeddings of trees. This topic has received considerable attention during the last 40 years. The class $\mathscr{T}_k$ consists of all trees of order k. One can ask which properties force a graph H to be $\mathscr{T}_k$-universal. Loebl, Komlós and Sós considered in [9] the median degree of H.

Conjecture 1.1 (LKS Conjecture)**.** *Let G be a graph of order n. If at least $n/2$ of the vertices of G have degree at least k, then $\mathscr{T}_{k+1} \subseteq G$.*

The main result of this thesis is to prove the LKS Conjecture for "k linear in n". For the exact statement see our main result, Theorem 1.4.

The bound on k of the minimal degree of high degree vertices cannot be decreased. Indeed, if G is a graph in which half of its vertices have degree exactly $k-1$, then it does not contain a star $K_{1,k}$. On the other hand, it is suspected that the number of vertices of degree at least k can be lowered a little bit. This was first raised by Zhao [22]. Discussion on the lower bound is given in Section 9.

There have been several partial results concerning the LKS Conjecture. In [4], Bazgan Li and Woźniak proved the conjecture for paths. Piguet and Stein [17] proved that the LKS Conjecture is true when restricted to the class of trees of diameter at most 5, improving upon a result of Barr and Johansson [3] and Sun [20]. There are several results proving the LKS Conjecture under additional assumptions on the hosting graph.

Soffer [19] showed that the conjecture is true if the hosting graph has girth at least 7, Dobson [7] proved the conjecture when the complement of the hosting graph does not contain $K_{2,3}$.

A special case of the LKS Conjecture is when $k = n/2$. This is often referred to in the literature as the ($n/2$-$n/2$-$n/2$) Conjecture, or the Loebl Conjecture. Zhao [22] proved the ($n/2$-$n/2$-$n/2$) Conjecture for large graphs.

Theorem 1.2. *There exists a number n_0 such that if a graph G of order $n > n_0$ has at least $n/2$ of the vertices of degrees at least $n/2$, then $\mathscr{T}_{\lfloor n/2 \rfloor +1} \subseteq G$.*

An approximate version of the LKS Conjecture was proven by Piguet and Stein [16].

Theorem 1.3. *For any $q > 0$ there exists a number n_0 and a function $f : \mathbb{N} \to \mathbb{R}$, $f \in o(1)$ such that for any $n > n_0$ and $k > qn$ the following holds. If G is a graph of order n with at least $(1/2 + f(n))n$ vertices of degree at least $(1 + f(n))k$, then $\mathscr{T}_{k+1} \subseteq G$.*

In this thesis we strengthen Theorem 1.3 by removing the $o(1)$ term.

Theorem 1.4 (Main Theorem)**.** *For any $q > 0$ there exists a number $n_0 = n_0(q)$ such that for any $n > n_0$ and $k > qn$ the following holds: if G is a graph of order n with at least $n/2$ vertices of degree at least k, then $\mathscr{T}_{k+1} \subseteq G$.*

In fact, the proof of Theorem 1.4 will yield that the requirement on the number of vertices of large degree can be relaxed in the case when n/k is far from being an integer.

Theorem 1.5. *For any $q_2 > q_1 > 0$ such that the interval $[1/q_2, 1/q_1]$ does not contain an integer, there exist numbers $\varepsilon = \varepsilon(q_1, q_2) > 0$ and n_0 such that for any $n > n_0$ and $k \in (q_1 n, q_2 n)$ the following holds: if G is a graph of order n with at least $(1/2 - \varepsilon)n$ vertices of degree at least k, then $\mathscr{T}_{k+1} \subseteq G$.*

We explicitly prove only Theorem 1.4 in the thesis. In Section 2 we sketch how the proof method can be revised to give Theorem 1.5. However, determining the correct value of $\varepsilon(q_1, q_2)$ remains open. Note also that Theorem 1.4 has slightly weaker assumptions on G than Theorem 1.2 when reduced to the case $k = \lfloor n/2 \rfloor$—when n is odd, the number of large vertices in Theorem 1.4 is smaller by one compared to Theorem 1.2.

Recently, we learned that Oliver Cooley announced an independent proof of Theorem 1.4.

The parameter which is considered in the LKS conjecture is the median degree. If we replace it by the average degree, we obtain a famous conjecture of Erdős and Sós, which dates back to 1963.

Conjecture 1.6 (ES Conjecture)**.** *Let G be a graph of order n with more than $(k-2)n/2$ edges. Then $\mathscr{T}_k \subseteq G$.*

If true, the conjecture is sharp. After several partial results on the problem, a breakthrough was achieved by Ajtai, Komlós, Simonovits and Szemerédi [1], who announced a proof of the Erdős-Sós Conjecture for large k.

Theorem 1.7. *There exists a number k_0 such that for any $k > k_0$ the following holds: if a graph G of order n has more than $(k-2)n/2$ edges, then $\mathscr{T}_k \subseteq G$.*

The proof of Theorem 1.7 by Ajtai et al. has two parts. One part settles the dense version of the problem; the statement is analogous to Theorem 1.4. The other part deals with the case when $k/n < q_0$ for some fixed value q_0. We have indications that the same approach might work for the LKS Conjecture. Thus our Theorem 1.4 may be one of two essential ingredients in a proof of the LKS Conjecture.

The current work utilizes techniques of Zhao [22] and of Piguet and Stein [16]. We postpone a detailed discussion of similarities between our approach and theirs, and of our own contribution until Section 2.

1.1 Ramsey number of a tree

We show in this section the connection between the LKS Conjecture and the Ramsey number of trees. For two graphs F and H we write $R(F,H)$ for the *Ramsey number* of the graphs F, H. This is the smallest number m such that in any red/blue edge-coloring of K_m there is a red copy of F or a blue copy of H. For two families of graphs $\mathscr{F}$ and $\mathscr{H}$ the Ramsey number $R(\mathscr{F},\mathscr{H})$ is the smallest number m such that in any red/blue edge-coloring of K_m the graph induced by the red edges is $\mathscr{F}$-universal, or the graph induced by the blue edges is $\mathscr{H}$-universal. We shall show how Theorem 1.4 implies an almost tight upper bound (up to an additive error of one) on the Ramsey number of trees, partially answering a question of Erdős, Füredi, Loebl and Sós [9].

For a fixed number $p \in (0,1/2)$ consider two numbers ℓ_1 and ℓ_2 such that $\ell_1/\ell_2 \in (p,1/p)$ and $\ell_1,\ell_2 > n_0$, where $n_0 = n_0(p/2)$ from Theorem 1.4. Consider any red/blue edge-coloring of the graph $K_{\ell_1+\ell_2}$. We say that a vertex $v \in V(K_{\ell_1+\ell_2})$ is red if it incident to at least ℓ_1 red edges. Similarly, $v \in V(K_{\ell_1+\ell_2})$ is blue if it incident to at least ℓ_2 blue edges. Each vertex of $K_{\ell_1+\ell_2}$ is either red or blue. Thus we have at least half of the vertices of $K_{\ell_1+\ell_2}$ that are red, or at least half of the vertices that are blue. Theorem 1.4 can be applied to the graph induced by the majority color. We conclude that $R(\mathscr{T}_{\ell_1+1},\mathscr{T}_{\ell_2+1}) \le \ell_1+\ell_2$.

For the lower bound, first consider the case when at least one of ℓ_1 and ℓ_2 is odd. It is a well-known fact that there exists a red/blue edge-coloring of $K_{\ell_1+\ell_2-1}$ such that the red degree of every vertex is ℓ_1-1. Neither a red copy of K_{1,ℓ_1} nor a blue copy of K_{1,ℓ_2} is contained in $K_{\ell_1+\ell_2-1}$ with this coloring. Thus $R(\mathscr{T}_{\ell_1+1},\mathscr{T}_{\ell_2+1}) > \ell_1+\ell_2-1$. A construction in a similar spirit shows that $R(\mathscr{T}_{\ell_1+1},\mathscr{T}_{\ell_2+1}) > \ell_1+\ell_2-2$, if ℓ_1 and ℓ_2 are even. We have

$$R(\mathscr{T}_{\ell_1+1},\mathscr{T}_{\ell_2+1}) = \ell_1+\ell_2\,, \quad \text{if } \ell_1 \text{ is odd or } \ell_2 \text{ is odd, and} \tag{1.1}$$

$$\ell_1+\ell_2-1 \le R(\mathscr{T}_{\ell_1+1},\mathscr{T}_{\ell_2+1}) \le \ell_1+\ell_2\,, \quad \text{otherwise.} \tag{1.2}$$

Let us note that an easy consequence of the ES Conjecture would be that the lower bound in (1.2) is attained.

Ramsey numbers of several other classes of trees have been investigated; the reader is referred to a survey of Burr [5] and to newer results in [8, 10, 12].

2 Outline of the proof

Theorem 1.4 is proved by iterating the following procedure in steps $i = 1,2,3,\ldots$. At each step i, we find a set $Q \subseteq V(G) \setminus \bigcup_{j\le i} V_j$ such that at least about half of the vertices in Q are large (i. e., of degree at least k). Using the Regularity Lemma, we try to embed a given tree $T \in \mathscr{T}_{k+1}$ in Q. If we do not succeed, then we can extract from Q a subset $V_{i+1} \subseteq Q$ of size approximately k, that is nearly isolated from the rest of the of the graph, and for which at least half of the vertices are large. If we cannot embed $T \in \mathscr{T}_{k+1}$ in any of the iterating steps (i. e., $V(G) \setminus \bigcup_i V_i \cong \emptyset$), we obtain a particular configuration of the graph G, called the *Extremal Configuration*. In this case, we prove that $T \subseteq G$, without the use of the Regularity Lemma.

In the remainder of the overview, we explain in more detail the proof of the part using the Regularity Lemma, as well as the part when G is in the Extremal configuration.

The Regularity Lemma Part. Before applying the Regularity Lemma itself, we first resolve two simple cases. The first one is when Q is close to a bipartite graph with one of its color-classes being the large vertices (see Proposition 4.2). The second case (see Proposition 4.3) is when the tree T is locally unbalanced (see definition on page 10). In both cases an easy argument shows that $T \subseteq G$.

We apply the Regularity Lemma to the graph G and obtain a cluster graph $\mathbf{G}$. We apply a Tutte-type proposition (Proposition 6.4) to the subgraph induced by clusters in Q, which guarantees the existence of one of two certain matching structures in $\mathbf{G}$. Both expose a matching M in the cluster graph, and two clusters A and B that are adjacent in $\mathbf{G}$ and that have high average degree to the matching M. These structures are called Case I and Case II. The principle of the embedding is to use the edges of M to embed parts of the tree in them, and use the clusters A and B to connect these parts.

The Extremal Case Configuration. In the Extremal case we are given disjoint sets $V_1, \ldots, V_i \subseteq V(G)$ such that each of them has size approximately k, contains at least nearly $k/2$ large vertices, and each set V_j is almost isolated from the rest of the graph.

If the sets $V_1, \ldots, V_i$ exhaust the whole graph G, we are able to show $T \subseteq G$. We find a set V_{i_0} so that most of T can be mapped to V_{i_0}. We may need to use the few edges that interconnect distinct sets V_j to distribute parts of the tree T outside V_{i_0}. The way of finding these “bridges” depends on the structure of the tree T.

If $V_1, \ldots, V_i$ do not exhaust G, the method remains the same. However, it has two possible outputs. Either we show that $T \subseteq G$ or we are able to exhibit a set $Q \subseteq V \setminus \bigcup_{j \leq i} V_j$ allowing the next step of the iteration.

Strengthening of Theorem 1.4—Theorem 1.5. The only place where we use the exact bound on the number of large vertices is the last step of the Extremal case. That is, the whole vertex set $V(G)$ is decomposed into sets V_j, each of them almost exactly of size k. But such a decomposition cannot exist when $k \in (q_1 n, q_2 n)$, $[1/q_2, 1/q_1] \cap \mathbb{N} = \emptyset$. This suffices to prove Theorem 1.5.

Relation to previous work. The proof of Theorem 1.4 is inspired by techniques used to prove Theorem 1.3 ([16]) and Theorem 1.2 ([22]). Both these papers build on a seminal paper of Ajtai, Komlós and Szemerédi [2] where an approximate version of the $(n/2-n/2-n/2)$-Conjecture is proven. In [2] the basic strategy is outlined.

In [22] the aproach of Ajtai, Komlós and Szemerédi is combined with the Stability method of Simonovits [18]. One extremal case is identified, and solved without the use of the Regularity Lemma.

The main contribution of [16] is a more general Tutte-type proposition, which is applicable even when $k/n < 1/2$.

In this thesis we further strengthen the Tutte-type proposition from [16]. The Extremal case is an extensive generalization of the Extremal case from [22].

Algorithmic questions. Let us remark that our proof of Theorem 1.4 yields a polynomial time algorithm for finding an embedding of any tree $T \in \mathscr{T}_{k+1}$ in G, given that k and G satisfy the conditions of Theorem 1.4. Indeed, it is easily checked that all existential results we use (Regularity Lemma, and various matching theorems) are known to have polynomial-time constructive algorithmic counterparts. We omit details.

3 Notation and preliminaries

For $n \in \mathbb{N}$ we write $[n] = \{1, 2, \dots, n\}$. The symbol $\div$ means the symmetric difference of two sets. The function $\mathrm{ci} : \mathbb{R} \to \mathbb{Z}$ is the *closest integer function* defined by $\mathrm{ci}(x) = \lfloor x \rfloor$ if $x - \lfloor x \rfloor < 0.5$, and $\mathrm{ci}(x) = \lceil x \rceil$ otherwise.

We use standard graph-theory terminology and notation, following Diestel's book [6]. We define here only those symbols which are not used there. The order of a graph H and the number of its edges are denoted by $v(H)$ and $e(H)$, respectively. We write $H[X,Y]$ for the bipartite graph induced by the disjoint vertex sets X and Y, and $E(X,Y)$ for the set of the edges with one end-vertex in X and the other in Y. We write $e(X,Y) = |E(X,Y)|$. For a vertex x and a vertex set X we define $\deg(x,X) = \deg_X(x) = e(\{x\},X)$. For two sets $X,Y \subseteq V(H)$ we define the *average degree* from X to Y by $\overline{\deg}(X,Y) = e(X,Y)/|X|$. We write $\overline{\deg}(X)$ as a short for $\overline{\deg}(X,V(H))$. We define two variants of the minimum degree of H. In the following, X and Y are arbitrary vertex sets.

$$\delta(X) = \min_{v \in X} \deg(v) \text{ , and}$$
$$\delta(X,Y) = \min_{v \in X} \deg(v,Y) \text{ .}$$

$\mathrm{N}(x)$ is the set of neighbors of the vertex x, $\mathrm{N}_X(x)$ is the neighborhood of x restricted to a set X, i. e., $\mathrm{N}_X(x) = \mathrm{N}(x) \cap X$, and $\mathrm{N}(X)$ is the set of all vertices in H which are adjacent to at least one vertex from X, i. e., $\mathrm{N}(X) = \bigcup_{v \in X} \mathrm{N}(v)$.

Let $P = v_1 v_2 \dots v_\ell$ be a path. For arbitrary sets of vertices $X_1, X_2, \dots, X_\ell$ we say that P is a *$X_1 \leftrightarrow X_2 \leftrightarrow \dots \leftrightarrow X_\ell$-path* if $v_i \in X_i$ for every $i \in [\ell]$. An edge xy is an $X \leftrightarrow Y$ edge if $x \in X$ and $y \in Y$ and a matching M is a $X \leftrightarrow Y$ matching if its every edge is an $X \leftrightarrow Y$ edge.

The *weighted graph* is a pair (H, ω), where H is a graph and $\omega : E(H) \to (0, +\infty)$ is its weight function. For two sets $X, Y \subseteq V(H)$ the *weight of the edges crossing from X to Y* is defined by $\overline{e}^{\omega}(X,Y) = \sum_{xy \in E(X,Y)} \omega(xy)$. Denote by $\overline{\deg}^{\omega}$ the weighted degree, $\overline{\deg}^{\omega}(v) = \sum_{u \in V(H), vu \in E(H)} \omega(vu)$. For a vertex v and a vertex set X we define $\overline{\deg}^{\omega}(v, X)$ analogously to $\deg(v, X)$.

We omit rounding symbols when this does not effect the correctness of calculations.

3.1 Trees

Let F be a rooted tree with a root $r \in V(F)$. We define a partial order $\preceq$ on $V(F)$ by saying that $a \preceq b$ if and only if the vertex b lies on the path connecting a with r. If $a \preceq b$ we say that *a is below b*. A vertex a is a *child of b* if $a \preceq b$ and $ab \in E(F)$. And, in the other way, the vertex b is a *parent of a*. $\mathrm{Ch}(b)$ denotes the set of children of b. The parent of a vertex a is denoted $\mathrm{Par}(a)$ (note that $\mathrm{Par}(a)$ is undefined if $a = r$). We extend the definitions of $\mathrm{Ch}(\cdot)$ and $\mathrm{Par}(\cdot)$ to an arbitrary set $U \subseteq V(F)$ by $\mathrm{Par}(U) = \bigcup_{u \in U} \mathrm{Par}(u)$ and $\mathrm{Ch}(U) = \bigcup_{u \in U} \mathrm{Ch}(u)$. We say that a tree $F_1 \subseteq F$ is *induced* by a vertex $x \in V(F)$ if $V(F_1) = \{v \in V(F) : v \preceq x\}$ and we write $F_1 = F(r, \downarrow x)$, or if the root is obvious from the context $F_1 = F(\downarrow x)$. A subtree F_0 of F is a *full-subtree with the root $y \in V(F)$*, if there exists a set $C \subseteq \mathrm{Ch}(y)$, $C \neq \emptyset$ such that $F_0 = F[\{y\} \cup \bigcup_{b \in C} \{v : v \preceq b\}]$. We never refer to y as to a leaf of the full subtree F_0, and of the tree F_1 induced by y, even though it may be a leaf of F_0 and of F_1 in the usual sense. A tree $F_2 \subseteq F$ is an *end subtree* if there exists a vertex $w \in V(F)$ such that $F_2 = F(\downarrow w)$. If a subtree $F_3 \subseteq F$ is not an end subtree, then we call it an *interior subtree*.

Fact 3.1. *Let (F, r) be a rooted tree of order m with ℓ leaves.*

1. *For any integer m_0, $0 < m_0 \leq m$, there exists a full-subtree F_0 of F of order $\tilde{m} \in [m_0/2, m_0]$.*

2. *For any integer ℓ_0, $0 < \ell_0 \leq \ell$, there exists a full-subtree F_0 of F with $\tilde{\ell}$ leaves, where $\tilde{\ell} \in [\ell_0/2, \ell_0]$.*

Proof. 1. We shall move sequentially the candidate r_0 for the root of F_0 downwards (in $\preceq$), starting with $r_0 = r$. In the first step we have $v(F(\downarrow r_0)) = m \geq m_0/2$. If $v(F(\downarrow c)) < m_0/2$ for every $c \in \mathrm{Ch}(r_0)$ then we can find a set $C \subseteq \mathrm{Ch}(r_0)$ of vertices such that the full-subtree $F_0 = F[\{r_0\} \cup \bigcup_{c \in C} \{v : v \preceq c\}]$ has order in the interval $[m_0/2, m_0]$. Otherwise, there exists a vertex $c \in \mathrm{Ch}(r_0)$ such that $v(F(\downarrow c)) \geq m_0/2$. We reset $r_0 = c$ and continue.

2. This is analogous.

□

Fact 3.1 is sometimes used without the root of the tree being specified. Then, any internal vertex of the tree can serve as a root.

For any tree F we write F_e and F_o for the vertices of its two color classes with F_e being the larger one. We define the *gap* of the tree F as $\mathrm{gap}(F) = |F_e| - |F_o|$. For a tree F, a partition of its vertices into sets U_1 and U_2 is called *semiindependent* if $|U_1| \leq |U_2|$ and U_2 is an independent set. Furthermore, the *discrepancy* of (U_1,U_2) is $\mathrm{disc}(U_1,U_2) = |U_2| - |U_1|$ and the discrepancy of F is

$$\mathrm{disc}(F) = \max\{\mathrm{disc}(U_1,U_2) \,:\, (U_1,U_2) \text{ is semiindependent}\} .$$

Clearly, $\mathrm{gap}(F) \leq \mathrm{disc}(F)$.

Fact 3.2. *Let (U_1,U_2) be a semiindependent partition of a tree F, $v(F) > 1$. Then U_2 contains at least $|U_2| - |U_1| + 1$ leaves.*

Proof. We root F at an arbitrary vertex $x \in U_1$. Let U_2' be the set of internal vertices in U_2. Since each vertex in U_2' has at least one child in $U_1 \setminus \{x\}$ and these children are (for distinct vertices in U_2') distinct, we obtain $|U_1 \setminus \{x\}| \geq |U_2'|$. Hence the number of leaves in U_2 is at least $|U_2| - |U_1| + 1$. □

Lemma 3.3. *Let r be a vertex of a tree T, and let (U_1,U_2) be any semiindependent partition of T. Let $\mathscr{K}$ be a subset of the components of the forest $T - \{r\}$. Then*

1. $||V(\mathscr{K}) \cap T_e| - |V(\mathscr{K}) \cap T_o|| \leq \mathrm{disc}(T) + 1$.
2. $|V(\mathscr{K}) \cap U_2| - |V(\mathscr{K}) \cap U_1| \leq \mathrm{disc}(T) + 1$.

Proof. We prove only Part 1, Part 2 being analogue. The statement is obvious when $|V(\mathscr{K}) \cap T_e| - |V(\mathscr{K}) \cap T_o| = 0$. Suppose that $|V(\mathscr{K}) \cap T_a| - |V(\mathscr{K}) \cap T_b| = \ell > 0$, where $a,b \in \{\mathrm{e},\mathrm{o}\}$, $a \neq b$ is a choice of color-classes. It is enough to exhibit a semiindependent partition (U_1,U_2) of the tree T with $|U_2| - |U_1| \geq ||V(\mathscr{K}) \cap T_e| - |V(\mathscr{K}) \cap T_o|| - 1$. Partition the components of the forest $T - \{r\}$ that are not included in $\mathscr{K}$ into two families $\mathscr{A}$ and $\mathscr{B}$ so that $\mathscr{A}$ contains those components $K \notin \mathscr{K}$ for which $|V(K) \cap T_a| \geq |V(K) \cap T_b|$, and $\mathscr{B}$ contains those components $K \notin \mathscr{K}$ for which $|V(K) \cap T_a| < |V(K) \cap T_b|$. Obviously, the partition below satisfies the requirements.

$$\begin{aligned} U_1 &= \{r\} \cup (V(\mathscr{K}) \cap T_b) \cup (V(\mathscr{A}) \cap T_b) \cup (V(\mathscr{B}) \cap T_a) , \\ U_2 &= (V(\mathscr{K}) \cap T_a) \cup (V(\mathscr{A}) \cap T_a) \cup (V(\mathscr{B}) \cap T_b) . \end{aligned}$$

□

Fact 3.4. *Let F be a tree with ℓ leaves. Then F has at most $\ell - 2$ vertices of degree at least three.*

Proof. We partition $V(F)$ into the set of leaves V_1, the set V_2 of vertices of degree two, and the set V_3 of vertices of degree at least three. The handshaking lemma applied to F yields that

$$2v(F)-2=\sum_v \deg(v)\geq |V_1|+2|V_2|+3|V_3|=2v(F)-\ell+|V_3|\ .$$

The statement readily follows. □

3.2 Greedy embeddings

Given a tree F and a graph H there are several situations when one can embed F in H *greedily*. For example, if $\delta(H)\geq v(F)-1$, then we embed the root of F in an arbitrary vertex of H and extend the embedding levelwise. An analogous procedure works if H is bipartite, $H=(V_1,V_2;E)$, and $\delta(V_1,V_2)\geq |F_{\mathrm{e}}|, \delta(V_2,V_1)\geq |F_{\mathrm{o}}|$. The fact stated below generalizes the greedy procedure.

Fact 3.5. *Let (U_1,U_2) be a semiindependent partition of a tree F. If there exist two disjoint sets of vertices V_1 and V_2 of a graph H such that $\min\{\delta(V_1,V_2),\delta(V_1,V_1),\delta(V_2,V_1)\}\geq |U_1|$ and $\delta(V_1)\geq v(F)-1$, then $F\subseteq H$.*

Proof. The statement is trivial when $v(F)=1$. In the rest, assume that $v(F)>1$. The set U_2^{l} denotes the leaves of U_2. By Fact 3.2, $|U_2\setminus U_2^{\mathrm{l}}|\leq |U_1|-1$. We embed greedily $F-U_2^{\mathrm{l}}$ in H, mapping the vertices from U_1 to V_1 and the vertices from $U_2\setminus U_2^{\mathrm{l}}$ to V_2. We argue that the greedy procedure works. If we have just embedded a vertex $u\in U_1$ then we can extend the embedding to all vertices $\mathrm{N}(u)\cap U_1$ since $\delta(V_1,V_1)\geq |U_1|$. The embedding can be extended to all vertices from $\mathrm{N}(u)\cap(U_2\setminus U_2^{\mathrm{l}})$ since $\delta(V_1,V_2)\geq |U_2\setminus U_2^{\mathrm{l}}|$. If we have just embedded a vertex $w\in U_2\setminus U_2^{\mathrm{l}}$ then we can extend the embedding to all vertices from $\mathrm{N}(w)$ since $\delta(V_2,V_1)\geq |U_1|$. The leaves U_2^{l} are embedded last, using high degrees of the vertices in V_1. □

3.3 Matchings

Let us state a simple corollary of Hall's Matching Theorem.

Proposition 3.6. *Let $K=(W_1,W_2;J)$ be a bipartite graph such that $\delta(K)\geq |W_1|/2$ and $|W_1|\leq |W_2|$. Then K contains a matching covering W_1.*

3.4 A number-theoretic proposition

Proposition 3.7. *Let I be a finite nonempty set, and let $a,b,\Delta>0$. For $i\in I$, let $\alpha_i,\beta_i\in(0,\Delta]$. Suppose that*

$$\frac{a}{\sum_{i\in I}\alpha_i}+\frac{b}{\sum_{i\in I}\beta_i}\leq 1\ .$$

Then I can be partitioned into two sets I_a and I_b so that $\sum_{i\in I_a}\alpha_i>a-\Delta$, and $\sum_{i\in I_b}\beta_i\geq b$.

Proof. The reader may find a straightforward proof in [16]. $\square$

3.5 Specific notation

A graph H is said to have the *LKS-property* (with parameter k) if at least half of its vertices have degrees at least k, i. e., we have $|L^H| \geq v(H)/2$, where $L^H = \{v \in V(H) : \deg_H(v) \geq k\}$.

When we refer to q, n_0, n, k or G in the rest of the thesis, we always refer to the objects from the statement of Theorem 1.4. The vertex set of G is denoted by V. We partition $V = L \cup S$, where $L = \{v \in V : \deg(v) \geq k\}$ and $S = \{v \in V : \deg(v) < k\}$. We call vertices from L *large* and vertices from S *small*. The hypothesis of Theorem 1.4 implies that $|L| \geq n/2$. Finally T denotes a tree of order $k+1$ which we want to embed in G.

Statements like "there exists a number $\gamma > 0$ such that a property $\mathscr{P}(\gamma)$ holds for any graph G" should read as "given $q > 0$, there exists a number $\gamma > 0$ such that a property $\mathscr{P}(\gamma)$ holds for any graph G of order at least $n_0(q)$".

4 Proof of the Main Theorem (Theorem 1.4)

We first need to state some auxiliary propositions. For the first proposition, we need to introduce the notion of (β, σ)-Extremality. For two numbers $\beta, \sigma \in (0,1)$, a decomposition of the vertex set $V = V_1 \cup V_2 \cup \ldots \cup V_\lambda \cup \tilde{V}$ is (β, σ)-*Extremal* if

- $\lambda \geq 1$.
- $(1-\beta)k < |V_i| < (1+\beta)k$ for each $i \in [\lambda]$.
- $\tilde{V} = \emptyset$ or $|\tilde{V}| > \sigma k$.
- $e(V_i, V \setminus V_i) < \beta k^2$ for each $i \in [\lambda]$, and $e(\tilde{V}, V \setminus \tilde{V}) < \beta k^2$.
- $(1/2 - \beta)k < |V_i \cap L|$ for each $i \in [\lambda]$.
- $|\tilde{V} \cap L| \leq (1/2 - \sigma)|\tilde{V}|$.

Proposition 4.1. *There exists a constant $c_{\mathbf{E}} > 0$ such that the following holds. If G admits a (β, σ)-Extremal partition $V_1, \ldots, V_\lambda, \tilde{V}$ for $\beta, \sigma \leq c_{\mathbf{E}}$, $\beta \ll \sigma$, then $\mathscr{T}_{k+1} \subseteq G$, or there exists a set $Q \subseteq \tilde{V}$ such that*

- $|Q| > k/2$.
- $|Q \cap L| > |Q|/2$.
- $e(Q, V \setminus Q) < \sigma k^2$.

Proposition 4.1 will be proved in Section 8. The next proposition is referred to as the Special Case.

Proposition 4.2. *For all $q, c_{\mathbf{E}} > 0$, there exists a number $c_{\mathbf{S}} > 0$, $c_{\mathbf{S}} \ll c_{\mathbf{E}}$ such that if there exists a set $\bar{V} \subseteq V$ with the following properties*

- $|\bar{V}| > \sqrt[4]{c_{\mathbf{S}}}k$,
- $e(\bar{V}, V \setminus \bar{V}) < c_{\mathbf{S}}k^2$,
- $(1/2 - c_{\mathbf{S}})|\bar{V}| < |\bar{V} \cap L|$, *and*
- $e(G[\bar{V} \cap L]) < c_{\mathbf{S}}k^2$,

then $\mathscr{T}_{k+1} \subseteq G$.

Proof of Proposition 4.2 is given in Section 5. The following proposition is will allow us to reduce trees which are locally unbalanced from further considerations. Let us introduce the notion (un)balanced forest now.

For a number $c \in (0, 1/2)$ we say that a family $\mathscr{C}$ of vertex disjoint subtrees of a tree $T \in \mathscr{T}_{k+1}$ is *c-balanced* if the forest formed by the trees $t \in \mathscr{C}$ with $|t_{\mathrm{o}}| > c \cdot v(t)$ is of order at least ck, i. e.,

$$\sum_{\substack{t \in \mathscr{C} \\ |t_{\mathrm{o}}| > cv(t)}} v(t) \geq ck \ .$$

The family $\mathscr{C}$ is *c-unbalanced* if it is not c-balanced.

Proposition 4.3. *Let $c_{\mathbf{S}}$ be given by Proposition 4.2. Then there exists a constant $c_{\mathbf{U}} > 0$ such that the following holds for any tree $T \in \mathscr{T}_{k+1}$. If there exists a set $W \subseteq V(T)$, $|W| < c_{\mathbf{U}}k$ such that the family $\mathscr{C}$ of all components of the forest $T - W$ is $c_{\mathbf{U}}$-unbalanced, then $T \subseteq G$.*

Proposition 4.3 will be proved in Section 6.2. The last auxiliary proposition (Proposition 4.4) will be proved in Section 7.

Proposition 4.4. *Suppose that $q, c_{\mathbf{S}}, c_{\mathbf{E}}$ and $c_{\mathbf{U}}$ are fixed positive numbers. For any $\sigma, \omega > 0$ with $\sigma \ll \omega \leq \min\{q, c_{\mathbf{S}}, c_{\mathbf{E}}, c_{\mathbf{U}}\}$, there exist $\beta > 0$ and $n_0 = n_0(\sigma, \omega)$ such that for any graph G on $n \geq n_0$ vertices satisfying the LKS-property (with $k \geq qn$) with a subset $\bar{V} \subseteq V$ having the following properties*

- $|\bar{V}| > \sqrt[4]{c_{\mathbf{S}}}k$,
- $e(\bar{V}, V \setminus \bar{V}) \leq \beta k^2$, *and*
- $|L \cap \bar{V}| \geq (1 - \sigma)|\bar{V}|/2$,

there exists a subset $V' \subseteq \bar{V}$ *such that*

- $\diamond$ $(1-\omega)k \leq |V'| \leq (1+\omega)k$ *,*
- $\diamond$ $|V' \cap L| \geq |V'|/2$ *, and*
- $\diamond$ $e(V', V \setminus V') \leq \omega k^2$ *,*

or $\mathscr{T}_{k+1} \subseteq G$*.*

Proof of Theorem 1.4. Let $c_{\mathbf{S}}, c_{\mathbf{U}}$, and $c_{\mathbf{E}}$ be given by Propositions 4.3, 4.2 and 4.1, respectively. Set $\ell = \lceil \frac{1}{q} \rceil$, $\omega_\ell = \min\{q, c_{\mathbf{S}}, c_{\mathbf{U}}, c_{\mathbf{E}}\}$, and $\sigma_\ell \ll \omega_\ell$. We find a sequence of parameters

$$0 < \beta_1 \ll \sigma_1 \ll \omega_1 = \beta_2 \ll \sigma_2 \ll \omega_2 = \beta_3 \ll \cdots \ll \omega_{\ell-1} = \beta_\ell \ll \sigma_\ell \ll \omega_\ell \,, \tag{4.1}$$

obtained by the following iterative procedure. In step $i = 1$ start by setting β_ℓ as the number given by Proposition 4.4 for input parameters σ_ℓ and ω_ℓ. Set $\omega_{\ell-1} = \beta_\ell$ and $\sigma_{\ell-1} \ll \omega_{\ell-1}$. In general, in step i we define $\beta_{\ell+1-i}$ as the number given by Proposition 4.4 for input parameters $\sigma_{\ell+1-1}$ and $\omega_{\ell+1-\ell}$. Set $\omega_{\ell-i} = \beta_{\ell+1-i}$ and $\sigma_{\ell-i} \ll \omega_{\ell-i}$. Repeat the procedure for ℓ steps. Set $n_0 = \max_{i=1,\ldots,\ell}\{n_0(\sigma_i, \omega_i)\}$, where $n_0(\sigma_i, \omega_i)$ is also from Proposition 4.4.

Let G be a graph satisfying the conditions of Theorem 1.4 (i.e., q is fixed, n is sufficiently large, and $k > qn$). We can make the following assumptions.

Assumption 4.5. $|L| \leq |S| + 1$.

Proof. Suppose that $|L| \geq |S| + 2$. If $e(L,S) = 0$, then any tree $T \in \mathscr{T}_{k+1}$ embeds in $G[L]$ greedily, and Theorem 1.4 is proven. Otherwise, there exists an edge $e \in E(L,S)$. The graph $G' = G - e$ is of order n and has the LKS-property. Indeed, at most one vertex of L has decreased its degree in G'. For a graph H, denote by L^H the vertices of H with degrees at least k and S^H the vertices of degree less than k, i. e., $L = L^G$. Then $|L^{G'}| \geq |L^G| - 1 \geq |S^G| + 2 - 1 \geq |S^{G'}|$. If $\mathscr{T}_{k+1} \subseteq G'$, then $\mathscr{T}_{k+1} \subseteq G$. We can repeat this procedure until $\mathscr{T}_{k+1} \subseteq G$ or obtain a spanning subgraph $G^* \subseteq G$ satisfying the LKS-property and such that $|L^{G^*}| \leq |S^{G^*}| + 1$. $\square$

Assumption 4.6. *The set S is independent.*

Proof. If Assumption 4.6 is not fulfilled, we erase in G all the edges induced by S. Clearly, the modified graph G' still has the LKS-property and fulfills Assumption 4.6. This does not disturb Assumption 4.5. Any tree that is subgraph of G' is also a subgraph of G. $\square$

Let $\vartheta = \mathrm{ci}(n/k)$. We iterate the following process for at most ϑ steps. In step i, $i \leq \vartheta$, we prove that $\mathscr{T}_{k+1} \subseteq G$ or we define a set $V_i \subseteq V \setminus \bigcup_{j<i} V_j$ such that the following conditions are fulfilled for each $j \in [i]$.

(P1)$_i$ $(1-\beta_i)k \le |V_j| \le (1+\beta_i)k$,

(P2)$_i$ $|L\cap V_j| \ge (1/2-\beta_i)k$, and

(P3)$_i$ $e(V_j, V\setminus V_j) \le \beta_i k^2$.

In the step $i=1$, we apply Proposition 4.4 with parameters $\overline{V}=V$, $\sigma=\sigma_1$, $\omega=\omega_1$ and obtain that $\mathscr{T}_{k+1}\subseteq G$, or there exists a set V_1 satisfying (P1)$_1$, (P2)$_1$, and (P3)$_1$. Suppose that in step i we have sets $V_1,\dots,V_{i-1}$ that satisfy the conditions (P1)$_{i-1}$, (P2)$_{i-1}$, and (P3)$_{i-1}$. Set $V^*=V\setminus\bigcup_{j<i}V_j$.

First assume that $|V^*|>\sqrt[4]{c_{\mathbf{S}}}k$. If $|L\cap V^*|\ge(1-\sigma_{i-1})|V^*|/2$, the graph G satisfies the conditions of the Proposition 4.4 (with $\overline{V}=V^*$). If $|L\cap V^*|<(1-\sigma_{i-1})|V^*|/2$, then the decomposition $V_1,\dots,V_{i-1},V^*$ is $(\beta_{i-1},\sigma_{i-1})$-Extremal. We first apply Proposition 4.1 and show that $\mathscr{T}_{k+1}\subseteq G$, or there exists a set $Q\subseteq V^*$ satisfying

- $|Q|>k/2$,
- $|Q\cap L|>|Q|/2$, and
- $e(Q,V\setminus Q)<\sigma_{i-1}k^2$.

It is enough to assume the latter case. Again, the graph G satisfies the conditions of Proposition 4.4 (with $\overline{V}=Q$). Proposition 4.4 yields that $\mathscr{T}_{k+1}\subseteq G$, or that there exists a set $V_i\subseteq Q$ satisfying Properties (P1)$_i$–(P3)$_i$.

It remains to deal with the case $|V^*|\le\sqrt[4]{c_{\mathbf{S}}}k$. Having found sets $V_1,\dots,V_\vartheta$ satisfying (P1)$_\vartheta$–(P3)$_\vartheta$, we redistribute the small amount of (at most $\sqrt[4]{c_{\mathbf{S}}}k$) vertices of $\tilde{V}$ equally between $V_1,\dots,V_\vartheta$. The thus defined partition is $(\sqrt[4]{c_{\mathbf{S}}},c_{\mathbf{E}})$-Extremal. Proposition 4.1 yields that $\mathscr{T}_{k+1}\subseteq G$ (as no new set Q can be found). □

5 Special case (proof of Proposition 4.2)

Proof of Proposition 4.2. Fix a set $L'\subseteq L\cap\overline{V}$ of size $|L'|=(1/2-c_{\mathbf{S}})|\overline{V}|$. Define $\tilde{L}=\{u\in L' : \deg(u,\overline{V}\setminus L')\ge(1-2\sqrt{c_{\mathbf{S}}})k\}$. It holds for any vertex $x\in L'\setminus\tilde{L}$ that $\deg(x,L')+\deg(x,V\setminus\overline{V})>2\sqrt{c_{\mathbf{S}}}k$, otherwise it would be included in $\tilde{L}$. Since $e(G[L'])+e(L'\setminus\tilde{L},V\setminus\overline{V})<2c_{\mathbf{S}}k^2$ we get that $|L'\setminus\tilde{L}|<2\sqrt{c_{\mathbf{S}}}k$ (each vertex of $L'\setminus\tilde{L}$ is incident with at least $2\sqrt{c_{\mathbf{S}}}k$ such edges). Consequently, $|\tilde{L}|>(1/2-3\sqrt{c_{\mathbf{S}}})|\overline{V}|$. Next we verify that the set $\tilde{S}$, defined as $\tilde{S}=\{u\in\overline{V}\setminus L' : \deg(u,\tilde{L})\ge(1-9\sqrt{c_{\mathbf{S}}})k\}$, covers almost the whole set $\overline{V}\setminus L'$. Indeed, not more than $c_{\mathbf{S}}k^2$ edges of $E[\tilde{L},\overline{V}\setminus L']$ are incident to some vertex $x\in\overline{L}$, where $\overline{L}$ is the set of vertices of $x\in\overline{V}\setminus L'$ with $\deg(x,\tilde{L})>k$. Observe that $\overline{L}\subseteq L$. Hence the number of edges in the bipartite graph $G[\tilde{L},\overline{V}\setminus(L'\cup\overline{L})]$ is at least

$$|\tilde{L}|(1-2\sqrt{c_{\mathbf{S}}})k-c_{\mathbf{S}}k^2>\frac{1}{2}|\overline{V}|k-4\sqrt{c_{\mathbf{S}}}|\overline{V}|k-c_{\mathbf{S}}k^2>\frac{1}{2}|\overline{V}|k-6\sqrt{c_{\mathbf{S}}}|\overline{V}|k\,.$$

Since no vertex from $\bar{V} \setminus (L' \cup \bar{L})$ receives more than k edges from $\tilde{L}$, it holds that

$$|(\bar{V} \setminus (L' \cup \bar{L})) \cap \tilde{S}| \geq \frac{\frac{1}{2}|\bar{V}|k - 6\sqrt{c_{\mathbf{S}}}|\bar{V}|k}{k} = \frac{1}{2}|\bar{V}| - 6\sqrt{c_{\mathbf{S}}}|\bar{V}| \,.$$

Obviously, $\bar{L} \subseteq \tilde{S}$ and thus, $|\bar{V} \setminus (L' \cup \tilde{S})| \leq 7\sqrt{c_{\mathbf{S}}}|\bar{V}|$ (recall that L' and $\tilde{S}$ are disjoint, and $|L'| = (1/2 - c_{\mathbf{S}})|\bar{V}|$). By the choice of $\tilde{L}$ and $\tilde{S}$ and the fact that $|\bar{V} \setminus (L' \cup \tilde{S})| \leq 7\sqrt{c_{\mathbf{S}}}|\bar{V}|$, the minimum degree of vertices in $\tilde{L}$ in the bipartite graph $G_1 = G[\tilde{L}, \tilde{S}]$ is at least $k - 9\sqrt[4]{c_{\mathbf{S}}}|\bar{V}|$, and of those in $\tilde{S}$ at least $(1 - 9\sqrt{c_{\mathbf{S}}})k$. By choosing sufficiently small $c_{\mathbf{S}}$ (as a function of q; recall $q > k/n$) we can guarantee that $\delta(G_1) > k/2$.

Let $T \in \mathscr{T}_{k+1}$ be an arbitrary tree. We write $T_{\mathrm{e}}^{\mathrm{n}}$ for the set of internal vertices of T which are contained in T_{e} and $T_{\mathrm{e}}^{\mathrm{l}}$ for the set of leaves in T_{e}. By Fact 3.2 it holds $|T_{\mathrm{e}}^{\mathrm{n}}| \leq |T_{\mathrm{o}}| \leq k/2$. We embed the subtree $T - T_{\mathrm{e}}^{\mathrm{l}}$ in G_1 using the greedy algorithm embedding the vertices from $T_{\mathrm{e}}^{\mathrm{n}}$ in $\tilde{S}$. The last step is to embed the leaves $T_{\mathrm{e}}^{\mathrm{l}}$. This can be done using the property of high degree of vertices in $\tilde{L}$ (note that $T_{\mathrm{e}}^{\mathrm{l}}$ may be mapped outside G_1 at this step). $\square$

6 Tools for the proof of Proposition 4.4

6.1 Szemerédi Regularity Lemma

In this section we recall briefly the Szemerédi Regularity Lemma [21] and establish related notation. The reader may find more on the Regularity Method in [14, 13].

Let $H = (V(H); E(H))$ be a graph of order m. For two nonempty disjoint sets $X, Y \subseteq V(H)$ we define *density* of the pair (X, Y) by

$$\mathrm{d}(X,Y) = \frac{e(X,Y)}{|X||Y|} \,.$$

For $\varepsilon > 0$ we say that a pair of vertex sets (A, B) is ε*-regular* if $|\mathrm{d}(A,B) - \mathrm{d}(X,Y)| < \varepsilon$ for every choice of X and Y, where $X \subseteq A$, $Y \subseteq B$, $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$. For an ε-regular pair (A, B) a set $X \subseteq A$, and a set $Y \subseteq B$ is called a *significant set* if $|X| > \varepsilon|A|$, and $|Y| > \varepsilon|B|$, respectively. For an ε-regular pair (A, B) we say that a vertex $v \in X$ is *typical* with respect to a significant set $W \subseteq Y$ if $\deg(v, B) \geq (\mathrm{d}(A,B) - 2\varepsilon)|W|$.

Fact 6.1. *1. Let (X, Y) be an ε-regular pair and $W \subseteq Y$ be a significant set. Then all but at most $\varepsilon|X|$ vertices of X are typical w.r.t. W.*

2. Let $X, Y_1, Y_2, \ldots, Y_\ell$ be disjoint sets of vertices, such that $(X, Y_1), (X, Y_2), \ldots, (X, Y_\ell)$ are ε-regular pairs. Suppose that we are given sets $W_i \subseteq Y_i$ which are significant for each $i \in [\ell]$. Then there are at most $\sqrt{\varepsilon}|X|$ vertices of X which are not typical with respect to at least $\sqrt{\varepsilon}\ell$ sets W_i.

Proof. 1. The proof is direct.

2. For a vertex $v \in X$, let $I_v \subseteq [\ell]$ be the set of those indices i for which v is not typical with respect to W_i. For contradiction, suppose that $|\{v \in X : |I_v| > \sqrt{\varepsilon}\ell\}| > \sqrt{\varepsilon}|X|$. Then

$$\sum_{i \in [\ell]} |\{v \in X : i \in I_v\}| = \sum_{v \in X} |I_v| > \varepsilon |X| \ell .$$

By averaging, there exists an index $i_0 \in [\ell]$ such that the set $U = \{v \in X : i_0 \in I_v\}$ is significant. Then,

$$\mathrm{d}(U, W_{i_0}) = \frac{\sum_{v \in U} \deg(v, W_{i_0})}{|U||W_{i_0}|} < \mathrm{d}(X, W_{i_0}) - 2\varepsilon \leq \mathrm{d}(X, Y_{i_0}) - \varepsilon ,$$

a contradiction to the regularity of the pair (X, Y_{i_0}).

□

A partition $V_0, V_1, \ldots, V_N$ of the vertex set $V(H)$ of the graph H is called (ε, N)-*regular* if

- $|V_0| < \varepsilon m$,
- $|V_i| = |V_j|$ for every $i, j \in [N]$, and
- all but at most εN^2 pairs (V_i, V_j) (for $i, j \in [N]$) are ε-regular.

The sets $V_1, \ldots, V_N$ are called *clusters*.

The Regularity Lemma we use deals with graphs with initial prepartitioning of the vertex set. Its proof follows the same lines as the proof of Szemerédi's original result [21].

Theorem 6.2 (Regularity Lemma, with initial partition)**.** *For every $\varepsilon > 0$ and every $m_0, r \in \mathbb{N}$, there exist numbers $M_0, N_0 \in \mathbb{N}$ such that every graph H of order $m \geq N_0$ whose vertex sets is partitioned into r sets $O_1 \cup O_2 \cup \ldots \cup O_r = V(H)$ admits an $(\varepsilon; N)$-regular partition $V_0, V_1, \ldots, V_N$ for some $m_0 \leq N \leq M_0$ such that for every $i \in [N]$ we have $V_i \subseteq O_j$ for some $j \in [r]$.*

6.2 Cutting the trees, and the (un)balanced trees

Let $T \in \mathscr{T}_{k+1}$ be a tree and $\ell \in \mathbb{N}, \ell < k$. The purpose of this section is to give constructive definitions of an ℓ-fine partition of T, and a switched ℓ-fine partition of T. The tree T is rooted in a vertex R. This gives us order $\preceq$ on $V(T)$.

For a tree $F \subseteq T$ such that $R \notin V(F)$ we define the *seed of* F as the unique vertex $v \in V(T) \setminus V(F)$ such that $F \subseteq T(R, \downarrow v)$ and v is adjacent to a vertex from F. We write $\mathrm{Seed}(F) = v$.

Set $T_0 = T$ and $i = 1$. We repeatedly (in step i) choose a vertex $x_i \in V(T_{i-1})$ such that $v(T_{i-1}(\downarrow x_i)) > \ell$ and such that x_i is $\preceq$-minimal among all such possible choices. We set $T_i =$

$T_{i-1} - (V(T_{i-1}(\downarrow x_i)) \setminus \{x_i\})$. If no such x_i exists we have $v(T_{i-1}) \leq \ell$. We then set $x_i = R$ and terminate. Since we deleted at least ℓ vertices in each step, we have $i \leq \lceil (k+1)/\ell \rceil$ at the moment of terminating. Set

$$A' = \{x_j : \mathrm{dist}(x_j, R) \text{ is even}\} \quad \text{and} \quad B' = \{x_j : \mathrm{dist}(x_j, R) \text{ is odd}\}\ .$$

Let $\mathscr{C}_A$ and $\mathscr{C}_B$ be those components t of the forest $T - (A' \cup B')$ which have $\mathrm{Seed}(t) \in A'$ and $\mathrm{Seed}(t) \in B'$, respectively. For a component t we write

$$\begin{aligned} X(t) &= V(t) \cap \mathrm{N}(B') \quad \text{for } t \in \mathscr{C}_A\text{, and} \\ X(t) &= V(t) \cap \mathrm{N}(A') \quad \text{for } t \in \mathscr{C}_B. \end{aligned}$$

Set $W_A = A' \cup \bigcup_{t \in \mathscr{C}_A} X(t)$ and $W_B = B' \cup \bigcup_{t \in \mathscr{C}_B} X(t)$. Observe that $\max\{|W_A|, |W_B|\} \leq |A'| + |B'|$. Let $\mathscr{D}_A$ and $\mathscr{D}_B$ be those components t of the forest $T - (W_A \cup W_B)$ which have $\mathrm{Seed}(t) \in W_A$ and $\mathrm{Seed}(t) \in W_B$, respectively. The *ℓ-fine partition of T* is the quaternary $\mathscr{D} = (W_A, W_B, \mathscr{D}_A, \mathscr{D}_B)$. The following properties of the ℓ-fine partition of T are obvious from the construction.

- $R \in W_A$.
- The distance from any vertex in W_A to any vertex in W_B is odd. The distance between any pair of vertices in W_A or between any pair of vertices in W_B is even.
- T is decomposed into vertices W_A, W_B, and into trees $\mathscr{D}_A$ and $\mathscr{D}_B$.
- No tree from $\mathscr{D}_A$ is adjacent to any vertex in W_B. No tree from $\mathscr{D}_B$ is adjacent to any vertex in W_A.
- $\max\{|W_A|, |W_B|\} \leq \frac{4k}{\ell}$.
- $v(t) \leq \ell$ for any tree $t \in \mathscr{D}_A \cup \mathscr{D}_B$.

The partition $\mathscr{D}$ will be further refined to get a switched ℓ-fine partition. Let $\mathscr{D}_A^*$ and $\mathscr{D}_B^*$ denote the end-trees from $\mathscr{D}_A$ and $\mathscr{D}_B$, respectively. In the following we assume that $\sum_{t \in \mathscr{D}_A^*} v(t) \geq \sum_{t \in \mathscr{D}_B^*} v(t)$. If this was not the case, we exchange the sets W_A, W_B, and $\mathscr{D}_A$, $\mathscr{D}_B$. For any tree $t \in \mathscr{D}_B \setminus \mathscr{D}_B^*$ set $Y(t) = V(t) \cap \mathrm{N}(W_B)$. Observe that $\sum_{t \in \mathscr{D}_B \setminus \mathscr{D}_B^*} |Y(t)| \leq 2|W_B|$. Define $W_A' = W_A \cup \bigcup_{t \in \mathscr{D}_B \setminus \mathscr{D}_B^*} Y(t)$. The *switched ℓ-fine partition of T* is the quaternary $\mathscr{D} = (W_A', W_B, \mathscr{D}_A', \mathscr{D}_B')$, where $\mathscr{D}_A'$ and $\mathscr{D}_B'$ are the sets of components of $T - (W_A' \cup W_B)$ with the seed in W_A' and W_B, respectively. The switched ℓ-fine partition of T satisfies the following properties.

- $R \in W_A' \cup W_B$.
- The distance from any vertex in W_A' to any vertex in W_B is odd. The distance between any pair of vertices in W_A' or between any pair of vertices in W_B is even.

- T is decomposed into vertices W'_A, W_B, and into trees $\mathscr{D}'_A$ and $\mathscr{D}'_B$.
- No tree from $\mathscr{D}'_A$ is adjacent to any vertex in W_B. No tree from $\mathscr{D}'_B$ is adjacent to any vertex in W'_A.
- $\max\{|W'_A|,|W_B|\} \le \frac{12k}{\ell}$.
- $v(t) \le \ell$ for any tree $t \in \mathscr{D}'_A \cup \mathscr{D}'_B$.
- $\mathscr{D}'_B$ contains no internal tree.
- We have
$$\sum_{\substack{t\in\mathscr{D}'_A\\ t \text{ end tree}}} v(t) \ge \sum_{t\in\mathscr{D}'_B} v(t)\ .$$

For an ℓ-fine partition (or a switched ℓ-fine partition) $\mathscr{D} = (W_A, W_B, \mathscr{D}_A, \mathscr{D}_B)$ the trees $t \in \mathscr{D}_A \cup \mathscr{D}_B$ are called *shrublets*.

The ℓ-fine partition and the switched ℓ-fine partition may not be unique, the construction depended on the choice of the root R. However, this is not a problem in the later setting; we only need that there exists at least one ℓ-fine partition $\mathscr{D}$ and one switched ℓ-fine partition $\mathscr{D}'$ of T satisfying the above properties.

Proof of Proposition 4.3. Set $c_{\mathbf{U}} = c_{\mathbf{S}}/4$.

If the set L induces less then $c_{\mathbf{S}}n^2$ edges then we have $T \subseteq G$ by Proposition 4.2. In the rest we assume that $G[L]$ contains at least $c_{\mathbf{S}}n^2$ edges. A well-known fact asserts that there exists a graph $G' \subseteq G[L]$ with minimum degree at least half of the average degree of $G[L]$, i. e., $\delta(G') \ge c_{\mathbf{S}}n \ge 4c_{\mathbf{U}}(k+1)$.

Let $\mathscr{C}' \subseteq \mathscr{C}$ be those trees $t \in \mathscr{C}$ for which $|t_{\mathrm{o}}| \le c_{\mathbf{S}}v(t)$. It holds that $\sum_{t\in\mathscr{C}'} v(t) > (1-4c_{\mathbf{U}})k$. We apply Fact 3.2 on each tree $t \in \mathscr{C}'$. Summing the bound on the number of leaves, given by Fact 3.2, we get that there are at least $(1-2c_{\mathbf{U}})(k+1)$ leaves in the trees of $\mathscr{C}'$. A leaf of a tree $t \in \mathscr{C}'$ is either a leaf of T or it is adjacent to a vertex in W. Root T at an arbitrary vertex r. The vertex r determines a partial order $\preceq$ with r being the maximal element. Let X be those vertices of T which are a leaf of some tree $t \in \mathscr{C}'$ but not a leaf of T. Each vertex in X is either a $\preceq$-minimal or a $\preceq$-maximal vertex of some tree $t \in \mathscr{C}$. Let $X_{\min} \subseteq X$ be the $\preceq$-minimal vertices and $X_{\max} = X \setminus X_{\min}$. (Note that $X_{\max}$ does not have to contain exactly the $\preceq$-maximum "fake" leaves of T; the vertices which come out from 1-vertex trees of $\mathscr{C}'$ are not included.) As each tree t has a unique $\preceq$-maximal vertex we get $|X_{\max}| \le h$, where h is the number of trees t in $\mathscr{C}'$ which have order more than 1. Observe, that each such tree t has at least $1/c_{\mathbf{U}}$ vertices and thus $h \le c_{\mathbf{U}}(k+1)$. For each $v \in X_{\min}$ we have $|\mathrm{Ch}(v) \cap W| \ge 1$. Since for each $u \in W$ it holds $|\mathrm{Par}(u) \cap X_{\min}| \le 1$ we have $|X_{\min}| \le |W| < c_{\mathbf{U}}k$. Summing the bounds we get $|X| < 2c_{\mathbf{U}}(k+1)$. Thus T has at least $(1-4c_{\mathbf{U}})(k+1)$ leaves. Let $T' \subseteq T$ be a subtree of T formed by its internal vertices. We have

$v(T') \leq 4c_{\mathbf{U}}(k+1)$. We embed T' in G' greedily. Then we extend the embedding also to the leaves of T, using the high degree of the images of $V(T')$. $\square$

6.3 A Tutte-type proposition

Graph H is called *factor critical* if for any its vertex v the graph $H - v$ has a perfect matching.

The following statement is a fundamental result in the Matching theory. See [15], for example.

Theorem 6.3 (Gallai-Edmonds Matching Theorem)**.** *Let H be a graph. Then there exist a set $Q \subseteq V(H)$ and a matching M of size $|Q|$ in H such that every component of $H - Q$ is factor critical and the matching M matches every vertex in Q to a different component of $H - Q$.*

The set Q in Theorem 6.3 is called a *separator*.

Proposition 6.4. *Let (H, ω) be a weighted graph of order N, with $\omega : E(H) \to (0, s]$. Let σ, K be two positive numbers with $1/(2N) < \sigma < \min\{K/(32Ns), 1/10\}$. Let $\mathscr{L}$ be an arbitrary set of vertices, such that*

- *$V(H) \setminus \mathscr{L}$ is an independent set,*
- *$|\mathscr{L}| > N/2 - \sigma N$,*
- *$\mathrm{d\bar{e}g}^{\omega}(u) \geq K$ for every $u \in \mathscr{L}$,*
- *the set $\mathscr{L}$ induces at least one edge in H,*
- *$\mathrm{d\bar{e}g}^{\omega}(u) < (1+\sigma)K$ for every $u \in V(H) \setminus \mathscr{L}$.*

Set $\mathscr{L}^* = \{u \in V(H) : \mathrm{d\bar{e}g}^{\omega}(u) \geq (1+\sigma)K/2\}$.

Then there exist a matching M and two adjacent vertices $A, B \in V(H)$ such that at least one of the following holds.

Case I For the vertex A it holds $\mathrm{d\bar{e}g}^{\omega}(A, V(M)) \geq K$ and for each edge $e \in M$ we have $|\mathrm{N}(A) \cap e| \leq 1$. For the vertex B it holds $\mathrm{d\bar{e}g}^{\omega}(B, V(M) \cup \mathscr{L}^*) \geq (1+\sigma)K/2$.

Case II There exists a set $\mathscr{X}' \subseteq V(H)$, with $\mathrm{d\bar{e}g}^{\omega}(x, V(M)) \geq \mathrm{d\bar{e}g}^{\omega}(x) - 2\sigma Ns$ for all vertices $x \in \mathscr{X}'$. Furthermore, $A, B \in \mathscr{X}' \cap \mathscr{L}$, and $|V(M') \setminus \mathscr{X}'| \leq 1$, where $M' = \{xy \in M : x, y \in \mathrm{N}(\mathscr{X}')\}$.

Moreover observe that each edge $e \in M$ intersects the set $\mathscr{L}$.

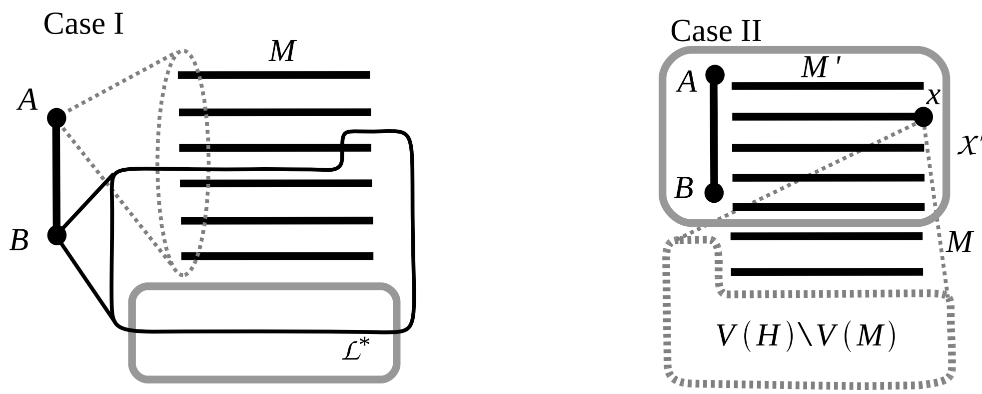

Figure 1: Two resulting matching structures from Proposition 6.4. Dashed lines represent no connections (in Case I), or sparse connections (in Case II).

Proof. Among all matchings satisfying the conclusion of the Gallai-Edmonds Matching Theorem, choose a matching M_0 that covers a maximum number of vertices from $V(H)\setminus\mathscr{L}^*$. Let Q be the corresponding separator. Recall that M_0 is a $Q\leftrightarrow(V(H)\setminus Q)$-matching. Set $L_0=\mathscr{L}\setminus Q$ and $\mathscr{S}=V(H)\setminus\mathscr{L}$.

We distinguish three cases.

- There exists an $L_0\leftrightarrow L_0$ edge.

Set $\mathscr{X}'=L_0\cup\mathrm{N}(L_0)\setminus Q$ and let A and B be vertices of any $L_0\leftrightarrow L_0$ edge. Then A and B lie in the same component C of $H-Q$. If $V(M_0)\cap V(C)\neq\emptyset$, then take $\{x\}=V(M_0)\cap V(C)$, and choose x arbitrarily in C, otherwise. Since C is factor critical, there exists a perfect matching M_1 in $C-x$. It is straightforward to check that the matching $M=M_0\cup M_1$ satisfies conditions of Case II.

- $L_0=\emptyset$.

Set $\mathscr{X}'=V(H)$ and $M=M_0$. Let A and B be end-vertices of an arbitrary $\mathscr{L}\leftrightarrow\mathscr{L}$ edge. It is clear that $V(M')\setminus\mathscr{X}'=\emptyset$. Since $Q\supseteq\mathscr{L}$, $|\mathscr{L}|\geq N/2-\sigma N$, and $|V(M)|=2|Q|$ it holds that all but at most $2\sigma N$ vertices of H are covered by M, thus for any vertex $x\in\mathscr{X}'$, we have that $\overline{\deg}^{\omega}(x,V(M))\geq\overline{\deg}^{\omega}(x)-2\sigma Ns$.

- L_0 is an independent set and $L_0\neq\emptyset$.

First we observe that each component C of $H-Q$ is a singleton. Indeed, since $\mathscr{S}$ and L_0 are independent all the edges in any matching in C are in the form $\mathscr{S}\leftrightarrow L_0$. Since C is factor critical, we have $|V(C-u)\cap L_0|=|V(C-u)\cap\mathscr{S}|$ for any vertex $u\in V(C)$. Thus $v(C)=1$. (Note that M_0 is thus maximum.) Set $M=M_0$.

Define $\tilde{L}=\{u\in\mathrm{N}(L_0)\,:\,\overline{\deg}^{\omega}(u)\geq K\}$. Observe that $\tilde{L}\subseteq Q$. We shall prove that

$$\tilde{L}\neq\emptyset \tag{6.1}$$

by contradiction. Assume that for every vertex $u \in \mathrm{N}(L_0)$ it holds $\mathrm{d\overline{eg}}^{\omega}(u) < K$. We get $|L_0|K \leq \overline{e}^{\omega}(L_0, \mathrm{N}(L_0)) < K|\mathrm{N}(L_0)|$ implying $|L_0| < |\mathrm{N}(L_0)|$. From $\tilde{L} = \emptyset$ it follows that $\mathrm{N}(L_0) \cap \mathscr{L} = \emptyset$ and thus every vertex in $\mathrm{N}(L_0)$ is matched by M to a distinct vertex in L_0, a contradiction.

We show that the graph $V(H)$ fulfills conditions of Case I. It suffices to find a vertex $B \in \mathrm{N}(L_0)$ such that $\mathrm{d\overline{eg}}^{\omega}(B, V(M) \cup \mathscr{L}^*) \geq (1+\sigma)K/2$. The pair (A,B), where $A \in \mathrm{N}(B) \cap L_0$, satisfies conditions of Case I.

Define $X = V(H) \setminus (V(M) \cup \mathscr{L}^*)$. For contradiction, assume that for every $B \in \tilde{L}$ we have

$$\mathrm{d\overline{eg}}^{\omega}(B, V(M) \cup \mathscr{L}^*) < (1+\sigma)K/2\,, \tag{6.2}$$

which yields

$$\mathrm{d\overline{eg}}^{\omega}(B, X) > (1-\sigma)K/2\,. \tag{6.3}$$

This implies that M does not contain any edge with both end-vertices in $\mathscr{L}$. Indeed, suppose that such an edge $xy \in M$ exists. Then $x \in L_0$ and $y \in \tilde{L}$. By (6.3), $\mathrm{d\overline{eg}}^{\omega}(y, X) > (1-\sigma)K/2$. In particular, there exists a vertex $p \in \mathrm{N}_X(y)$. The matching $M_1 = \{yp\} \cup M_0 \setminus \{xy\}$ is a matching as in Gallai-Edmonds Matching Theorem (with separator Q) which covers more vertices of $V(H) \setminus \mathscr{L}^*$ than M_0 does. This contradicts the choice of M_0. Observe that for any vertex $u \in X$, we have $\mathrm{d\overline{eg}}^{\omega}(u, V(M)) = \mathrm{d\overline{eg}}^{\omega}(u) < (1+\sigma)K/2$ and thus $\mathrm{d\overline{eg}}^{\omega}(u, \tilde{L}) < (1+\sigma)K/2$. We bound $\overline{e}^{\omega}(\tilde{L}, X)$ from both sides.

$$(1-\sigma)|\tilde{L}|\frac{K}{2} \leq \overline{e}^{\omega}(\tilde{L}, X) \leq (1+\sigma)|X|\frac{K}{2}\,,$$

which yields

$$|\tilde{L}| \leq \frac{1+\sigma}{1-\sigma}|X|\,. \tag{6.4}$$

We use (6.2) to obtain bounds on $\overline{e}^{\omega}(Q, L_0)$.

$$\begin{aligned} |L_0|K \leq \overline{e}^{\omega}(Q, L_0) &= \overline{e}^{\omega}(\tilde{L} \cup (Q \setminus \tilde{L}), L_0) \\ &\leq (1+\sigma)\frac{K}{2}\left(|\tilde{L}| + |Q \setminus \mathscr{L}|\right) \\ &\leq (1+\sigma)\frac{K}{2}|\tilde{L}| + K|Q \setminus \mathscr{L}|\,, \end{aligned}$$

which gives

$$2|L_0| \leq (1+\sigma)|\tilde{L}| + 2|Q \setminus \mathscr{L}|\,. \tag{6.5}$$

Every vertex in $Q \setminus \mathscr{L}$ is matched to a vertex in L_0, and conversary, if a vertex in L_0 is matched, then it is matched to a vertex in $Q \setminus \mathscr{L}$. Therefore, $|Q \setminus \mathscr{L}| = |L_0 \cap V(M)|$. Combined with (6.5) we have that $2|L_0 \setminus V(M)| \le (1+\sigma)|\tilde{L}|$. Plugging (6.4) we obtain

$$2|L_0 \setminus V(M)| \le \frac{(1+\sigma)^2}{1-\sigma}|X| \,. \tag{6.6}$$

From $|\mathscr{L}| > |V(H) \setminus \mathscr{L}| - 2\sigma N$ we get $|L_0 \setminus V(M)| \ge |X| - 2\sigma N$ (Recall that any edge of M has one end-vertex in $\mathscr{L}$ and the other one in $V(H) \setminus \mathscr{L}$). Together with (6.6) we obtain

$$\frac{(1+\sigma)^2}{1-\sigma}|X| \ge 2|X| - 4\sigma N \,,$$

yielding

$$\frac{4\sigma N}{1-3\sigma} \ge |X| \,.$$

A contradiction with (6.3), (6.1), and the bound on σ. □

6.4 Embedding lemmas

In this section, we introduce some tools for embedding a forest in one regular pair. Similar results are folklore, however we prove them tailed to our needs. Lemma 6.5 describes sufficient conditions for embedding a rooted tree in a regular pair.

Lemma 6.5. *Let (t,r) be a rooted tree, and $d > 2\varepsilon > 0$. Let (X,Y) be an ε-regular pair with $|X| = |Y| = s$ and density $\mathrm{d}(X,Y) \ge d$. Let $P' \subseteq P \subseteq X$ and $Q' \subseteq Q \subseteq Y$ be such that $\min\{|P|,|Q|\} \ge \Delta$ and $\max\{|P'|,|Q'|\} \ge \Delta$, where $\Delta = \frac{\varepsilon s + v(t)}{d-2\varepsilon}$. Then there exists an embedding ϕ of t in $P \cup Q$ such that the root r is mapped to $P' \cup Q'$. The following two further requirements can be also fulfilled.*

1. *If $|P \setminus P'| \ge \Delta$, we can ensure that $\phi(V(t) \setminus \{r\}) \cap P' = \emptyset$, and similarly, if $|Q \setminus Q'| \ge \Delta$, we can ensure that $\phi(V(t) \setminus \{r\}) \cap Q' = \emptyset$.*
2. *If $|P'| \ge \Delta$ we can can prescribe the vertex r to be mapped to P'. If $|Q'| \ge \Delta$ we can can prescribe the vertex r to be mapped to Q'.*

Proof. Without loss of generality assume that $|P'| \ge \Delta$. Choose an auxiliary set $S_P \subseteq P$ with $|S_P| = \Delta$ subject to $|S_P \cap P'|$ being minimal. In particular, we have $S_P \subseteq P \setminus P'$, if $|P \setminus P'| \ge \Delta$. Similarly, choose a set $S_Q \subseteq Q$ with $|S_Q| = \Delta$ with respect to $|S_Q \cap Q'|$ being minimal. The sets S_P and S_Q are significant. Choose a vertex $v \in P'$ which is typical w. r. t. S_Q. There are at least $|P'| - \varepsilon s > 1$ such vertices. Set $\phi(r) = v$.

We inductively extend the embedding ϕ, so that every vertex of t which was mapped to P is typical w. r. t. S_Q, and that every vertex which was mapped to Q is typical w. r. t. S_P. We illustrate the inductive step by describing how to embed the neighborhood of a vertex u which was already embedded in P. The case when $\phi(u) \in Q$ is analogous. Let $N \subseteq \mathrm{N}(u)$ be the yet unembedded neighbors of u. The vertex $\phi(u)$ has at least $(d-2\varepsilon)\Delta \geq \varepsilon s + v(t)$ neighbors in S_Q. At least $|N|$ of them are typical w. r. t. S_P and not yet used by ϕ. We then map N to these vertices.

Clearly, Part 1. was satisfied. In addition, Part 2. can also be fulfilled. Indeed, we only need to observe that if $|P'| \geq \Delta$, there is at least one vertex in P' which is typical w. r. t. S_Q. This vertex will be used for embedding the root r. The second condition of Part 2 is analogous. $\square$

For the proof of Proposition 4.4 (which is the key tool for proving Theorem 1.4), we need to embed the shrublets of the tree T in an efficient way. To this end, we try to fill the clusters of the regular pair with the same speed. The following definition of i-packness formalizes this.

Let $i \in \{1,2\}$ and $X,Y,Z \subseteq V(G)$ be three disjoint subsets. We say that $U \subseteq X \cup Y$ is *i-packed* (with parameters λ, τ) with respect to the *head set* Z and with respect to the *embedding sets* X and Y, if

$$\min\{|X \cap U|, |Y \cap U|\} \geq \min\{i\mu, v\} - \lambda,$$

or

$$||X \cap U| - |Y \cap U|| \leq \tau,$$

where

$$\mu = \min\{\overline{\deg}(Z,X), \overline{\deg}(Z,Y)\}, \quad \text{and} \quad v = \max\{\overline{\deg}(Z,X), \overline{\deg}(Z,Y)\}.$$

If U represents the vertices used by an embedding, then to keep U 1-packed means that we have roughly the same amount of used vertices on both sides of X and Y until we have embedded roughly 2μ vertices. If we manage to keep U 2-packed, we have this "balance" for even longer.

The following embedding lemma allows us to "fill-up" a regular pair with a rooted forest. The lemma is divided into three parts to satisfy different embedding requirements of the proof of Proposition 4.4. The most important one is the "saving" Part 3. Having a cluster Z and a regular pair (X,Y), Part 1 ensures the embedding of a rooted forest (F,R) mapping R to Z and $F-R$ to $X \cup Y$, provided that the order of F is slightly less than $\overline{\deg}(Z, X\cup Y)$. Part 3 allows us to embed even a larger forest F, under certain additional conditions.

Lemma 6.6. *Let (F,R) be a rooted tree with root R such that each component of $F-R$ has order at most τ. Let X,Y,Z be three disjoint vertex sets, with $|X| = |Y| = s$, forming three ε-regular pairs. Assume that $\frac{e(X,Y)}{s^2} \geq d > 2\varepsilon$ and $\mathrm{d}(Z,X), \mathrm{d}(Z,Y) \in \{0\} \cup [d,1]$. Set $\Delta = \frac{\varepsilon s + \tau}{d - 2\varepsilon}$. Let $U \subseteq X \cup Y$. In the following we write F_1 and F_2 for the vertices of $F-R$ with odd and even distance from R, respectively.*

1. *If $v(F) + |U| \leq \overline{\deg}(Z, X\cup Y) - \lambda_1 - \Delta - 2\varepsilon s$, where $\lambda_1 = \Delta + \tau + 3\varepsilon s$, U is 1-packed w. r. t. Z (with parameters λ_1 and τ), and R is mapped to a vertex $r \in Z$ that is typical w. r. t. X and w. r. t. Y, then the mapping of R can be extended to an embedding φ of F such that*

   (c1) $\varphi(V(F-R)) \subseteq (X \cup Y) \setminus U$,

(c2) each vertex of F_1 is mapped to a vertex which has at least $(d-2\varepsilon)|Z|$ neighbors in Z, and

(c3) the set $U \cup \varphi(V(F-R))$ is 1-packed (with parameters λ_1 and τ) w. r. t. the head set Z and the embedding sets X and Y.

2. *If $\max\{|F_1|,|F_2|\}+|X\cap U| \le \mathrm{d\,\bar{e}g}(Z,X)-\lambda_1-\Delta-\varepsilon s$, U is 1-packed (with parameters $\lambda_1 = \Delta+\tau+3\varepsilon s$ and τ) w. r. t. the head set Z and the embedding sets X and Y, and R is mapped to a vertex $r\in Z$ that is typical w. r. t. X and w. r. t. Y, then the mapping of R can be extended to an embedding φ of F such that (c1), (c2), and (c3) hold.*

3. *If $\mathrm{d\,\bar{e}g}(Z,X)\in[\eta s,(1-\eta)s]$, where $\eta s \ge 12\lambda_2$, and $\lambda_2 = 2\Delta+7\varepsilon s+4\tau$, U is 2-packed w. r. t. Z (with parameter λ_2 and τ), each component of $F-R$ has at least two vertices, R is mapped to a vertex $r\in Z$ that is typical w. r. t. $X\setminus U$ and w. r. t. $Y\setminus U$, and*

$$v(F)+|U| \le \mathrm{d\,\bar{e}g}(Z,X\cup Y)+\frac{\eta s}{4}\,, \tag{6.7}$$

then the mapping of R can be extended to an embedding φ of F such that (c1), (c2), and

(d) $U\cup\varphi(V(F-R))$ is 2-packed w. r. t. Z (with parameters λ_2 and τ)

hold.

Proof. Set $\mu = \min\{\mathrm{d\,\bar{e}g}(Z,X),\mathrm{d\,\bar{e}g}(Z,Y)\}$ and $\nu = \max\{\mathrm{d\,\bar{e}g}(Z,X),\mathrm{d\,\bar{e}g}(Z,Y)\}$. We split the embedding of the forest $F-R$ into ℓ steps, where ℓ is the number of components of $F-R$. In each step i, we embed a component t_i of $F-R$ in $(X\cup Y)\setminus(U\cup U_i)$, where $U_i = \varphi(\bigcup_{j<i}V(t_j))$ is the image of trees embedded in previous steps. The component t_i is a tree, we write r_i for its root, $\{r_i\} = V(t_i)\cap \mathrm{N}(R)$. Moreover, we assume that the trees t_i are ordered so that $t_1,\dots,t_{\ell_1}$ are trees of order at most two, $t_{\ell_1+1},\dots,t_{\ell_2}$ are stars of order at least three with their centers in the roots of the components and $t_{\ell_2+1},\dots,t_\ell$ are trees which are not stars centered in the roots r_i. This ordering is unnecessarily in the proof of Parts 1, 2, we only use it in the embedding described in Part 3. Observe that the assumptions of Part 3 assert that all tree t_i, $i\in[\ell_1]$ have order exactly two. For step i, set $P_i = X\setminus(U\cup U_i\cup B)$, and $Q_i = Y\setminus(U\cup U_i\cup B)$, where B is the set of vertices in $X\cup Y$ which are not typical w. r. t. the set Z. We have $\max\{|X\cap B|,|Y\cap B|\}\le\varepsilon s$. Define $P_i' = P_i\cap\mathrm{N}(r)$ and $Q_i' = Q_i\cap\mathrm{N}(r)$.

Part 1. In each step i, the embedding will satisfy conditions (c1)$_i$, (c2)$_i$, and (c3)$_i$. These conditions are modified versions of (c1), (c2), and (c3), where we consider $U\cup U_i$ instead of U and $\varphi(t_i)$ instead of $\varphi(V(F-R))$. Conditions (c1)$_0$, (c2)$_0$, and (c3)$_0$ are clearly met. We shall verify (c1)$_i$, (c2)$_i$, and (c3)$_i$ inductively at the end of each step i. First we claim that $\max\{|P_i'|,|Q_i'|\}\ge\Delta$. This is implied by the following chain of inequalities.

$$|P_i'\cup Q_i'| = \deg(r,P_i\cup Q_i) \ge \mathrm{d\,\bar{e}g}(Z,X\cup Y)-|U\cup U_i|-|B|-4\varepsilon s \ge \lambda_1+\Delta-3\varepsilon s > 2\Delta\,. \tag{6.8}$$

Second, we claim that $\min\{|P_i|,|Q_i|\} \geq \Delta$. If this is not the case,

$$\max\{|X\cap(U\cup U_i)|,|Y\cap(U\cup U_i)|\} \geq s-\Delta-\varepsilon s \geq \nu-\Delta-\varepsilon s\ .$$

Now as $U\cup U_i$ is 1-packed,

$$\min\{|X\cap(U\cup U_i)|,|Y\cap(U\cup U_i)| \geq \mu-\lambda_1\ ,$$

or

$$\min\{|X\cap(U\cup U_i)|,|Y\cap(U\cup U_i)| \geq \nu-\Delta-\varepsilon s-\tau\ .$$

In both cases, we obtain that $|U\cup U_i| > \bar{\deg}(Z,X\cup Y)-\lambda_1-\Delta-\varepsilon s$, a contradiction. Thus by Lemma 6.5, we can embed the tree t_i. If $\min\{|P_i'|,|Q_i'|\} \geq \Delta$, we embed t_i in $P_i\cup Q_i$ using Lemma 6.5, Part 2, so that

$$||X\cap(U\cup U_{i+1})|-|Y\cap(U\cup U_{i+1})|| \leq \max\{||X\cap(U\cup U_i)|-|Y\cap(U\cup U_i)||,\tau\}. \tag{6.9}$$

Inequality (6.9) ensures that Property (c3)$_i$ holds. There is nothing to prove if

$$\min\{|X\cap(U\cup U_{i+1})|,|Y\cap(U\cup U_{i+1})|\} \geq \min\{\bar{\deg}(Z,X),\bar{\deg}(Z,Y)\}-\lambda_1\ . \tag{6.10}$$

So, suppose that (6.10) does not hold. We show that $\min\{|P_i'|,|Q_i'|\} \geq \Delta$. Then by (6.9) and by the fact that $U\cup U_i$ is 1-packed, we obtain that $||X\cap(U\cup U_{i+1})|-|Y\cap(U\cup U_{i+1})|| \leq \tau$. Assume for contradiction and without loss of generality that $|P_i'| < \Delta$. Then

$$|X\cap(U\cup U_i)| \geq \deg(r,X)-\Delta-|B\cap X| \geq \mu-\lambda_1+\tau.$$

As $U\cup U_i$ is 1-packed, we obtain (6.10), a contradiction to our assumption. Properties (c1)$_i$ and (c2)$_i$ follow from the fact that P_i is disjoint from $U\cup U_i$ and B.

Part 2. The proof goes in a similar spirit as in Part 1. We embed sequentially the components t_i of $F-R$ using Lemma 6.5. In each step, vertices of $V(t_i)\cap F_1$ are mapped to $N(A)\cap(X\cup Y)\setminus(U\cup U_i)$ so that $U\cup U_i$ remains 1-balanced.

Part 3. In each step i of the embedding we require the following four invariants to hold.

(P1) $U\cup U_{i+1}$ is 2-packed (with parameters λ_2 and τ).

(P2) If $|P_i\setminus P_i'| > \Delta$, then the tree t_i is embedded so that $\varphi(V(t_i)\setminus\{r_i\})\cap \mathrm{N}(r)\cap X=\emptyset$. Similarly, if $|Q_i\setminus Q_i'| > \Delta$, then $\varphi(V(t_i)\setminus\{r_i\})\cap \mathrm{N}(r)\cap Y=\emptyset$.

(P3) If $\min\{|P_i'|,|Q_i'|\} \geq \Delta$, then $||(U\cup U_{i+1})\cap X|-|(U\cup U_{i+1})\cap Y|| \leq \max\{\tau,||(U\cup U_i)\cap X|-|(U\cup U_i)\cap Y||\}$.

(P4) If $\min\{|(U\cup U_{i+1})\cap X|,|(U\cup U_{i+1})\cap Y|\} < \min\{2\mu,\nu\}-\lambda_2$, then $\min\{|P_{i+1}'|,|Q_{i+1}'|\} \geq \Delta$.

Properties **(P1)**, **(P2)**, **(P3)**, and **(P4)** are clearly met at step $i = 0$. Assume that **(P1)**, **(P2)**, **(P3)**, and **(P4)** hold in the step $i-1$. We first prove the following auxiliary claims

(α) $\max\{|P'_i|, |Q'_i|\} \geq \Delta$, and

(β) $\min\{|P_i|, |Q_i|\} \geq \Delta$.

We prove (α) by contradiction. Suppose that $\max\{|P'_i|, |Q'_i|\} < \Delta$. We claim that

$$\min\{|X \setminus (U \cup U_i \cup \mathrm{N}(r))|, |Y \setminus (U \cup U_i \cup \mathrm{N}(r))|\} \geq \Delta + \varepsilon s \,. \tag{6.11}$$

Suppose that (6.11) does not hold. Assume without loss of generality that $|X \setminus (U \cup U_i \cup \mathrm{N}(r))| < \Delta + \varepsilon s$. Recall that $|P'_i| < \Delta$. Thus we have $|X \cap (U \cup U_i)| > s - 2\Delta - 2\varepsilon s$. The fact that $U \cup U_i$ is 2-packed implies that $|U \cup U_i| \geq s + \min\{2\mu, \nu\} - \lambda_2 - 2\Delta - 2\varepsilon s > \overline{\mathrm{deg}}(Z, X \cup Y) + \frac{\eta s}{2}$, a contradiction. Inequality (6.11) implies by **(P2)** that only the roots of the trees t_j ($j < i$) were embedded in $\mathrm{N}(r)$ and thus $|U_i \cap \mathrm{N}(r)| \leq |U_i|/2 \leq \nu(F)/2$ (recall that $\nu(t_j) \geq 2$ for all $j < i$). We have thus

$$\begin{aligned}
|P'_i| + |Q'_i| &\geq \mathrm{d}(Z,X)|X \setminus U| + \mathrm{d}(Z,Y)|Y \setminus U| - |U_i \cap \mathrm{N}(r)| - 6\varepsilon s \\
&\geq \overline{\mathrm{deg}}(Z, X \cup Y) - \frac{\nu(F)}{2} - \mathrm{d}(Z,X)|X \cap U| - \mathrm{d}(Z,Y)|Y \cap U| - 6\varepsilon s \\
&\overset{(6.7)}{\geq} (\mathrm{d}(Z,X) + \mathrm{d}(Z,Y))\frac{s}{2} + (1/2 - \mathrm{d}(Z,X))|X \cap U| + (1/2 - \mathrm{d}(Z,Y))|Y \cap U| - \frac{\eta s}{6} \,.
\end{aligned} \tag{6.12}$$

We write *RHS* to denote the right-hand side of (6.12). We bound *RHS* in two cases separately, based on the value of $\mathrm{d}(Z,Y)$.

- $\mathrm{d}(Z,Y) \geq 1/2$.

$$\begin{aligned}
RHS &\geq (\mathrm{d}(Z,X) + \mathrm{d}(Z,Y))s/2 + (1/2 - \mathrm{d}(Z,X))|X \cap U| + (1/2 - \mathrm{d}(Z,Y))s - \frac{\eta s}{6} \\
&= (\mathrm{d}(Z,X) - \mathrm{d}(Z,Y))s/2 + (1/2 - \mathrm{d}(Z,X))|X \cap U| + s/2 - \frac{\eta s}{6} \\
&= \frac{1}{2}\mathrm{d}(Z,X)|X \setminus U| + \frac{1}{2}(1 - \mathrm{d}(Z,X))|X \cap U| + (1 - \mathrm{d}(Z,Y))s/2 - \frac{\eta s}{6} \\
&\geq \frac{\eta s}{12} \,,
\end{aligned}$$

a contradiction.

- $\mathrm{d}(Z,Y) \le 1/2$.

$$\begin{aligned} RHS &\ge \mathrm{d}(Z,X)s/2 + (1/2 - \mathrm{d}(Z,X))|X \cap U| - \frac{\eta s}{6} \\ &= \frac{1}{2}(1 - \mathrm{d}(Z,X))|X \cap U| + \frac{1}{2}\mathrm{d}(Z,X)|X \setminus U| - \frac{\eta s}{6} \\ &\ge \frac{\eta s}{12}, \end{aligned}$$

a contradiction.

We now turn to proving (β). If (β) does not hold, then $\max\{|X \cap (U \cup U_i)|, |Y \cap (U \cup U_i)|\} \ge s - \Delta - \varepsilon s$. As $U \cup U_i$ is 2-packed $\min\{|X \cap (U \cup U_i)|, |Y \cap (U \cup U_i)|\} \ge s - \Delta - \varepsilon s - \tau$, or $\min\{|X \cap (U \cup U_i)|, |Y \cap (U \cup U_i)|\} \ge \min\{2\mu, \nu\} - \lambda_2$. In both cases, we obtain

$$\begin{aligned} |U \cup U_i| &\ge s + \min\{2\mu, \nu\} - \Delta - \varepsilon s - \lambda_2 \\ &\ge \overline{\deg}(Z, X \cup Y) + \eta s - \Delta - \varepsilon s - \lambda_2, \end{aligned}$$

a contradiction with the bound (6.7), as $\eta s - \Delta - \varepsilon s - \lambda_2 > \frac{\eta s}{4}$.

Having proved that (α) and (β) hold, we may use Lemma 6.5 in order to embed t_i in $P_i \cup Q_i$. If $\min\{|(U \cup U_i) \cap X|, |(U \cup U_i) \cap Y|\} \ge \min\{2\mu, \nu\} - \lambda_2$ we use only Part 1. If $\min\{|(U \cup U_i) \cap X|, |(U \cup U_i) \cap Y|\} < \min\{2\mu, \nu\} - \lambda_2$, we use Parts 1 and 2. Property **(P4)** for $i-1$ implies that we have the choice or mapping r_i to P'_i or to Q'_i. We choose the side so that $||(U \cup U_{i+1}) \cap X| - |(U \cup U_{i+1}) \cap Y|| \le \max\{\tau, ||(U \cup U_i) \cap X| - |(U \cup U_i) \cap Y||\}$, and if $v(t_i) = 2$, we map r_i to the opposite cluster to the one where lies $\varphi(r_{i-1})$.

The embedding of t_i clearly satisfies **(P1)**, **(P2)** and **(P3)**. To prove that the embedding of t_i satisfies also **(P4)**, we need the following auxiliary claim.

Claim. *If* $\min\{|(U \cup U_i) \cap X|, |(U \cup U_i) \cap Y|\} < \min\{2\mu, \nu\} - \lambda_2$, *then* $|\varphi(\{r_1, \dots, r_i\}) \cap X| \le |U_{i+1} \cap X|/2 + \tau + 1$ *and* $|\varphi(\{r_1, \dots, r_i\}) \cap Y| \le |U_{i+1} \cap Y|/2 + \tau + 1$.

The proof of the claim is postponed to the end of the inductive step.

We prove Property **(P4)** by contradiction, so assume that $\min\{|(U \cup U_{i-1}) \cap X|, |(U \cup U_{i-1}) \cap Y|\} < \min\{2\mu, \nu\} - \lambda_2$ and that $|P'_{i+1}| < \Delta$ (the case when $|Q'_{i+1}| < \Delta$ is proved analogously). We claim that

$$|P_{i+1} \setminus P'_{i+1}| \ge \Delta + s - \min\{2\mu, \nu\} + 6\varepsilon s + 3\tau > \Delta. \tag{6.13}$$

Indeed, otherwise $|X \cap (U \cup U_{i+1})| > s - |P_{i+1} \setminus P'_{i+1}| - \Delta - \varepsilon s \ge \min\{2\mu, \nu\} - \lambda_2 + \tau$. Property **(P1)** implies that $\min\{|(U \cup U_{i+1}) \cap X|, |(U \cup U_{i+1}) \cap Y|\} > \min\{2\mu, \nu\} - \lambda_2$, a contradiction with our assumption. This settles (6.13). The property **(P2)**, together with Inequality (6.13) and Part 1 of Lemma 6.5, implies that only the roots of the trees t_j, $j \le i$ were mapped to $X \cap \mathrm{N}(r)$, i. e.,

$U_{i+1} \cap X \cap \mathrm{N}(r) = \varphi(\mathrm{N}(R)) \cap X$. By the auxiliary claim, we obtain

$$|U_{i+1} \cap X \cap \mathrm{N}(r)| = |\varphi(\{r_1, \dots, r_i\}) \cap X| \leq |U_{i+1} \cap X|/2 + \tau + 1. \tag{6.14}$$

On the other hand, using (6.13), we obtain

$$\begin{aligned} |U_{i+1} \cap X| &\leq |X \setminus U| - |P_{i+1} \setminus P'_{i+1}| \\ &\leq \min\{2\mu, \nu\} - |X \cap U| - \Delta - 6\varepsilon s - 3\tau \\ &\leq 2\mathrm{d}(Z,X)|X \setminus U| - \Delta - 6\varepsilon s - 3\tau. \end{aligned}$$

Together with the assumption $|P'_{i+1}| < \Delta$, this yields the following inequality.

$$\begin{aligned} |U_{i+1} \cap X \cap \mathrm{N}(r)| &\geq |\mathrm{N}(r) \cap (X \setminus U)| - \Delta - \varepsilon s \\ &\geq \mathrm{d}(Z,X)|X \setminus U| - \Delta - 3\varepsilon s \\ &> |U_{i+1} \cap X|/2 + \tau + 1, \end{aligned}$$

a contradiction to (6.14). Let us now prove the auxiliary claim.

Proof of the auxiliary claim. We alternated the embedding of the roots r_j, $j \leq \min\{i, \ell_1\}$ between X and Y. This ensures that for $j \leq \min\{i, \ell_1\}$ we have

$$\begin{aligned} &|\varphi(\{r_1, \dots, r_j\}) \cap X| \leq |U_{\min\{i,\ell_1\}+1} \cap X|/2 + 1 \text{ and} \\ &|\varphi(\{r_1, \dots, r_j\}) \cap Y| \leq |U_{\min\{i,\ell_1\}+1} \cap Y|/2 + 1, \end{aligned} \tag{6.15}$$

proving the claim for $i \leq \ell_1$. Thus we assume that $i > \ell_1$. Denote by Γ_i the roots of the the trees t_j for $j \in \{\ell_1 + 1, \dots, \min\{i, \ell_2\}\}$. Then set $X_1 = X \cap \varphi(\Gamma_i)$, $X_2 = X \cap \varphi(\mathrm{N}_T(\Gamma_i)) \cap V(T(\downarrow \Gamma_i))$, and similarly $Y_1 = Y \cap \varphi(\Gamma_i)$ and $Y_2 = Y \cap \varphi(\mathrm{N}_T(\Gamma_i))V(T(\downarrow \Gamma_i))$. Thus the sets X_1, X_2, Y_1, Y_2 form a partition of the set $U_{\min\{i,\ell_2\}+1} \setminus U_{\ell_1+1}$. As all trees under consideration have order at least 3, observe that $2|X_1| \leq |Y_2|$ and $2|Y_1| \leq |X_2|$. As U and $U_{\min\{i,\ell_2\}+1}$ are 2-packed and $|U_{\ell_1} \cap X| = |U_{\ell_1} \cap Y|$, we know that $||X_1 \cup X_2| - |Y_1 \cup Y_2|| \leq 2\tau$. Then

$$|X_1| + |X_2| + 2\tau \geq |Y_1| + |Y_2| \geq |Y_2| \geq 2|X_1|.$$

This implies that $|X_2| + 2\tau \geq |X_1|$. The same holds for Y_1 and Y_2. Together with (6.15), this leads to the desired inequalities, if $i \leq \ell_2$. To see that the claim also holds for $i > \ell_2$, it is enough to realize that for $j > \ell_2$, when embedding the root r_j of the tree t_j in a set $C \in \{X, Y\}$, at least one vertex of $t_j - r_j$ is also mapped to C. $\square$

It remains to check whether the embedding φ of $F - R$ satisfies (c1), (c2), and (d). Each component was mapped to $P_i \cup Q_i$, which is disjoint with the set U and contains only vertices typical w. r. t. Z. This ensures Properties (c1) and (c2). Property (d) follows from the way we utilized property **(P4)** during embedding via Lemma 6.5 Part 2.

□

7 Proof of Proposition 4.4

Proof. Set η so that $\sigma \ll \eta \ll \omega$, and β, γ, α so that

$$0 < \beta \ll \gamma \ll \alpha \ll \sigma.$$

Let n_0 (the minimal order of the graph) and Π_1 (the upper bound for the number of clusters) be the numbers given by the Regularity Lemma 6.2 for input parameters β (for precision), $\Pi_0 = 2/\beta$ (for minimum number of clusters) and 4 (for the number of pre-partition classes).

Let G be a graph of order $n \geq n_0$ and the set $\bar{V} \subseteq V$ satisfying the assumptions of Proposition 4.4.

Prepartition the vertex-set V into $\bar{V} \cap L, \bar{V} \cap S, L \setminus \bar{V}$, and $S \setminus \bar{V}$. By the Regularity Lemma 6.2, there exists a partition $V = C_0 \cup C_1 \cup \cdots \cup C_N$ satisfying the following.

- $\Pi_0 \leq N \leq \Pi_1$,
- $|C_i| = |C_j| = s$, for any $i, j \in [N]$,
- $|C_0| \leq \beta n$,
- all but at most βN^2 pairs (C_i, C_j) are β-regular,
- if $C_i \cap L \neq \emptyset$, then $C_i \subseteq L$, for any $i \in [N]$, and
- if $C_i \cap \bar{V} \neq \emptyset$, then $C_i \subseteq \bar{V}$, for any $i \in [N]$.

Let G_γ denote the subgraph of G obtained from G by deleting the edges incident to C_0, contained in some C_i, lying between $V \setminus \bar{V}$ and $\bar{V}$, or between pairs that are irregular or of density smaller than $\gamma^2/2$. Let $(\mathbf{G}, \mathrm{d\bar{e}g}_{G_\gamma}(\cdot,\cdot))$ denote the weighted cluster graph induced by G_γ, i. e., $\mathbf{G}$ has order N, with vertex-set $V(\mathbf{G}) = \{C_1, \ldots, C_N\}$ and edge-set

$$E(\mathbf{G}) = \{CD : (C,D) \text{ is an } \beta\text{-regular pair with density at least } \gamma^2/2\},$$

with the weight function $\mathrm{d\bar{e}g} : E(\mathbf{G}) \to \mathbb{R}$, defined by $\mathrm{d\bar{e}g}(CD) = \mathrm{d\bar{e}g}_{G_\gamma}(C,D)$. Denote by $\mathscr{L}$ the set of clusters contained in $L \cap \bar{V}$ which have large average degree in $\bar{V}$,

$$\mathscr{L} = \{C \in V(\mathbf{G}) : C \subseteq L \cap \bar{V},\ \mathrm{d\bar{e}g}_{G_\gamma}(C, \bar{V}) \geq k - \gamma n\}.$$

We write $\bar{N}$ to denote the number of clusters in $\bar{V}$. Observe that $|\mathscr{L}| \geq (1-\sigma)\bar{N}/2 - \gamma N \geq \bar{N}/2 - \sigma\bar{N}$. Most of the clusters $V(\mathbf{G})$ formed by vertices of $L \cap \bar{V}$ are in $\mathscr{L}$. From Assumption 4.6, there are at most

$$2\gamma N \tag{7.1}$$

clusters $C \in V(\mathbf{G}) \setminus \mathscr{L}$ with $C \subseteq \bar{V}$ such that $\mathrm{d\bar{e}g}_{G_\gamma}(C, V(\mathbf{G}) \setminus \mathscr{L}) > \gamma n$. Let $\mathbf{H}$ be the subgraph of $\mathbf{G}$ induced by clusters contained in $\bar{V}$ such that all edges induced by the set $\{C \in \mathbf{G} : C \subseteq \bar{V} \setminus \bigcup_{D \in \mathscr{L}} D\}$ are removed. The weights of the edges in $\mathbf{H}$ are inherited from $\mathbf{G}$.

7.1 Matching structure in the cluster graph

If G satisfies the Special Case with parameter $c_{\mathbf{S}}$ (considering the set $\bar{V}$), then $\mathscr{T}_{k+1} \subseteq G$ by Proposition 4.2. In the rest of the proof, we thus assume that $e(G[\bar{V} \cap L]) \geq c_{\mathbf{S}} n^2$, and thus $e(G_\gamma[\bar{V} \cap L]) \geq \frac{c_{\mathbf{S}}}{2} n^2$, implying that $\mathscr{L}$ induces at least one edge in $\mathbf{G}$. This edge is an edge in $\mathbf{H}$ also. The weighted graph $(\mathbf{H}, \mathrm{d\bar{e}g}_{G_\gamma})$ satisfies all the conditions of Proposition 6.4 (with parameters σ and $K = k - \gamma n$). This ensures that one of the two specific matching structures in $\mathbf{H}$ exists. Together with (7.1), this yields the existence of one of the following two configurations in the cluster graph $\mathbf{G}$.

Case I: There are two adjacent clusters A, B and a matching M in $\mathbf{G}$ such that

- $\mathrm{d\bar{e}g}_{G_\gamma}(A, V(M)) \geq k - \gamma n$,
- each edge $e \in M$ intersects the neighbourhood of A in at most one cluster, and
- $\mathrm{d\bar{e}g}_{G_\gamma}(B, V(M) \cup \mathscr{L}^*) \geq (1+\sigma/2)\frac{k}{2}$, where $\mathscr{L}^* = \{C \in V(\mathbf{G}) : \mathrm{d\bar{e}g}_{G_\gamma}(C) \geq (1+\sigma/2)\frac{k}{2}\}$.

Case II: There exist a set of clusters $\mathscr{X}' \subseteq V(\mathbf{G})$, two adjacent clusters A, B, and a matching M in $\mathbf{G}$ such that

- $A, B \in \mathscr{X}' \cap \mathscr{L}$,
- $|V(M') \setminus \mathscr{X}'| \leq 1$, where $M' = \{CD \in M : C, D \in \mathrm{N}(\mathscr{X}')\}$,
- all but at most $3\gamma N$ clusters $C \in \mathscr{X}'$ satisfy $\mathrm{d\bar{e}g}_{G_\gamma}(C, V(M)) \geq \mathrm{d\bar{e}g}_{G_\gamma}(C) - 3\sigma n$,
- and each edge $e \in M$ intersects $\mathscr{L}$.

In the rest of the thesis the average degree $\mathrm{d\bar{e}g}$ will always be associated with the underlying graph G_γ, i.e., $\mathrm{d\bar{e}g}$ is an abbraviation for $\mathrm{d\bar{e}g}_{G_\gamma}$.

Let $\tilde{M} \subseteq M$ be the maximal submatching of M not covering A nor B. Let $T \in \mathscr{T}_{k+1}$ be any tree with k edges. Trivially, $|\tilde{M}| \geq |M| - 2$. Choose a root $R \in V(T)$ and cut the tree T as in Section 6.2 in order to obtain a switched τ-fine partition $(W_A, W_B, \mathscr{D}_A, \mathscr{D}_B)$, with $\tau = \beta k/\Pi_1$.

7.2 Case I

Denote by $\mathscr{T}_F$ the components of $\mathscr{D}_A$ consisting of interior subtrees and by $\mathscr{T}_A$ the ones consisting of end subtrees of $\mathscr{D}_A$. Denote by T_F the forest induced by the components in $\mathscr{T}_F$, by T_A the forest induced by the components in $\mathscr{T}_A$ and by T_B the forest induced by the components in $\mathscr{D}_B$. Recall that $\mathscr{D}_B$ consists only of end subtrees. If $\mathscr{D}_A \cup \mathscr{D}_B$ is $c_{\mathbf{U}}$-unbalanced, then $T \subseteq G$, as shown by Proposition 4.3. Thus we may assume that $\mathscr{T}_F \cup \mathscr{T}_A \cup \mathscr{D}_B$ is $c_{\mathbf{U}}$-balanced.

We partition each cluster $C \in V(M) \cup \mathscr{L}^*$ so that the partition defines two disjoint sets M^F and M^B of vertices of G, such that $M^F, M^B \subseteq \bigcup\{C \in V(\tilde{M})\}$. The embedding $\varphi : V(T) \to V$ of the tree T is defined in three phases. In the first phase, we embed the subtree $T' = T[W_A \cup W_B \cup V(T_F \cup T_B^M)]$, where $T_B^M \subseteq T_B$ will be defined later. The forest T_F is embedded in M^F and the forest T_B^M in M^B. In the second phase, we embed $T_B^L = T_B - V(T_B^M)$ in $\bigcup\{C \in (\mathscr{L}^* \setminus V(M)) \cup \mathrm{N}(\mathscr{L}^*)\}$. In the last phase we embed T_A in $\bigcup\{C \in V(\tilde{M})\}$. Thus we complete the embedding of T.

The difference between the presented proof of Theorem 1.4 and its approximate version Theorem 1.3 is that in the proof of Theorem 1.4 we have to fight to gain back small loses caused by the use of the Regularity Lemma. However, this is not necessary when we have the matching structure of Case I. Then, we are able to reduce the situation to the “approximate version”, i.e., to the setting of similar nature as in Theorem 1.3.

We partition each cluster $C \in V(M) \cup \mathscr{L}^*$ into C^F and C^B in an arbitrary way so that $|C^F| = (1-y)|C|$ and $|C^B| = y|C|$, where

$$y = \frac{v(T_A \cup T_B)}{k} \cdot \frac{1}{1+\sigma/4} + \alpha \geq \frac{2v(T_B)}{k} \cdot \frac{1}{1+\sigma/4} + \alpha. \tag{7.2}$$

Set

$$M^B = \bigcup_{C \in V(\tilde{M})} C^B, \quad M^F = \bigcup_{C \in V(\tilde{M})} C^F, \text{ and } \quad \mathscr{L}^B = \bigcup_{C \in \mathscr{L}^* \setminus V(M)} C^B.$$

Observe that $y \in (\alpha, 1-\alpha)$. Thus, for each $C \in V(M) \cup \mathscr{L}^*$, the sets C^B and C^F are significant. Observe also that the pairs (C^F, D^F) and (C^B, D^B) are β/α-regular for every $C, D \in V(M) \cup \mathscr{L}^*$. Now,

$$\begin{aligned}
\overline{\deg}(B, M^B \cup \mathscr{L}^B) &\geq y(1+\sigma/2)\frac{k}{2} - \beta n - 4s \\
&\overset{(7.2)}{\geq} \frac{1+\sigma/2}{1+\sigma/4} v(T_B) + \alpha \frac{k}{2} - \beta n - 4s \\
&> v(T_B) + \alpha \frac{k}{4}\,.
\end{aligned} \tag{7.3}$$

A similar calculation shows that for any cluster $D \in \mathscr{L}^*$, we have

$$\mathrm{d}\,\overline{\mathrm{eg}}(D, V \setminus (M^F \cup A \cup B)) \geq v(T_B) + \alpha \frac{k}{4}. \tag{7.4}$$

For cluster A, we obtain

$$\begin{aligned} \mathrm{d}\,\overline{\mathrm{eg}}(A, M^F) &\geq (1-y)(k - \gamma n) - \beta n - 4s \\ &\overset{(7.2)}{\geq} k - v(T_A \cup T_B)/(1+\sigma/4) - \alpha k - \gamma n - \beta n - 4s \\ &\geq v(T_F) + v(T_A \cup T_B)\sigma/8 - 2\alpha n \\ &\geq \max\{|V(T_F) \cap T_{\mathrm{o}}|, |V(T_F) \cap T_{\mathrm{e}}|\} + \sigma c_{\mathbf{U}}^2 k/32 - 2\alpha n, \end{aligned} \tag{7.5}$$

where the last inequality follows from the fact that $\mathscr{T}_F$ is $c_{\mathbf{U}}/2$-balanced, or $\mathscr{T}_A \cup \mathscr{D}_B$ is. Let $\mathscr{T}_B^M \subseteq \mathscr{D}_B$ be a maximal subset of $\mathscr{D}_B$ such that

$$\sum_{t \in \mathscr{T}_B^M} v(t) \leq \mathrm{d}\,\overline{\mathrm{eg}}(B, M^B) - \frac{\alpha k}{8}. \tag{7.6}$$

Let T_B^M be the forest formed by the trees of $\mathscr{T}_B^M$, let $\mathscr{T}_B^L = \mathscr{D}_B \setminus \mathscr{T}_B^M$ and T_B^L be the forest formed by the trees in $\mathscr{T}_B^L$. Recall that $T' = T[W_A \cup W_B \cup V(T_F) \cup V(T_B^M)]$.

Phase 1. In this phase, we embed the subtree T'. The embedding of T' is devided into $w = |W_A \cup W_B|$ steps. We label the vertices of $W_A \cup W_B$ as $x_1, \ldots, x_w$, indexing from the root R downwards, i.e., in such way that $j_1 \leq j_2$ whenever $x_{j_1} \succeq_R x_{j_2}$. In step $i \geq 1$, we shall take the vertex x_i and define the embedding for x_i and the shrublets hanging from x_i, i. e., we embed the tree T_i,

$$T_i = T\left[\{x_i\} \cup \bigcup_{\iota \in [c_i]} V(P_\iota)\right],$$

where $P_1, \ldots, P_{c_i}$ denotes the components P of $T_F \cup T_B^M$ such that $\mathrm{Ch}(x_i) \cap V(P) \neq \emptyset$. The tree T_i is a union of trees $t_i^\iota = T[\{x_i\} \cup V(P_\iota)]$ ($\iota \in [c_i]$). Set $V_i = \bigcup_{j<i} V(T_j)$ and $U_i = \varphi(V_i)$.

If $i > 1$, let $p_i = \mathrm{Par}(x_i)$. During the embedding process we will keep the following three invariants in every step i.

(I1) The $U_i \cap (C^F \cup D^F)$ is 1-packed with parameters

$$\lambda_F = \frac{\beta s'/\alpha + \tau}{\gamma^2/2 - 2\beta/\alpha} + \tau + 3\beta s'/\alpha \text{ and } \tau, \text{ where } s' = (1-y)s,$$

with respect to the embedding sets C^F and D^F and the head set A for each edge $CD \in \tilde{M}$,

(I2) The $U_i \cap (C^B \cup D^B)$ is 1-packed with parameters

$$\lambda_B = \frac{\beta s''/\alpha + \tau}{\gamma^2/2 - 2\beta/\alpha} + \tau + 3\beta s''/\alpha \text{ and } \tau \text{ , where } s'' = ys \text{ ,}$$

with respect to the embedding sets C^B and D^B and the head set B for each edge $CD \in \tilde{M}$, and

(I3) if $i > 1$, then the vertex p_i was already embedded in some previous step so that $|\mathrm{N}(\varphi(p_i)) \cap A| \geq \gamma^2 s/4$ (if $x_i \in W_A$), or $|\mathrm{N}(\varphi(p_i)) \cap B| \geq \gamma^2 s/4$ (if $x_i \in W_B$).

Say that a vertex is *A-typical*, if it is typical w. r. t. all but at most $\sqrt{\beta}N$ sets C^F, $C \in V(\tilde{M})$, w. r. t. all but at most $\sqrt{\beta}N$ clusters $C \in V(\tilde{M})$, and w. r. t. the cluster B. All but at most $3\sqrt{\beta}|A|$ vertices of cluster A are A-typical. Say that a vertex is *B-typical*, if is is typical w. r. t. all but at most $\sqrt{\beta}N$ sets C^B, $C \in V(\tilde{M})$, w. r. t. $\mathscr{L}^B$, and w. r. t. the cluster A. All but at most $3\sqrt{\beta}|B|$ vertices of cluster B are B-typical. The embedding φ will be defined in such a way that $\varphi(W_A) \subseteq A$ and $\varphi(W_B) \subseteq B$. From the property of the switched τ-fine partition $(W_A, W_B, \mathscr{D}_A, \mathscr{D}_B)$ we have $\max\{|W_A|, |W_B|\} \leq 12k/\tau \ll \gamma^2 s/4$. Thus if the predecessor of a vertex $x_i \in W_A$ has at least $\gamma^2 s/4$ neighbours in A, then we have have enough candidates to choose an unused A-typical vertex from as $\varphi(x_i)$.

To define the embedding of the tree T_i we first choose $\varphi(x_i)$. If $i = 1$ then $x_i = R$, and we map x_i to an arbitrary A-typical vertex in A (if $R \in W_A$), or on an arbitrary B-typical vertex in B (if $R \in W_B$). If $i > 1$ choose for $\varphi(x_i)$ any A-typical vertex in $A \cap \mathrm{N}(\varphi(p_i))$ (if $x_i \in W_A$), or any B-typical vertex in $B \cap \mathrm{N}(\varphi(p_i))$ (if $x_i \in W_B$). This is possible by **(I3)**.

Assume that $x_i \in W_A$. Then $V(T_i) \subseteq V(T_F)$. Set $\mathscr{C}_i = \{C \in V(\tilde{M}) \cap \mathrm{N}(A) : \varphi(x_i) \text{ is typical w. r. t. } C^F\}$. We deduce that

$$\begin{aligned}
\sum_{C \in \mathscr{C}_i} \overline{\mathrm{deg}}(A, C^F) - |U_i \cap M^F| &\geq \overline{\mathrm{deg}}(A, M^F) - \sqrt{\beta}n - |V_i \cap V(T_F)| \\
&\overset{(7.5)}{\geq} \max\{|V(T_F) \cap V(T_\mathrm{o})|, |V(T_F) \cap V(T_\mathrm{e})|\} - |V_i \cap V(T_F)| + \frac{\sigma}{8}\left(\frac{c_{\mathbf{U}}}{2}\right)^2 \cdot k - 2\alpha n - \sqrt{\beta}n \\
&\geq \max\{|V(T_i) \cap V(T_\mathrm{o})|, |V(T_i) \cap V(T_\mathrm{e})|\} + \alpha k. \qquad (7.7)
\end{aligned}$$

We consider an auxiliary mapping $\zeta : [c_i] \to \tilde{M}$ which has the property that for any $XY \in \tilde{M}$, $X \in \mathscr{C}_i$ it holds

$$\sum_{\iota \in \zeta^{-1}(XY)} v(P_\iota) + |U_i \cap (X^F \cup Y^F)| \leq \overline{\mathrm{deg}}(A, X^F \cup Y^F) - \lambda_F \text{ .} \qquad (7.8)$$

From (7.7) such mapping ζ exists.

We embed the trees t_i^ι, $\iota = 1, \ldots, c_i$ using Lemma 6.6 Part 2. The setting for applying Lemma 6.6 is the following. The root of t_i^ι is the vertex x_i. The head set is the cluster A and the embedding sets are the sets X^F, Y^F, where $XY = \zeta(\iota)$. The set of "forbidden vertices" is $U_{i,\iota} =$

$(U_i \cup \bigcup_{\ell<\iota} \varphi(t_i^\ell) \cap (X^F \cup Y^F)$. The set $U_{i,\iota}$ is 1-packed with parameters λ and τ, by induction. Now, Lemma 6.6 Part 1 allows us to embed the tree t_i^ι so that

- $\varphi(t_i^\iota) \subseteq (X^F \cup Y^F) \setminus U_{i,\iota}$,
- each vertex in $V(t_i^\iota)$ with odd distance from x_i has at least $\gamma^2 s/4$ neighbors in A,
- the set $(U_i \cup \bigcup_{\ell \le \iota} \varphi(t_i^\ell)) \cap (X^F \cup Y^F)$ is 1-packed with parameters λ and τ.

Observe that the last property is sufficient for our inductive assumption on the sets $U_{i,\iota}$, and also to prove invariant **(I1)**. The second property ensures invariant **(I3)** to hold. Property **(I2)** is preserved.

In the case that $x_i \in W_B$, set $M_i = \{C^B D^B : CD \in \tilde{M},\ \varphi(x_i) \text{ is typical w.r.t. both } C^B \text{ and } D^B\}$. Similar calculations as above give

$$\sum_{C^B D^B \in M_i} \overline{\deg}(B, (C^B \cup D^B) \setminus U_i) \ge v(T_i) + \alpha k/16.$$

We embed the trees t_i^ι, $\iota = 1, \ldots, c_i$ using Lemma 6.6 Part 1 in the sets $C^B \cup D^B$ ($CD \in M_i$) so that invariants **(I1)**, **(I1)**, and **(I3)** hold.

Phase 2. In this phase, we embed the yet unembedded shrublets adjacent to W_B (i.e. T_B^L). We label the shrublets of $\mathscr{T}_B^L$ as $t_1, \ldots, t_{|\mathscr{T}_B^L|}$. In step $i \ge 1$, we define the embedding for shrublet t_i in a suitable edge $CD \in E(\mathbf{G})$. Set $U_i = \varphi(V(T_F \cup T_B^M) \cup \bigcup_{j<i} V(t_j))$. Let $x_i \in W_B$ be the parent of the root of the shrublet t_i. The vertex $\varphi(x_i)$ is typical w.r.t. $\mathscr{L}^B$ and hence by (7.3) and (7.6),

$$\begin{aligned}
\deg(\varphi(x_i), \mathscr{L}^B) &\ge \overline{\deg}(B, \mathscr{L}^B) - 2\beta n \\
&= \overline{\deg}(B, M^B \cup \mathscr{L}^B) - \overline{\deg}(B, M^B) - 2\beta n \\
&\ge v(T_B) + \alpha k/4 - v(T_B^M) - \alpha k/8 - 2\beta n \\
&\ge v(T_B^L) + \alpha k/16\,.
\end{aligned}$$

Thus there is a cluster $D \in \mathscr{L}^* \setminus V(M)$ containing a large unused neighbourhood of $\varphi(x_i)$. That is

$$|\mathrm{N}(\varphi(x_i)) \cap D \setminus U_i| \ge \frac{\alpha k}{16N} \ge \frac{\beta s + \tau}{\gamma^2/2 - 2\beta}\,.$$

From (7.4) we obtain that

$$\overline{\deg}(D, V \setminus U_i) \ge \overline{\deg}(D, V \setminus (M^F \cup A \cup B)) - |\varphi(V(T_B)) \cap U_i| \ge v(t_i) + \alpha k/4\,.$$

Thus there is a cluster $C \in \mathrm{N}(D)$ with $|C \setminus U_i| \ge \frac{\beta s + \tau}{\gamma^2/2 - 2\beta}$. Use Lemma 6.5 to embed t_i in $(C \cup D) \setminus U_i$ so that the root r_i of the shrublet t_i is mapped to $\mathrm{N}(\varphi(x_i)) \cap D \setminus U_i$.

Phase 3. In this phase, we finish the embedding of the tree by embedding the end shrublets adjacent to W_A (i. e. T_A). We label the shrublets of $\mathscr{T}_A$ as $t_1,\dots,t_{|\mathscr{T}_A|}$.

First assume that $\mathscr{T}_F \cup \mathscr{D}_B$ is $c_{\mathbf{U}}/2$-balanced. The embedding will be defined for steps $i \in [|\mathscr{T}_A|]$. In step i for a cluster $X \in V(\tilde{M})$ denote by X_{U_i} the set of vertices in X used by the embedding of $T_F \cup T_B$ and of $\bigcup_{j<i} t_j$. We find a suitable edge $CD \in \tilde{M}$ in which we embed the tree t_i. Let $x_i \in W_A$ be the parent of the root of t_i. By Lemma 6.5, the shrublet t_i can be embedded in unused vertices of an edge $CD \in \tilde{M}$, $C \in \mathrm{N}(A)$ in such a way that the root of t_i is mapped to a neighbor of $\varphi(x_i)$, whenever CD satisfies

$$\Upsilon^i_{CD} = \min\{|\mathrm{N}(\varphi(x_i)) \cap C \setminus C_{U_i}|, |D \setminus D_{U_i}|\} \geq v(t_i) + \alpha s \,. \tag{7.9}$$

Thus we are able to finish the embedding of T if we can find an every step i an edge $CD \in \tilde{M}$ satisfying (7.9). Suppose that at some step $i \geq 1$ there are no edges in $\tilde{M}$ with this property. Denote by $M_i \subseteq \tilde{M}$ the submatching of $\tilde{M}$ induced by the clusters $\{X \in V(\tilde{M}) : \varphi(x_i) \text{ is typical w. r. t. } X\}$. Then $\Upsilon^i_{CD} < v(t_i) + \alpha s$ for any $CD \in M_i$. The non-existence of a suitable matching edge implies that

$$\sum_{CD\in\tilde{M}} \Upsilon^i_{CD} < \sum_{CD\in\tilde{M}} (\tau + \alpha s) \leq \frac{1}{2} N(\tau + \alpha s) < \alpha n \,.$$

On the other hand,

$$\begin{aligned}\sum_{\substack{CD\in\tilde{M}\\ C\in\mathrm{N}(A)}} \Upsilon^i_{CD} \geq & \sum_{\substack{CD\in M_i\\ C\in\mathrm{N}(A)}} \left(|\mathrm{N}(\varphi(x_i)) \cap C| - \max\{|C_{U_i}|, |D_{U_i}|\}\right) \\ \geq & k - \gamma n - \sqrt{\beta} n - (v(T_F \cup T_B) - c_{\mathbf{U}}^2 k/4) - v(T_A) \\ \geq & \alpha n \,, \end{aligned}$$

a contradiction.

If $\mathscr{T}_F \cup \mathscr{D}_B$ is $c_{\mathbf{U}}/2$-unbalanced, then $\mathscr{T}_A$ is $c_{\mathbf{U}}/2$-balanced implying that $\max\{|V(T_A \cap T_{\mathrm{e}})|, |V(T_A \cap T_{\mathrm{o}})|\} \leq v(T_A) - (c_{\mathbf{U}}/2)^2 k$. Similarly as above, we find a suitable edge $CD \in \tilde{M}$, $C \in \mathrm{N}(A)$ with

$$\Upsilon^i_{CD} = \min\{|\mathrm{N}(\varphi(x_i)) \cap C \setminus C_{U_i}|, |D \setminus D_{U_i}|\} \geq \max\{|V(t_i) \cap T_{\mathrm{o}}|, |V(t_i) \cap T_{\mathrm{e}}|\} + \alpha s \,.$$

The calculations that such an edge exists are left to the reader. We use Proposition 6.5 to embed t_i in $(C \setminus C_{U_i}) \cup (D \setminus D_{U_i})$ with the root of t_i mapped to $C \cap \mathrm{N}(\varphi(x_1))$.

7.3 Case II

This case follows the lines of part of the proof from [22]. For completeness, and to adjust the setting, we prove this part in all detail.

Denote by T_A the forest induced by the components in $\mathscr{D}_A$ and by T_B the forest induced by the components in $\mathscr{D}_B$. Observe that $v(T_B) \le v(T_A)$. If $\mathscr{D}_A \cup \mathscr{D}_B$ is $c_{\mathbf{U}}$-unbalanced, then $T \subseteq G$, as shown by Proposition 4.3. Thus we may assume that $\mathscr{D}_A \cup \mathscr{D}_B$ is $c_{\mathbf{U}}$-balanced. In the first part of this section, after auxiliary Lemmas 7.1 and 7.2, we show in Lemma 7.3 that $T \subseteq G$ or the clusters A and B are very densely connected to their respective neighbourhood. In the second part, we prove in Lemma 7.7 that if V', the neighbourhood of the cluster A, is well connected to $V \setminus V'$, then $T \subseteq G$. If V' is poorly connected to $V \setminus V'$, then we show that V' satisfies the properties required by the statements of Proposition 4.4.

Let $\tilde{M}$ be the maximum submatching of M not containing the clusters A and B. With a slight abuse of notation, we can write $\tilde{M} = M \setminus \{e_A, e_B\}$, where e_A and e_B are the matching edges containing A, and B respectively (the edges e_A, e_B may be not defined, though). Observe that

$$\min\{\mathrm{d\bar{e}g}(A, V(\tilde{M})), \mathrm{d\bar{e}g}(B, V(\tilde{M}))\} \ge k - 4\sigma n \, . \tag{7.10}$$

PART I: Defining V'.

Lemma 7.1. *Suppose that $v(T_B) \ge \sqrt[4]{\sigma}k$. Then $\sum_{e \in M} |\mathrm{d\bar{e}g}(A,e) - \mathrm{d\bar{e}g}(B,e)| < 9\sqrt[4]{\sigma}k$, or $T \subseteq G$.*

Proof. Assume that $v(T_B) \ge \sqrt[4]{\sigma}k$ and $\sum_{e \in M} |\mathrm{d\bar{e}g}(A,e) - \mathrm{d\bar{e}g}(B,e)| \ge 9\sqrt[4]{\sigma}k$. Then $\sum_{e \in \tilde{M}} |\mathrm{d\bar{e}g}(A,e) - \mathrm{d\bar{e}g}(B,e)| \ge 8\sqrt[4]{\sigma}k$. We show that then $T \subseteq G$. Set $M^1 = \{e \in \tilde{M} : \mathrm{d\bar{e}g}(A,e) \ge \mathrm{d\bar{e}g}(B,e)\}$ and $M^2 = \tilde{M} \setminus M^1$. Without loss of generality, we may assume that

$$\mathrm{d\bar{e}g}(A, V(M^1)) - \mathrm{d\bar{e}g}(B, V(M^1)) \ge 4\sqrt[4]{\sigma}k \, . \tag{7.11}$$

Label the edges of $\tilde{M}$ as $\{e_1, \dots, e_{|\tilde{M}|}\}$ so that for any $i < j$, it holds that

$$\frac{\mathrm{d\bar{e}g}_{e_i}(A)}{\mathrm{d\bar{e}g}_{e_i}(B)} \ge \frac{\mathrm{d\bar{e}g}_{e_j}(A)}{\mathrm{d\bar{e}g}_{e_j}(B)} \, ,$$

with the convention that $\frac{x}{0} = +\infty$, for any $x \ge 0$. As $v(T_B) \ge \sqrt[4]{\sigma}k$, there exists an index ℓ such that

$$v(T_A) + \alpha k \le \sum_{i \le \ell} \mathrm{d\bar{e}g}_{e_i}(A) < v(T_A) + \alpha k + 2s \overset{(7.10)}{<} \mathrm{d\bar{e}g}(A, V(\tilde{M})) \, . \tag{7.12}$$

Set $M_A = \{e_1, \dots, e_\ell\}$ and $M_B = \tilde{M} \setminus M_A$. We claim that

$$\mathrm{d\bar{e}g}(B, V(M_B)) \ge v(T_B) + \alpha k \, . \tag{7.13}$$

We prove (7.13) by case analysis. If $\mathrm{d\overline{eg}}(B,V(M_A)) < k/4$, then

$$\begin{aligned}
\mathrm{d\overline{eg}}(B,V(M_B)) &= \mathrm{d\overline{eg}}(B,V(\tilde{M})) - \mathrm{d\overline{eg}}(B,V(M_A)) \\
&\overset{(7.10)}{>} k - 4\sigma n - k/4 > k/2 + \alpha k \\
&\geq v(T_B) + \alpha k \;.
\end{aligned}$$

If $\mathrm{d\overline{eg}}(A,V(M_A)) - \mathrm{d\overline{eg}}(B,V(M_A)) \geq \frac{\sqrt{\sigma}k}{4}$, then

$$\begin{aligned}
\mathrm{d\overline{eg}}(B,V(M_B)) &= \mathrm{d\overline{eg}}(B,V(\tilde{M})) - \mathrm{d\overline{eg}}(B,V(M_A)) \\
&\overset{(7.10)}{\geq} k - 4\sigma n - \mathrm{d\overline{eg}}(A,V(M_A)) + \sqrt{\sigma}k/4 \\
&\overset{(7.12)}{\geq} k - v(T_A) + \sqrt{\sigma}k/4 - 4\sigma n - \alpha k - 4s \\
&\geq v(T_B) + \alpha k \;.
\end{aligned}$$

Hence, we may assume in the rest of the proof of (7.13), that

$$\mathrm{d\overline{eg}}(B,V(M_A)) \geq k/4 \;, \text{ and} \tag{7.14}$$

$$\mathrm{d\overline{eg}}(A,V(M_A)) - \mathrm{d\overline{eg}}(B,V(M_A)) < \frac{\sqrt{\sigma}k}{4} \;. \tag{7.15}$$

First, we consider the case when $e_\ell \in M^2$. We deduce from (7.11) and (7.15) that

$$\mathrm{d\overline{eg}}(B,V(M_A \setminus M^1)) - \mathrm{d\overline{eg}}(A,V(M_A \setminus M^1)) \geq (4\sqrt[4]{\sigma} - \sqrt{\sigma}/4)k \geq 2\sqrt[4]{\sigma}qn \;.$$

Hence there is at least one matching edge $e_a \in M_A \setminus M^1$ for which

$$\mathrm{d\overline{eg}}(B,e_a) - \mathrm{d\overline{eg}}(A,e_a) > 2\sqrt[4]{\sigma}qn/|M_A \setminus M^1| \geq 4\sqrt[4]{\sigma}qn/N \;.$$

Therefore, for the number $\rho_\ell = \mathrm{d\overline{eg}}(B,e_\ell)/\mathrm{d\overline{eg}}(A,e_\ell)$ it holds,

$$\rho_\ell \geq \frac{\mathrm{d\overline{eg}}(B,e_a)}{\mathrm{d\overline{eg}}(A,e_a)} \geq \frac{4\sqrt[4]{\sigma}qn}{2sN} + 1 \geq 2\sqrt[4]{\sigma}q + 1 \;, \tag{7.16}$$

and thus

$$\begin{array}{rcl}
\mathrm{d\,\overline{eg}}(B,V(M_B)) & = & \sum_{e\in M_B,\mathrm{d\,\overline{eg}}(A,e)=0}\mathrm{d\,\overline{eg}}(B,e)+\sum_{e\in M_B,\mathrm{d\,\overline{eg}}(A,e)\neq 0}\frac{\mathrm{d\,\overline{eg}}(B,e)}{\mathrm{d\,\overline{eg}}(A,e)}\deg(A,e)\\
& \geq & \rho_\ell\cdot\mathrm{d\,\overline{eg}}(A,V(M_B))\\
& = & \rho_\ell\cdot(\mathrm{d\,\overline{eg}}(A,V(\tilde{M}))-\mathrm{d\,\overline{eg}}(A,V(M_A)))\\
& \overset{(7.10)\&(7.12)}{\geq} & \rho_\ell\cdot(v(T_B)-5\sigma n)\\
& \overset{(7.16)}{\geq} & 2\sqrt[4]{\sigma}q(\sqrt[4]{\sigma}k-5\sigma n)+v(T_B)-5\sigma n\\
& \geq & v(T_B)+\alpha k\,.
\end{array}$$

Now, assume that $e_\ell\in M^1$. From

$$\frac{\mathrm{d\,\overline{eg}}(A,V(M_A))}{\mathrm{d\,\overline{eg}}(B,V(M_A))}\overset{(7.15)}{<}\frac{\sqrt{\sigma}k}{4\cdot\mathrm{d\,\overline{eg}}(B,V(M_A))}+1\overset{(7.14)}{\leq}\sqrt{\sigma}+1\,.$$

we deduce that there exists an edge $e_b\in M_A$ such that $\mathrm{d\,\overline{eg}}(A,e_b)<(\sqrt{\sigma}+1)\cdot\mathrm{d\,\overline{eg}}(B,e_b)$. For any $j\geq\ell$ it holds

$$\frac{\mathrm{d\,\overline{eg}}(A,e_j)}{\mathrm{d\,\overline{eg}}(B,e_j)}\leq\frac{\mathrm{d\,\overline{eg}}(A,e_b)}{\mathrm{d\,\overline{eg}}(B,e_b)}<\sqrt{\sigma}+1\,. \tag{7.17}$$

If $\mathrm{d\,\overline{eg}}(B,V(\tilde{M}))<3k$, then

$$\begin{array}{rcl}
4\sqrt[4]{\sigma}k & \overset{(7.11)}{\leq} & \sum_{e\in M^1}(\mathrm{d\,\overline{eg}}(A,e)-\mathrm{d\,\overline{eg}}(B,e)))\\
& = & \sum_{i\leq\ell}(\mathrm{d\,\overline{eg}}(A,e_i)-\mathrm{d\,\overline{eg}}(B,e_i))+\sum_{\substack{j>\ell\\ e_j\in M^1}}\left(\mathrm{d\,\overline{eg}}(A,e_j)-\mathrm{d\,\overline{eg}}(B,e_j)\right)\\
& \overset{(7.17)}{\leq} & \mathrm{d\,\overline{eg}}(A,V(M_A))-\mathrm{d\,\overline{eg}}(B,V(M_A))+\sqrt{\sigma}\cdot\mathrm{d\,\overline{eg}}(B,V(M^1\setminus M_A))\\
& \overset{(7.15)}{<} & \sqrt{\sigma}k/4+\sqrt{\sigma}3k\\
& < & 4\sqrt{\sigma}k\,,
\end{array}$$

a contradiction. It remains to consider the case when $\mathrm{d\,\overline{eg}}(B,V(\tilde{M}))\geq 3k$. As $e_\ell\in M^1$, we obtain

$$\begin{aligned}
\mathrm{d\,\overline{eg}}(B,V(M_B)) &= \mathrm{d\,\overline{eg}}(B,V(\tilde{M}))-\mathrm{d\,\overline{eg}}(B,V(M_A))\\
&\geq 3k-\mathrm{d\,\overline{eg}}(A,V(M_A))\\
&\geq k-v(T_A)+2k-\alpha k-2s\\
&\geq v(T_B)+\alpha k\,.
\end{aligned}$$

We have thus proved that Inequality (7.13) holds in all cases.

We say that a vertex is A-typical if it is typical w. r. t. cluster B and typical w. r. t. all but at most $\sqrt{\beta}N$ clusters of $V(M_A)$. We say that a vertex is B-typical if it is typical w. r. t. cluster A and typical

w. r. t. all but at most $\sqrt{\beta}N$ clusters of $V(M_B)$.

Label the vertices of W_A as $a_1,\dots,a_{|W_A|}$ so that $i \le j$ whenever $a_i \succeq_R a_j$. Similarly, label the vertices of W_B as $b_1,\dots,b_{|W_B|}$ in a non-$\preceq_R$-increasing way. We embed the tree T in the graph G using the standard embedding procedure. We start the embedding process with the root R and proceed downwards in the $\preceq_R$ order. We embed the vertices of W_A in A-typical vertices of the cluster A and the vertices of B in B-typical vertices of the cluster B. The shrublets of $\mathscr{D}_A$ are embedded in edges of M_A and the shrublets of $\mathscr{D}_B$ are embedded in edges of M_B. Adjacencies between the vertices of W_A and W_B, and between the shrublets $\mathscr{D}_A \cup \mathscr{D}_B$ and the seeds $W_A \cup W_B$ are preserved during the embedding. We use Lemma 6.6 Part 1 in order to embed the shrublets. It remains to set up enviroment for Lemma 6.6. In the first step we embed the root R in an A-typical vertex in A (if $R \in W_A$) or in a B-typical vertex in B (if $R \in W_B$). Suppose that vertex $a_i \in W_A$ was embedded in a A-typical vertex in A and we want to extend the embedding to the unembedded neighbors of a_i. Let $\mathscr{D}_A^{(a_i)} \subseteq \mathscr{D}_A$ be the set of shrublets below a_i which neighbor a_i. Set $W_B^{(a_i)} = W_B \cap \mathrm{N}(a_i) \cap T(\downarrow a_i)$ and $W_A^{(a_i)} = \mathrm{N}(V(\bigcup \mathscr{D}_A^{(a_i)})) \cap T(\downarrow a_i)$. The shrublets of $\mathscr{D}_A^{(a_i)}$ and the vertices $W_A^{(a_i)} \cup W_B^{(a_i)}$ will be embedded in this step. Let $M_A^{(a_i)}$ contain those edges e of M_A such that the image of a_i is typical with respect to both end-clusters of e. Define an auxiliary mapping $\zeta^{(a_i)} : \mathscr{D}_A^{(a_i)} \to M_A^{(a_i)}$ in such a way that

$$\overline{\deg}(A,e) \ge \sum_{t \in \left(\zeta^{(a_i)}\right)^{-1}(e)} v(t) + |U^{(a_i)} \cap \bigcup e| + 2\Delta + \tau + 5\beta s\,, \quad \text{for each } e \in M_A^{(a_i)},$$

where $U^{(a_i)}$ is the set of vertices of G used by the embedding in the previous steps, and $\Delta = (\beta s + \tau)/(\gamma^2/2 - 2\beta)$. It follows from(7.12) and from the A-typicality of the image of the vertex a_i that such an mapping $\zeta^{(a_i)}$ exists. Lemma 6.6 Part 1 ensures that we can embed each each shrublet $t \in \mathscr{D}_A^{(a_i)}$ in the edge $\zeta^{(a_i)}(t)$. Moreover, the embedding of $\mathscr{D}_A^{(a_i)}$ is such, that all the vertices of $W_A^{(a_i)}$ can be mapped to A-typical vertices in A. It is easy to embed the vertices of $W_B^{(a_i)}$ in B-typical vertices of B. This finishes the inductive step for $a_i \in W_A$. The case of extending the neighborhood of the vertex $b_j \in W_B$ is analogous. □

Lemma 7.2. *Let $M^* \subseteq M$ be a matching such that $\eta N \le |M^*| \le qN/8$, let $\{U_r\}_{r \in W_A}$ be a system of sets of vertices of G such that for every $r \in W_A$ it holds $U_r \subseteq \bigcup V(M)$, and let $\varphi : W_A \to A$ be a mapping that maps every vertex $r \in W_A$ to a vertex which is typical w. r. t. all but at most $\sqrt{\beta}N$ sets of $\{C \setminus U_r : C \in V(M^*)\}$. Let $\mathscr{D}^* \subseteq \mathscr{D}_A$ be such that*

$$v(T^*) \ge \overline{\deg}(A, V(M^*)) + \frac{\eta s}{20}|M^*|\,,$$

where T^* is the forest induced by the trees in $\mathscr{D}^*$.

If the mapping can be extended to an embedding of the subforest $T[W_A \cup V(T^*)]$ so that $\varphi(V(T^*)) \subseteq \bigcup V(M^*)$, then $T \subseteq G$.

Moreover, the same holds if we interchange the roles of W_A with W_B, and $\mathscr{D}_A$ with $\mathscr{D}_B$.

Proof. Label the edges of $\tilde{M} \setminus M^*$ as $\{e_1, \dots, e_m\}$, where $m = |\tilde{M} \setminus M^*|$, so that, if $i < j$, then

$$\frac{\overline{\deg}(B, e_i)}{\overline{\deg}(A, e_i)} \geq \frac{\overline{\deg}(B, e_j)}{\overline{\deg}(A, e_j)} .$$

Fix $\ell \in [m]$ so that the matching $M_B = \{e_1, \dots, e_\ell\} \subseteq \tilde{M} \setminus M^*$ satisfies

$$v(T_B) + \alpha k \leq \overline{\deg}(B, V(M_B)) \leq v(T_B) + \alpha k + 2s . \tag{7.18}$$

The choice of ℓ is possible from the bound $|M^*| \leq qN/8$. Set $M_A = \tilde{M} \setminus (M_B \cup M^*)$. We claim that

$$\overline{\deg}(A, V(M_A)) \geq |V(T_A - T^*)| + \alpha k . \tag{7.19}$$

To prove (7.19), first assume that $v(T_B) \geq \sqrt[4]{\sigma} k$. From Lemma 7.1, we may assume that

$$|\overline{\deg}(A, V(M_B)) - \overline{\deg}(B, V(M_B))| \leq \sum_{e \in M} |\overline{\deg}(A, e) - \overline{\deg}(B, e)| < 9\sqrt[4]{\sigma} k ,$$

since otherwise $T \subseteq G$. This implies that

$$\begin{aligned} \overline{\deg}(A, V(M_A)) &\geq \overline{\deg}(A, V(\tilde{M})) - \overline{\deg}(B, V(M_B)) - 9\sqrt[4]{\sigma} k - \overline{\deg}(A, V(M^*)) \\ &\overset{(7.10)\&(7.18)}{\geq} k - 4\sigma n - v(T_B) - \alpha k - 2s - 9\sqrt[4]{\sigma} k - v(T^*) + \tfrac{\eta s}{20}|M^*| \\ &> v(T_A - T^*) + \alpha k . \end{aligned}$$

Now, we consider the case when $v(T_B) < \sqrt[4]{\sigma} k$. If $2 \geq \overline{\deg}(A, e_\ell)/\overline{\deg}(B, e_\ell)$, then

$$\begin{aligned} \overline{\deg}(A, V(M_A)) &= \overline{\deg}(A, V(\tilde{M})) - \overline{\deg}(A, V(M^*)) - \overline{\deg}(A, V(M_B)) \\ &\overset{(7.10)}{\geq} k - 4\sigma n - v(T^*) + \tfrac{\eta^2 n}{20} - \overline{\deg}(B, V(M_B)) \cdot \overline{\deg}(A, V(M_B))/\overline{\deg}(B, V(M_B)) \\ &\geq k + \tfrac{\eta^2 n}{20} - 4\sigma n - v(T^*) - (v(T_B) + \alpha k + 2s) \cdot \overline{\deg}(A, e_\ell)/\overline{\deg}(B, e_\ell) \\ &\geq k + \tfrac{\eta^2 n}{20} - 4\sigma n - v(T^*) - v(T_B) - \sqrt[4]{\sigma} k - 2\alpha k - 4s \\ &\geq v(T_A - T^*) + \alpha k . \end{aligned}$$

On the other hand, if $\overline{\deg}(A, e_\ell)/\overline{\deg}(B, e_\ell) \geq 2$, then

$$\begin{aligned} \overline{\deg}(A, V(M_A)) &\geq 2 \cdot \overline{\deg}(B, V(M_A)) \\ &\geq 2 \cdot (\overline{\deg}(B, V(\tilde{M})) - 2s|M^*| - \overline{\deg}(B, V(M_B))) \\ &\overset{(7.10)}{\geq} 2(k - 4\sigma n - sqN/4 - \sqrt[4]{\sigma} k - \alpha k - 2s) \\ &\geq v(T_A - T^*) + \alpha k . \end{aligned}$$

For a set $U \subseteq \bigcup_{C\in V(M^*)} C$, say that a vertex is (A,U)-typical if it is typical w. r. t. the cluster B, typical w. r. t. all but at most $\sqrt{\beta}N$ clusters of $V(M_A)$, and typical to all but at most $\sqrt{\beta}N$ sets $C\setminus U$, $C \in V(M^*)$. Say that a vertex is B-typical, if it is typical w. r. t. cluster A and typical w. r. t. all but at most $\sqrt{\beta}N$ cluster of $V(M_B)$.

We embed the tree T, starting with the root R and progressing downwards in the $\preceq_R$-order. We embed the vertices $r \in W_A$ in (A,U_r)-typical vertices of the cluster A, and embed the vertices of W_B in B-typical vertices of the cluster B. According to the hypothesis of lemma, the shrublets of $\mathscr{D}^*$ are embedded in the edges of M^*. Then the shrublets of $\mathscr{D}_A \setminus \mathscr{D}^*$ are embedded in M_A, and the ones of $\mathscr{D}_B \setminus \mathscr{D}^*$ in M_B. The embeddings of $\mathscr{D}_A \setminus \mathscr{D}^*$ and of $\mathscr{D}_B$ are ensured by Lemma 6.6 Part 1, in a standard way. It remains to check whether the conditions of the Lemma 6.6 Part 1 are matched. If we denote by M^i the submatching of M_A such that $v_i \in \varphi(W_A)$ is typical to all its clusters, then $\deg(A,V(M^i)) \geq \overline{\deg}(A,V(M_A)) - 2\sqrt{\beta}n \geq v(T_A - T^*) + \alpha k - 2\sqrt{\beta}n$. We can thus partition the set $\mathscr{D}_A \setminus \mathscr{D}^* = \bigcup_{v_i\in\varphi(W_A)} \bigcup_{e\in M^i} \mathscr{D}^*_{i,e}$ in a suitable way so that each partition class $\mathscr{D}^*_{i,e}$ embeds in the edges e of M^i using Lemma 6.6 Part 1. Similar calculations hold for M_B.

We briefly sketch the "moreover" part of the statement, with the roles of W_A with W_B, and $\mathscr{D}_A$ with $\mathscr{D}_B$ interchanged. Consider the subforest T^* of T_B composed by components of $\mathscr{D}_B$ with

$$v(T^*) \geq \overline{\deg}(A,V(M^*)) + \frac{\eta s}{20}|M^*| \ .$$

Observe that we need to check only the case when $v(T_B) \geq \sqrt[4]{\sigma}k$. Similarly as before, we can find a submatching $M_B \subseteq \tilde{M} \setminus M^*$ so that

$$v(T_B - T^*) + \alpha k \leq \overline{\deg}(A,V(M_B)) \leq v(T_B - T^*) + \alpha k + 2s \ .$$

Set $M_A = \tilde{M} \setminus (M_B \cup M^*)$. From Lemma 7.1, we obtain that $T \subseteq G$, or we deduce that

$$\overline{\deg}(B,V(M_A)) \geq v(T_A) + \alpha k \ .$$

We use Lemma 6.6 to map the vertices $r \in W_B$ to vertices in A that are typical w. r. t. B, typical w. r. t. all but al most $\sqrt{\beta}N$ clusters of $V(M_B)$, and typical w. r. t. all but al most $\sqrt{\beta}N$ sets $C \setminus U_r, C \in V(M^*)$; we map W_A to vertices in B that are typical w. r. t. A, and typical w. r. t. all but at most $\sqrt{\beta}N$ clusters of $V(M_A)$. Embed T^* in M^*, $T_B - T^*$ in M_B, and T_A in M_A. $\square$

We consider the following submatchings of M. For a cluster $X \in V(\mathbf{G})$, set

$$\begin{aligned}
M_1^X &= \{CD \in M : \overline{\deg}(X,C) < \eta s \text{ and } \overline{\deg}(X,D) > (1-\eta)s\} \ , \\
M_2^X &= \{CD \in M : \overline{\deg}(X,C) \in [\eta s,(1-\eta)s] \text{ or } \overline{\deg}(X,D) \in [\eta s,(1-\eta)s]\} \ , \\
M_3^X &= \{CD \in M : \overline{\deg}(X,C\cup D) < 2\eta s\} \ , \text{ and} \\
M^-(X) &= M_1^X \cup M_2^X \cup M_3^X \ .
\end{aligned}$$

Lemma 7.3. *It holds* $\max\{|M_1^A|,|M_1^B|,|M_2^A|,|M_2^B|\} < 2\eta N$*, or* $T \subseteq G$ *.*

Proof. We prove only that if $\max\{|M_1^A|,|M_2^A|\} \geq 2\eta N$, then $T \subseteq G$. The case when $\max\{|M_1^B|,|M_2^B|\} \geq 2\eta N$ is analogous. Assume that $|M_1^A| \geq 2\eta N$ (resp. $|M_2^A| \geq 2\eta N$). Choose a submatching $M^* \subseteq M_1^A$ (resp. $M^* \subseteq M_2^A$) of size $2\eta N$. We know that $\mathscr{D}_A \cup \mathscr{D}_B$ is $c_{\mathbf{U}}$-balanced. Hence $\mathscr{D}_A$ is $c_{\mathbf{U}}/2$-balanced or $\mathscr{D}_B$ is $c_{\mathbf{U}}/2$-balanced. Suppose first that $\mathscr{D}_A$ is $c_{\mathbf{U}}/2$-balanced. Consider a minimal subset $\mathscr{D}^* \subseteq \mathscr{D}_A$ such that it induces a forest of order at least $\overline{\deg}(A,V(M^*)) + \eta^2 n/10$, and such that if $t \in \mathscr{D}^*$, then $\min\{|V(t)\cap T_{\mathrm{o}}|,|V(t)\cap T_{\mathrm{e}}|\} \geq c_{\mathbf{U}}/2\cdot v(t)$. Let T^* be the forest induced by the components of $\mathscr{D}^*$. We use Lemma 7.2 to show that $T \subseteq G$. To this end, it is enough to extend a mapping $\varphi : W_A \to A$ satisfying the conditions of Lemma 7.2 to an embedding of T^*. We label the vertices of W_A as $r_1, r_2, \ldots, r_{|W_A|}$ so that if $r_i \prec_R r_j$ then $i > j$. Set $\mathscr{D}_i^* = \{t \in \mathscr{D}^* : V(t) \cap \mathrm{Ch}(r_i) \neq \emptyset\}$. At each step $i \geq 1$ set $U_i = \varphi(\bigcup_{j<i} V(\mathscr{D}_j^*)) \subseteq V(M^*)$ for the set of used vertices used for the embedding in previous steps. Observe that $U_1 \cap (C \cup D) = \emptyset$ for all $CD \in M^*$ and thus it is 1-packed (resp. 2-packed) with any parameter and with respect to the embedding sets C, D, and the head set A. Set

$$M^*(r_i) = \{CD \in M^* : r_i \text{ is typical w. r. t. both } C \setminus U_{r_i} \text{ and } D \setminus U_{r_i}\}\ ,$$

where $U_{r_i} = \emptyset$ if $M^* \subseteq M_A^1$, and $U_{r_i} = U_i$ if $M^* \subseteq M_A^2$ (we define U_{r_i} inductively, as the embedding of T is always defined step by step in the $\preceq_R$ order). The embedding is extended separately for $M^* \subseteq M_A^1$ and $M^* \subseteq M_A^2$. Set $\Delta = \frac{\beta s+\tau}{\gamma^2/2-2\beta}$.

First consider the case when $M^* \subseteq M_1^A$. We shall use Lemma 6.6 Part 2. For $i > 1$, the set U_i is 1-packed (with parameter λ_1 and τ) by induction for any pair of embedding sets (C,D), where $CD \in M^*$. Set $\lambda_1 = \Delta + \tau + 3\beta s$. By the choice of $\mathscr{D}^*$, we know that

$$\begin{aligned}
\max\{|V(\mathscr{D}_i^*)\cap T_{\mathrm{o}}|,|V(\mathscr{D}_i^*)\cap T_{\mathrm{e}}|\} + &\sum_{\substack{CD\in M^* \\ \overline{\deg}(A,D)\geq(1-\eta)s}} |D \cap U_i| \\
&\leq (1-\frac{c_{\mathbf{U}}}{2})\left|\bigcup_{j\leq i} V(\mathscr{D}_j^*)\right| \\
&\leq (1-\frac{c_{\mathbf{U}}}{2})\left(\overline{\deg}(A,V(M^*)) + \frac{\eta^2 n}{10} + \tau\right) \\
&\leq \sum_{\substack{CD\in M^*(r_i) \\ \overline{\deg}(A,D)\geq(1-\eta)s}} \overline{\deg}(A,D) + 2\sqrt{\beta}n + 7\eta^2 n - c_{\mathbf{U}}\eta n \\
&\leq \sum_{\substack{CD\in M^*(r_i) \\ \overline{\deg}(A,D)\geq(1-\eta)s}} \overline{\deg}(A,D) - |M^*(r_i)|(\tau+\lambda_1+\Delta+\beta s)\ .
\end{aligned}$$

Thus we can partition the set $\mathscr{D}_i^*$ in sets $\mathscr{D}_{i,e}^*$ for each edge $e \in M^*(r_i)$ satisfying the conditions of Lemma 6.6 Part 2 (for $Z = A$, $U = U_i$ and for $e = CD$, we have $X = D$, where $\overline{\deg}(A,D) \geq (1-\eta)s$ and $Y = C$). We thus embed the forest $\mathscr{D}_{i,e}^*$ in the edge $e \in M^*(r_i)$.

Now consider the case when $M^* \subseteq M_A^2$. We shall use Lemma 6.6 Part 3. The set $U_{r_i} \cap (C \cup D)$, is 2-packed (with parameters λ_2 and τ) by induction, for all $CD \in M^*$. Set $\lambda_2 = 2\Delta + 7\beta s + 4\tau$. Observe that each tree of $\mathscr{D}^*$ has at least two vertices.

$$\begin{aligned}\left|\bigcup_{j\le i} V(\mathscr{D}_j^*)\right| &\le \left(\overline{\mathrm{deg}}(A, V(M^*)) + \frac{\eta^2 n}{10} + \tau\right) \\ &\le \sum_{CD\in M^*(r_i)} \overline{\mathrm{deg}}(A, C\cup D) + \sqrt{\beta} n + \frac{\eta^2 n}{10} + \tau \\ &\le \sum_{CD\in M^*(r_i)} \overline{\mathrm{deg}}(A, C\cup D) + N(\frac{\eta s}{4} - \tau) \,.\end{aligned}$$

Thus we can partition the set $\mathscr{D}_i^*$ in sets $\mathscr{D}_{i,e}^*$, $e \in M^*(r_i)$ satisfying the conditions of the Lemma 6.6 Part 3, for $Z = A$, $U = U_{r_i}$ and for $e = CD$ we have $X = C$ and $Y = D$. We thus embed each forest $\mathscr{D}_{i,e}^*$ in the edge e.

If $\mathscr{D}_B$ is $c_{\mathbf{U}}/2$-balanced, we interchange the role of $\mathscr{D}_A$ and $\mathscr{D}_B$, and of W_A and W_B in the above.

□

The pair of clusters (A, B) was characterized by the following properties:

- $AB \in E(\mathbf{G})$,
- $A, B \in \mathscr{X}' \cap \mathscr{L}$.

Thus, any pair of clusters (X, Y), such that $XY \in E(\mathbf{G})$, and $X, Y \in \mathscr{X}' \cap \mathscr{L}$ can play the same role as the clusters A and B, in particular Lemmas 7.1, 7.2, and 7.3 can be applied to any such pair of clusters (X, Y) to obtain $T \subseteq G$, or $\max\{|M_1^X|, |M_1^Y|, |M_2^X|, |M_2^Y|\} < 2\eta N$. Thus in the following it is enough to consider the latter case. Then, for any $C \in \mathscr{X}' \cap \mathscr{L} \cap \mathrm{N}(\mathscr{X}' \cap \mathscr{L})$ we have

$$\overline{\mathrm{deg}}(C, V(M^-(C))) \le 10\eta n \,. \tag{7.20}$$

Choose $M^*(A) \subseteq \tilde{M} \setminus M^-(A)$ maximal such that for $V' = \bigcup_{CD\in M^*(A)} C \cup D$ we have $|V'| \le k + 2s$. We claim that

$$|L \cap V'| \ge |V'|/2 \text{ , and} \tag{7.21}$$

$$|V'| \ge \overline{\mathrm{deg}}(A, V') \ge k - 10.5\eta n \,. \tag{7.22}$$

For property (7.21) it is enough to observe that at least half of the vertices in any edge $CD \in M^*(A)$ are large. Property (7.22) is proved by analysing two cases. If $M^*(A) = \tilde{M} \setminus M^-(A)$, then

$$\overline{\mathrm{deg}}(A, V') \ge \overline{\mathrm{deg}}(A, V(\tilde{M})) - \overline{\mathrm{deg}}(A, V(M^-(A))) \overset{(7.10)\&(7.20)}{\ge} k - 4\sigma n - 10\eta n \ge k - 10.5\eta n \,.$$

If $M^*(A) \neq \tilde{M} \setminus M^-(A)$, then $\mathrm{d\bar{e}g}(A,V') \geq (1-\eta)k > k - 10.5\eta n$.

Observe that for any $X \in \mathscr{X}' \cap \mathscr{L} \cap \mathrm{N}(\mathscr{X}' \cap \mathscr{L})$, similarly as above, we obtain

$$\mathrm{d\bar{e}g}(C, V(\tilde{M} \setminus M^-(C))) \overset{(7.10)\&(7.20)}{\geq} k - 10.5\eta n \; . \tag{7.23}$$

If $e_{G_\gamma}(V', V \setminus V') \leq \omega n^2/2$, then $e_G(V', V \setminus V') \leq \omega n^2$, as by cleaning the cluster graph $\mathbf{G}$ we deleted at most $2\gamma n^2$ edges, and $e_G(\bar{V}, V \setminus \bar{V}) \leq \beta n^2$ (recall that $\beta \ll \gamma \ll \omega$). The set V' satisfies the requirements of the Proposition 4.4.

PART II: Escaping from V'. In the rest of the proof, we assume that

$$e_{G_\gamma}(V', V \setminus V') \geq \omega n^2/2 \; . \tag{7.24}$$

Under this assumption, we show that $T \subseteq G$. We use the edges between V' and $V \setminus V'$ in order to "escape" from V'. More precisely, we save space in the neighbourhood of A by embedding part of the forest T_A in $V \setminus V'$.

Set $\mathscr{T}^{\geq 3} = \{t \in \mathscr{D}_A : |V(t) \setminus \mathrm{N}(W_A)| \geq 2\}$ and $\mathscr{T}_*^{\geq 3} = \{t \in \mathscr{D}_A \setminus \mathscr{T}^{\geq 3} : v(t) \geq 3\}$. For $i = 1,2$ set $\mathscr{T}^i = \{t \in \mathscr{D}_A : v(t) = i\}$, and by T^i the forest induced by $\mathscr{T}^i$. Observe that $\mathscr{T}^{\geq 3}$, $\mathscr{T}_*^{\geq 3}$, $\mathscr{T}^2$, and $\mathscr{T}^1$ partition $\mathscr{D}_A$. Since the distance between any two vertices in W_A is even, for each tree $t \in \mathscr{T}^1 \cup \mathscr{T}^2$, only the root of t is adjacent to W_A.

Lemma 7.4. *$|V(\bigcup\{t \in \mathscr{T}^{\geq 3}\})| < 36\eta n$, or $T \subseteq G$.*

Proof. Suppose that $|V(\bigcup\{t \in \mathscr{T}^{\geq 3}\})| \geq 36\eta n$. We show that $T \subseteq G$. Choose a maximal forest T_A^* of order at most $36\eta(1-2\eta)n$ formed by components of $\mathscr{T}^{\geq 3}$. Then $v(T_A^*) \geq 36\eta(1-2\eta)n - \tau$. This forest contains relatively few vertices adjacent to W_A, more precisely

$$|\mathrm{N}(W_A) \cap V(T_A^*)| \leq 12(1-2\eta)\eta n + |W_A| \; . \tag{7.25}$$

As $e_{G_\gamma}(V', V \setminus V') \geq \omega n^2/2$, for at least $\omega N/4$ clusters $C \in V(\mathbf{G})$, $C \subseteq V'$, it holds $\mathrm{d\bar{e}g}(C, V \setminus V') \geq \omega n/4$. All but at most $3\gamma N$ of these clusters have the property that $\mathrm{d\bar{e}g}(C, V(\tilde{M})) \geq \mathrm{d\bar{e}g}(C) - 3\sigma n - 4s > \mathrm{d\bar{e}g}(C) - 4\sigma n$ (from the assumptions of Case II). Thus

$$\mathrm{d\bar{e}g}(C, V(\tilde{M} \setminus M^*(A))) \geq \frac{\omega n}{4} - 4\sigma n \; . \tag{7.26}$$

Let $\mathscr{C}$ be a set of $12\eta N$ such clusters. We shall use the clusters in $\mathscr{C}$ as bridges to embed part of T_A^* outside of V'. In $\mathscr{C}$, we shall embed the vertices of T_A^* that are adjacent to W_A, and the rest $V(T_A^*)$ will be mapped to $V \setminus V'$. We cannot then use the clusters that are matched with $\mathscr{C}$ anymore, however this loss is overcompensated by the amount of vertices of T_A^* that we are able to embed in $V \setminus V'$.

Set $M^* = \{CD \in M^*(A) : \{C,D\} \cap \mathscr{C} \neq \emptyset\}$. Then,

$$\max\{\overline{\deg}(A, V(M^*)), \overline{\deg}(B, V(M^*))\} \leq 24\eta n \tag{7.27}$$

and thus

$$\begin{aligned} \overline{\deg}(A, V(M^*(A) \setminus M^*)) &\geq \overline{\deg}(A, V') - 24\eta n \overset{(7.22)}{\geq} k - 35\eta n \\ &\geq v(T) - v(T_A^*) + \eta n/2 \; . \end{aligned} \tag{7.28}$$

We claim that there are disjoint submatchings M_A and M_B of $\tilde{M} \setminus M^*$ such that

$$\overline{\deg}(A, V(M_A)) \geq v(T_A) - v(T_A^*) + \eta n/8 \; , \text{ and} \tag{7.29}$$

$$\overline{\deg}(B, V(M_B)) \geq v(T_B) + \eta n/8 \; . \tag{7.30}$$

To prove the existence of M_A and M_B satisfying (7.29) and (7.30), we consider two cases based on the order of T_B.

(**♣1**) First assume that $v(T_B) \geq \sqrt[4]{\sigma} k$. Lemma 7.1 implies that that

$$\overline{\deg}(B, V') \geq \overline{\deg}(A, V') - 9\sqrt{\sigma} k \overset{(7.22)}{\geq} k - 11\eta n \; .$$

Similarly as in (7.28), we obtain $\overline{\deg}(B, V(M^*(A) \setminus M^*)) \geq v(T) - v(T_A^*) + \eta n/2$. Requirements (7.29) and (7.30) follow by application of Proposition 3.7. Indeed, setting $\Delta = 2s, a = v(T_A) - v(T_A^*) + \eta n/8, b = v(T_B) + \eta n/8, I = M^*(A) \setminus (A) \setminus M^*$ and for $e \in I$ setting $\alpha_e = \overline{\deg}(A, e)$ and $\beta_e = \overline{\deg}(B, e)$, we infer that the matching $\tilde{M} \setminus M^*$ can be partitioned into two submatchings M_A and M_B satisfying (7.29) and (7.30).

(**♣2**) Now assume that $v(T_B) < \sqrt[4]{\sigma} k$. Then

$$\begin{aligned} \overline{\deg}(B, V(\tilde{M} \setminus (M^-(B) \cup M^*))) &\overset{(7.23)\&(7.27)}{\geq} k - 10.5\eta n - 24\eta n \\ &\geq v(T_B) + \eta n/8 \; . \end{aligned}$$

Let $M_B \subseteq \tilde{M} \setminus (M^-(B) \cup M^*)$ be such that $v(T_B) + \eta n/8 \leq \overline{\deg}(B, V(M_B)) \leq v(T_B) + \eta n/8 + 2s$. Equation (7.30) holds. Recall that B is densely connected to $M \setminus M^-(B)$, thus

$$\begin{aligned} 2s \cdot |M_B| &\leq (v(T_B) + \eta n/8 + 2s)/(1 - \eta) \\ &\leq 2\sqrt[4]{\sigma} k + (\eta n/8 + \eta^2 n/4) + 4s \\ &< \eta n/4 \; . \end{aligned} \tag{7.31}$$

Set $M_A = M^*(A) \setminus (M^* \cup M_B)$. Then,

$$\begin{aligned} \overline{\deg}(A, V(M_A)) &\geq \overline{\deg}(A, V(M^*(A) \setminus M^*)) - 2s \cdot |M_B| \\ &\overset{(7.28)\&(7.31)}{\geq} v(T) - v(T_A^*) + \eta n/2 - \eta n/4 \\ &> v(T_A) - v(T_A^*) + \eta n/8 \,, \end{aligned}$$

implying (7.29).

In both cases, observe that for each cluster $C \in \mathscr{C}$ we obtain

$$\overline{\deg}(C, V(\tilde{M} \setminus (M_B \cup M^*(A))) \overset{(7.26)}{\geq} \omega n/4 - 10\eta n - 4s - 2s|M_B \setminus M^*(A)| \overset{(7.31)}{>} \omega n/8 \,. \tag{7.32}$$

Say that a vertex is A-typical if it is typical w.r.t. cluster B, typical w.r.t. $\mathscr{C}$, typical w.r.t. all but at most $\sqrt{\beta}N$ clusters of $V(M_A)$. Say that a vertex is B-typical if it is typical w.r.t. cluster A, and typical w.r.t. all but at most $\sqrt{\beta}N$ clusters of $V(M_B)$.

We embed the tree T in the graph G starting with the root R and progressing downwards in the $\preceq_R$-order. We embed the vertices of W_A in A-typical vertices of the cluster A, and embed the vertices of W_B in B-typical vertices of the cluster B. The forest $T_A - T_A^*$ is embedded in M_A and the forest T_B in M_B. The set $\mathrm{N}(W_A) \cap V(T_A^*)$ is mapped to vertices in $\mathscr{C}$ that are typical w.r.t. all but at most $\sqrt{\beta}N$ clusters of $V(\tilde{M} \setminus (M^*(A) \cup M_B))$, and the forest $T_A^* - \mathrm{N}(W_A)$ is embedded in $\tilde{M} \setminus (M^*(A) \cup M_B)$. Adjacencies are preserved. To embed $T_A - T_A^*$, T_B and $T_A^* - \mathrm{N}(W_A)$, we shall use Lemma 6.6 Part 1.

Let v be any vertex in $\varphi(W_A)$, and let the set M_A^v consist of the edges $XY \in M_A$ such that v is typical to both X and Y. Similarly define M_B^v for a vertex $v \in \varphi(W_B)$ and $(M \setminus (M^*(A) \cup M_B))^v$ for a vertex $v \in \varphi(\mathrm{N}(W_A) \cap V(T_A^*))$. Then,

$$\deg(A, V(M_A^v)) \geq |V(T_A) \setminus V(T_A^*)| + \eta k/4 - 2\sqrt{\beta}Ns \geq |V(T_A) \setminus V(T_A^*)| + \alpha k \,.$$

For $v \in \varphi(W_A)$ by (7.25) it holds

$$\begin{aligned} \deg(v, \mathscr{C}) &\geq \overline{\deg}(A, \mathscr{C}) - \beta s |\mathscr{C}| \\ &\geq (1 - \eta - \beta) 12\eta n \\ &\geq |\mathrm{N}(W_A) \cap V(T_A^*)| + \alpha k \,. \end{aligned}$$

Similarly, we obtain $\overline{\deg}(B, V(M_B^v)) \geq v(T_B) + \alpha k$ for $v \in \varphi(W_B)$, and

$$\overline{\deg}(C, (\tilde{M} \setminus (M^*(A) \cup M_B))^v) \geq \omega n/8 - 2\sqrt{\beta}n \geq v(T_A^*) + \alpha k \,,$$

for $v \in \varphi(\mathrm{N}(W_A) \cap V(T_A^*))$. For each $r \in W_A$, we extend its mapping to an embedding of the components of $T_A - T_A^*$, with root in $\mathrm{Ch}(r)$. This is done by filling up the clusters C and D, for every $CD \in M_A^{\varphi(r)}$. Lemma 6.6 Part 1 ensures that we can embed in $CD \in M_A^{\varphi(r)}$ components of total order of at least $\mathrm{d\bar{e}g}(A, C \cup D) - \alpha k/2$ (the set U denotes the set of used vertices; it is 1-packed by induction). The embedding of T_B and of $T_A^* - \mathrm{N}(W_A)$ are treated similarly. $\square$

Now we have the tools to prove Lemma 7.5. It considers the situation when a substantial portion of the edges between V' and $V \setminus V'$ does not emanate from $\mathscr{L}$. Set $\tilde{\mathscr{S}} = \{C : CD \in M^*(A), C \notin \mathscr{L}\}$ and $\tilde{S} = \bigcup_{C \in \tilde{\mathscr{S}}} C$.

Lemma 7.5. *It holds $e_{G_\gamma}(\tilde{S}, V \setminus V') < 32\eta n^2$, or $T \subseteq G$.*

Proof. Assume that $e_{G_\gamma}(\tilde{S}, V \setminus V') \geq 32\eta n^2$. We show that $T \subseteq G$. For this, we consider three cases. The first case **(C1)** deals with the case when there are many leaves of T adjacent to vertices of W_A. As such leaves can be embedded at the end in a greedy way, it is enough to embed a significantly smaller tree. The second possibility **(C2)** deals with the case when the set $\mathscr{D}_A$ contains many 'large' components. This case was treated in the Lemma 7.4. In the last part of the proof we consider the remaining case **(C3)**, when most of the trees in $\mathscr{D}_A$ are paths of length 2.

(C1) If $|\bigcup_{t \in \mathscr{T}^1} V(t)| \geq 2\eta n$, then consider the subgraph $T' = T - V(T^1)$ obtained from T after deleting all leaves adjacent to W_A. Observe that T' is a tree.

$$v(T') + \eta n \leq k - \eta n \leq \min\{\mathrm{d\bar{e}g}(A, V(\tilde{M})), \mathrm{d\bar{e}g}(B, V(\tilde{M}))\} .$$

By Proposition 3.7, there exists a partition $\tilde{M} = M_A \cup M_B$ such that $\mathrm{d\bar{e}g}(A, V(M_A)) \geq |V(T_A) \setminus V(T^1)| + \eta n/4$ and $\mathrm{d\bar{e}g}(B, V(M_B)) \geq v(T_B) + \eta n/4$. We then define the embedding of T' in a standard way. The trees of $\mathscr{T}^1$ are leaves whose parent vertices are mapped to L, and can be embedded greedily. This implies that $T \subseteq G$.

(C2) By Lemma 7.4, if $|\bigcup_{t \in \mathscr{T}^{\geq 3}} V(t)| \geq 36\eta n$, then $T \subseteq G$.

(C3) If $|\bigcup_{t \in \mathscr{T}^{\geq 3}} V(t)| < 36\eta n$ and $|\bigcup_{t \in \mathscr{T}^1} V(t)| < 2\eta n$, then the trees from $\mathscr{D}_A \setminus (\mathscr{T}^{\geq 3} \cup \mathscr{T}^1 \cup \mathscr{T}^2)$ consist only of trees of order at least 3 that contain only one vertex not adjacent to W_A.

$$\begin{aligned}
\left| \bigcup_{t \in \mathscr{T}^2} V(t) \right| &= v(T_A) - |\bigcup_{t \in \mathscr{T}^{\geq 3}} V(t)| - v(T^1) - |\bigcup_{t \in \mathscr{T}_*^{\geq 3}} V(t)| \\
&\geq k/2 - |W_A \cup W_B| - 36\eta n - 2\eta n - 3|W_A| \\
&> 26\eta n .
\end{aligned}$$

Let T_A^* be a maximal forest of order at most $26\eta n$ formed by trees from $\mathscr{T}^2$. Observe that $26\eta n - \tau \leq v(T_A^*) \leq 26\eta n$.

There are at least $16\eta N$ clusters $C \in \tilde{\mathscr{S}}$ for which $\overline{\deg}(C, M \setminus M^*(A)) \geq 16\eta n$. Let $\mathscr{C}$ be a set of size $7\eta N$ formed by such clusters contained in different edges of M. Set

$$M^* = \{CD \in M^*(A) : \{C, D\} \cap \mathscr{C} \neq \emptyset\} .$$

From $\overline{\deg}(A, V(M^*)) \leq 14\eta n$ we deduce that

$$\begin{aligned}\overline{\deg}(A, V(M^*(A) \setminus M^*)) &\geq k - 11\eta n - 14\eta n \geq k - 25\eta n \\ &\geq v(T) - v(T_A^*) + \eta n .\end{aligned}$$

We claim that there exist disjoint submatchings M_A and M_B of $\tilde{M} \setminus M^*$ such that $\overline{\deg}(A, V(M_A)) \geq v(T_A) - v(T_A^*) + \eta n/8$ and $\overline{\deg}(B, V(M_B)) \geq v(T_B) + \eta n/8$. We consider two cases, depending on $v(T_B)$.

(♠**1**) First assume that $v(T_B) \geq \sqrt[4]{\sigma}k$. Then, similarly as above and by Lemma 7.1, we have that $T \subseteq G$, or

$$\overline{\deg}(B, V(M^*(A) \setminus M^*)) \geq v(T) - (T_A^*) + \eta n .$$

Using Proposition 3.7, we partition $M^*(A) \setminus M^*$ in two submatchings M_A and M_B so that $\overline{\deg}(A, V(M_A)) \geq |V(T_A) \setminus V(T_A^*)| + \eta n/8$ and $\overline{\deg}(B, V(M_B)) \geq v(T_B) + \eta n/8$.

(♠**2**) If $v(T_B) < \sqrt[4]{\sigma}k$, then choose a submatching $M_B \subseteq \tilde{M} \setminus (M^-(B) \cup M^*)$ so that

$$v(T_B) + \eta n/8 \leq \overline{\deg}(B, V(M_B)) \leq v(T_B) + \eta n/8 + 2s .$$

It follows that $2s \cdot |M_B| \leq (v(T_B) + \eta n/8 + 2s)/(1 - \eta) \leq \eta n/4$. Set $M_A = M^*(A) \setminus (M^* \cup M_B)$. Then,

$$\overline{\deg}(A, V(M_A)) \geq v(T) - v(T_A^*) + \eta n - 2s \cdot |M_B| > v(T_A - T_A^*) + \eta n/8 .$$

Say that a vertex is A-typical if it is typical w. r. t. cluster B, typical w. r. t. $\mathscr{C}$, typical w. r. t. $V(M^*) \setminus \mathscr{C}$, typical w. r. t. all but at most $\sqrt{\beta}N$ clusters of $V(M_A)$. A vertex is B-typical if it is typical w. r. t. cluster A, typical w. r. t. all but at most $\sqrt{\beta}N$ clusters of M_B.

We embed T progressing downwards in the $\preceq_R$-order. We embed the vertices of W_A in A-typical vertices of the cluster A, and embed the vertices of W_B in B-typical vertices of the cluster B. The forest $T_A - T_A^*$ is embedded in M_A, and the forest T_B in M_B. The roots of half of the forest T_A^* are mapped to vertices in $\mathscr{C}$ that are typical w. r. t. $V(M \setminus (M^*(A) \cup M_B))$, and the neighbours of such roots are mapped to the set $V \setminus V'$. The left-over roots of T_A^* are mapped to vertices of $V(M^*) \setminus \mathscr{C}$, and their respective neighbours are embedded greedily. This is possible, as vertices in $V(M^*) \setminus \mathscr{C}$ are large vertices. We use Lemma 6.6 Part 1 in a standard way in order to embed the components of the forest in the respective matching edges. Adjacencies are preserved. Details are left to the reader. □

Set $M_L = \{CD \in M^*(A) : \{C, D\} \subseteq \mathscr{L}\}$. In the same spirit as above, we prove the following auxiliary lemma.

Lemma 7.6. *It holds $|M_L| < 7\eta N$, or $T \subseteq G$.*

Proof. The proof is analogue to the one of Lemma 7.5 and thus we provide only a short sketch of it. Assume that $|M_L| \geq 7\eta N$. We choose $M^* \subseteq M_L$ of order $7\eta N$. We partition $\tilde{M} \setminus M^* = M_A \cup M_B$ as before. The set W_A is mapped to vertices that are typical w. r. t. cluster B, typical w. r. t. $V(M^*)$ and typical w. r. t. all but at most $\sqrt{\beta}N$ clusters of $V(M_A)$. The set W_B, the forest $T_A \setminus T_A^*$, and the forest T_B are embedded as above; the roots of T_A^* are mapped to vertices in $\bigcup V(M^*) \subseteq L$; the left-over leaves are embedded greedily. □

Lemma 7.7. *Under the above assumptions, it holds $T \subseteq G$.*

Proof. Assume that $e_{G_\gamma}(V' \setminus \tilde{S}, V \setminus V') \geq \omega n^2/4$ and that $|M_L| < 7\eta N$. We show that then $e_{G_\gamma}(\tilde{S}, V \setminus V') \geq 32\eta n^2$ and by Lemma 7.5, this implies that $T \subseteq G$.

For at least $\omega N/4$ clusters C of $V(M^*(A)) \setminus \tilde{\mathscr{S}}$ it holds that $\overline{\deg}(C, V \setminus V') \geq \omega n/4$. As such clusters are in $\mathrm{N}(A) \cap \mathscr{L}$, at least $\omega N/4 - 1 \geq \omega N/8$ of them are in $\mathscr{X}' \cap \mathscr{L}$ (see Proposition 6.4). Denote this set by $\mathscr{C}$. By (7.20), we obtain for $C \in \mathscr{C}$ that $\overline{\deg}(C, V(M_C)) \geq \omega n/4 - 11\eta n$, where $M_C = \tilde{M} \setminus (M^-(C) \cup M^*(A))$. At least nearly half of the weight from C to M_C goes to clusters that are in $\mathscr{L}$, as all matching edges are incident to $\mathscr{L}$ and the degrees to both end-clusters cannot differ too much. Also all but at most one cluster of $V(M_C) \cap \mathscr{L}$ are in $\mathscr{X}'$. Therefore $\overline{\deg}(C, V(M_C) \cap \mathscr{X}' \cap \mathscr{L}) > \omega n/10$.

Set $\mathscr{D} = \bigcup_{C \in \mathscr{C}} V(M_C) \cap \mathscr{X}' \cap \mathscr{L}$. Then $|\mathscr{D}| > \omega N/10$. We deduce that $e_{G_\gamma}(\bigcup \mathscr{C}, \bigcup \mathscr{D}) \geq (s \cdot \omega N/8) \cdot \omega n/10 = \omega^2 n^2/80$. From (7.20), we infer that each $D \in \mathscr{D}'$ sends at most $11\eta ns$ edges in $M^-(D)$. So $\overline{\deg}(D, \mathscr{C} \setminus V(M^-(D))) \geq \omega^2 n/80 - 11\eta n > \omega^2 n/100$. The cluster D has also large degree to the clusters which are matched to $\mathscr{C} \setminus V(M^-(D))$ by $M^*(A)$. As $|M_L| < 7\eta N$, nearly all those clusters are in $\tilde{\mathscr{S}}$. We deduce that $\overline{\deg}(D, \tilde{S}) \geq (1-\eta)\omega^2 n/100 - 7\eta n > \omega^2 n/200$ and thus

$$e_{G_\gamma}\left(V \setminus V', \tilde{S}\right) \geq e_{G_\gamma}\left(\bigcup \{D \in \mathscr{D}\}, \tilde{S}\right) > \frac{\omega N s}{10} \cdot \frac{\omega^2 n}{200} > 32\eta n^2 ,$$

what we wanted to show. □

This finishes the proof of the Proposition 4.4.

□

8 Extremal case (proof of Proposition 4.1)

Let γ be such that $\beta \ll \gamma \ll \sigma \ll 1$. Throughout this section we write $\vartheta = \mathrm{ci}(n/k)$. It holds $\lambda \leq \vartheta$. The sets V_i, $i \in [\lambda]$ are called *clusters*[1].

[1]The notion of "cluster" in Section 8 is very different from the one used in other sections of the thesis. There, a cluster is a vertex set obtained by the Regularity Lemma.

Suppose that G admits a (β,σ)-Extremal partition $V_1,\dots,V_\lambda,\tilde{V}$. In any cluster V_i most of the vertices of $V_i\cap L$ are adjacent to almost all vertices of the cluster. Likewise, almost every vertex in $V_i\cap S$ is adjacent to almost all large vertices of the cluster. We make these statements precise in the following claim, however throughout the rest of the section we just refer to (β,σ)-Extremality to use similar properties.

Claim (Properties of a cluster in a (β,σ)-Extremal partition)**.** *For any $i\in[\lambda]$ and any $c>0$ the following holds.*

1. *For all but at most $\sqrt{\beta}k/c$ vertices $v\in V_i\cap L$ it holds that $\deg(v,V_i)\geq k-c\sqrt{\beta}k$.*
2. *For all but at most $2\sqrt{\beta}k/c$ vertices $v\in V_i\cap S$ it holds that $\deg(v,V_i\cap L)\geq |V_i\cap L|-c\sqrt{\beta}k$.*

Proof. 1. Let $U=\{v\in V_i\cap L : \deg(v,V_i)<k-c\sqrt{\beta}k\}$. Since every vertex $v\in U$ sends at least $c\sqrt{\beta}k$ edges outside V_i, we deduce from $e(V_i,V\setminus V_i)<\beta k^2$ that $|U|\leq\sqrt{\beta}k/c$.

2. Let $W=\{v\in V_i\cap S : \deg(v,V_i\cap L)<|V_i\cap L|-c\sqrt{\beta}k\}$. From

$$\begin{aligned}
e(V_i\cap L,V_i\cap S) &> |V_i\cap L|k-|V_i\cap L|^2-\beta k^2>|V_i\cap L||V_i\cap S|-2\beta k^2\text{ , and}\\
e(V_i\cap L,V_i\cap S) &= e(V_i\cap L,W)+e(V_i\cap L,V_i\cap S\setminus W)\\
&\leq (|V_i\cap L|-c\sqrt{\beta}k)|W|+|V_i\cap L|(|V_i\cap S|-|W|)\\
&= |V_i\cap L||V_i\cap S|-c\sqrt{\beta}k|W|
\end{aligned}$$

we infer that $|W|<2\sqrt{\beta}k/c$.

□

(Using the above claim with $c=1$ will be sufficient for our purposes.)

For each $i\in[\lambda]$ we set $L^i=\{u\in L : \deg(u,V_i)>(1-\gamma/2)k\}$. Observe that $|L^i|\geq(1-\gamma/2)\frac{k}{2}$, and that $\delta(G[L^i,A])\geq|A|-\gamma k$ for every $A\subseteq V_i$.

The (β,σ)-Extremal partition has two subcases. It is *abundant* if there exists $i\in[\lambda]$ with $|L^i|\geq(k+1)/2$, and it is *deficient* if $|L^i|<(k+1)/2$ for all $i\in[\lambda]$.

For each $i\in[\lambda]$ we set $S^i_\diamond=\{v\in S\cap V_i : \deg(v,L^i)>|L^i|-\gamma k/2\}$. Observe that the sets $S^i_\diamond$ are pairwise disjoint, and that $|L^i\cup S^i_\diamond|\geq(1-\gamma/2)k$.

The goal of this section is to prove Proposition 4.1. That is, given a (β,σ)-Extremal decomposition $V_1,\dots,V_\lambda,\tilde{V}$ of V (with $\beta\ll\sigma$) we have to show that $\mathscr{T}_{k+1}\subseteq G$, or there exists a set $Q\subseteq\tilde{V}$ such that

- $|Q|>k/2$.
- $|Q\cap L|>|Q|/2$.

- $e(Q, V \setminus Q) < \sigma k^2$.

The proof of Proposition 4.1 is decomposed into two separate statements, Proposition 8.1 and Proposition 8.2, according the number of leaves of the tree $T \in \mathscr{T}_{k+1}$ considered.

Proposition 8.1. *Let $T \in \mathscr{T}_{k+1}$ be a tree that has at most $60\gamma k$ leaves. Furthermore, suppose that G admits a (β, σ)-Extremal partition $V_1, \ldots, V_\lambda, \tilde{V}$. Then $T \subseteq G$, or there exists a set $Q \subseteq \tilde{V}$ such that*

- $|Q| > k/2$.
- $|Q \cap L| > |Q|/2$.
- $e(Q, V \setminus Q) < \sigma k^2$.

Proposition 8.2. *Let $T \in \mathscr{T}_{k+1}$ be a tree that has more than $60\gamma k$ leaves. Furthermore, suppose that G admits a (β, σ)-Extremal partition $V_1, \ldots, V_\lambda, \tilde{V}$. Then $T \subseteq G$.*

The proofs of Propositions 8.1, 8.2 occupy Sections 8.1, and 8.2, respectively.

Let us first rule out some easy configuration from further considerations.

Lemma 8.3. *Suppose that G admits a (β, σ)-Extremal partition $V_1, \ldots, V_\lambda, \tilde{V}$. Any tree $T \in \mathscr{T}_{k+1}$ with discrepancy at least $2\gamma k$ is a subgraph of G.*

Proof. Choose $L^* \subseteq L^i$ with $|L^*| = (1-\gamma/2)\frac{k}{2}$, and set $S^* = (L^i \cup S^i_\diamond) \setminus L^*$. Observe that $|S^*| \geq (1-\gamma/2)\frac{k}{2}$, and thus

$$\min\{\delta(G[L^*, S^*]), \delta(G[S^*, L^*]), \delta(G[L^*, L^*])\} \geq (1-\gamma/2)k/2 - \gamma k/2 \geq (1-3\gamma/2)k/2 .$$

Take the semiindependent partition (U_1, U_2) of T witnessing that $\mathrm{disc}(T) \geq 2\gamma k$. Denote by W the set of leaves of T. Since by Fact 3.2

$$|U_2 \setminus W| \leq |U_1| \leq (k+1-(2\gamma k))/2 < (1-3\gamma/2)k/2 ,$$

we may apply Fact 3.5 to embed T in G using the sets L^* and S^*. □

Lemma 8.4. *1. The sets $\{L^i\}_{i \in [\lambda]}$ are mutually disjoint, or $\mathscr{T}_{k+1} \subseteq G$.*

2. Suppose that $\tilde{V} = \emptyset$. If there exists a vertex $u \in L \setminus (\bigcup_i L^i)$, then $\mathscr{T}_{k+1} \subseteq G$.

Proof. For each $i \in [\lambda]$ fix $A_i \subseteq L^i$ a set of size $(1/2 - \gamma/4)k$, and set $B_i = (L^i \cup S^i_\diamond) \setminus A_i$.

1. Suppose that there exist distinct indices $i,j \in [\lambda]$ and a vertex $u \in L^i \cap L^j$. Let $T \in \mathscr{T}_{k+1}$ be arbitrary. By Lemma 8.3 we can assume in the following that $\mathrm{disc}(T) < 2\gamma k$. Since $e(V_i,V_j) < \beta k^2$, it holds that $|L^i \cap L^j| < \gamma k$. By Fact 3.1 there exists a full-subtree $\tilde{T} \subseteq T$ rooted at a vertex r such that $v(\tilde{T}) \in [k/6,k/3]$. We map r to u, and the tree $\tilde{T}$ to $G[A_i,B_i]$ greedily (this is possible since $\max\{|T_\mathrm{e} \cap V(\tilde{T})|,|T_\mathrm{o} \cap V(\tilde{T})|\} < v(\tilde{T})/2+2\gamma k$, by Lemma 3.3). By Lemma 3.3 it holds $\min\{|T_\mathrm{e} \cap V(T-\tilde{T})|,|T_\mathrm{o} \cap V(T-\tilde{T})\}| > v(T-\tilde{T})/2-2\gamma k$, and we infer that $\max\{|T_\mathrm{e} \cap V(T-\tilde{T})|,|T_\mathrm{o} \cap V(T-\tilde{T})|\} < 5k/12+2\gamma k$, we can embed $T-\tilde{T}$ in $G[A_j,B_j]$ greedily (avoiding the previously used vertices of $L^i \cap L^j$).

2. Suppose that there exists a vertex $u \in L \setminus \bigcup_i L^i$. By Part 1 of the lemma, we may assume that the sets L^i are pairwise disjoint.

   We saw in the proof of Part 1 of the lemma that the graphs $G[A_i,B_i]$ are suitable for embedding a tree whose both color-classes have sizes at most $(1/2-2\gamma)k$, and of a tree with substantial discrepancy. We shall consider sets $X_i \subseteq A_i$ and $Y_i \subseteq B_i$ which have even better embedding properties. Define

$$X_i = \{u \in A_i : \deg(v,V_i) > (1-\gamma/(13\vartheta))k\}\ , \text{ and}$$
$$Y_i = \{u \in B_i : \deg(v,L^i) > |X_i| - \gamma k/(13\vartheta)\}\ .$$

   It holds that
$$|V_i \setminus (X_i \cup Y_i)| < \gamma k/(3\vartheta^2)\ . \tag{8.1}$$

   As $X_i \subseteq L^i$ and $Y_i \subseteq S^i_\diamond$, all the sets X_i and Y_i are pairwise disjoint. Let $T \in \mathscr{T}_{k+1}$ be arbitrary. Analogously as in the proof of Lemma 8.3 it holds $T \subseteq G$ if $\mathrm{disc}(T) \geq \gamma k/(6\vartheta)$. Therefore we assume that $\mathrm{disc}(T) < \gamma k/(6\vartheta)$. By Fact 3.1 there exists a full-subtree $\tilde{T} \subseteq T$ rooted in a vertex r such that $v(\tilde{T}) \in [0.3k, 0.6k]$. We will embed the whole tree T in G, mapping r to u. Let D be the set of leaves of T in $\mathrm{N}_T(u)$. We first embed the tree $T-D$. The embedding is then extended to an embedding of T using the property of high degree of u.

   A 2^+-*component* is a component of the forest $T-r$ of order at least two. Let $\mathscr{C}$ be the family of all 2^+-components. For any subfamily $\mathscr{C}'$ it holds by Lemma 3.3 and the assumption $\mathrm{disc}(T) \leq \gamma k/(6\vartheta)$ that

$$\max\{V(\mathscr{C}') \cap T_\mathrm{o}, V(\mathscr{C}') \cap T_\mathrm{e}\} < |V(\mathscr{C}')|/2 + \gamma k/(12\vartheta) + 1\ . \tag{8.2}$$

   By (8.1) at most $\gamma k/(3\vartheta)$ vertices of the graph G are not contained in $\bigcup_i (X_i \cup Y_i)$. Thus, $\deg(u, \bigcup_i (X_i \cup Y_i)) \geq (1-\gamma/(3\vartheta))k$. We shall assign each 2^+-component $C \in \mathscr{C}$ an index $i_C \in [\vartheta]$. The idea is that each 2^+-component will be mapped to the cluster V_{i_C}. Thus the following requirement on the assignment for each $j \in [\vartheta]$ is natural:

$$\deg(u, X_j \cup Y_j) \geq |\{C \in \mathscr{C} \mid i_C = j\}| \text{ , and} \tag{8.3}$$
$$\sum_{\substack{C \in \mathscr{C} \\ i_C = j}} v(C) \leq (1 - 2\gamma/3)k \, . \tag{8.4}$$

We argue that such an assignment exists. We order the 2^+-components in an arbitrary way as $C_1, \ldots, C_{|\mathscr{C}|}$. Without loss of generality, we assume that $\deg(u, X_1 \cup Y_1) \leq \ldots \leq \deg(u, X_\vartheta \cup Y_\vartheta)$. For $j = 1, 2, \ldots, \vartheta$ we sequentially assign the yet unassigned 2^+-components C the index j (i.e., we set $i_C = j$) as long as (8.3) and (8.4) hold. If one of the conditions is to be violated (for step j) we proceed with assigning the components the index $j+1$. It remains to check that there are no unassigned 2^+-components left when we finish the step $j = \vartheta$. Indeed, if all steps were terminated because of condition (8.3) then we are done. Otherwise, suppose that we assigned 2^+-components $C_1, \ldots, C_{\kappa-1}$ the indices $1, \ldots, j-1$ in such a way that the terminating rule performed was (8.3), and then the 2^+-components $C_\kappa, C_{\kappa+1}, \ldots, C_{\kappa+w-1}$ were assigned the index j, and we were not able to assign component $C_{\kappa+w}$ the index j even though $\deg(u, X_j \cup Y_j) < w$. Then $\sum_{\ell=\kappa}^{\kappa+w} v(C_\ell) > (1 - 2\gamma/3)k$. Since $\deg(u, X_j \cup Y_j) < (1 - 2\gamma/3)k$ we have that

$$\deg\left(u, \bigcup_{\ell \neq j} (X_\ell \cup Y_\ell)\right) > \sum_{\ell=1}^{\kappa-1} v(C_\ell) + \sum_{\ell=\kappa+w}^{|\mathscr{C}|} v(C_\ell) \, .$$

Thus the remaining 2^+-components can be assigned an index, not violating (8.3) Observe, that (8.4) is not be violated in any future step, since the 2^+-components of total order at least $k/6 - 2\gamma k/3$ were embedded in $X_j \cup Y_j$ (no 2^+-component is larger than $5k/6$ by the way the root r was found).

We embed the tree T as follows. The vertex r is mapped to u. For each component $C \in \mathscr{C}$ we embed its root $r_C \in V(C) \cap \mathrm{N}_T(r)$ in one vertex from $(X_{i_C} \cup Y_{i_C}) \cap \mathrm{N}_G(u)$ (so that distinct roots are mapped to distinct vertices). We denote the image of the root r_C by $\varphi(r_C)$. Then the embedding of the roots is extended to an embedding of all 2^+-components. This can be done greedily since each of the graphs $G[X_i, Y_i]$ has minimum degree at least $(1/2 - \gamma/(12\vartheta))k + 1$, and by (8.2) it holds by a double application of (8.2) that

$$\sum_{\substack{C \in \mathscr{C} \\ \varphi(r_C) \in X_i}} |V(C) \cap T_\mathrm{e}| + \sum_{\substack{C \in \mathscr{C} \\ \varphi(r_C) \in Y_i}} |V(C) \cap T_\mathrm{o}| < (1 - 2\gamma/3)k/2 + 2(\gamma k/(12\vartheta) + 1) \leq \delta(G[X_i, Y_i]) \text{ , and}$$
$$\sum_{\substack{C \in \mathscr{C} \\ \varphi(r_C) \in X_i}} |V(C) \cap T_\mathrm{o}| + \sum_{\substack{C \in \mathscr{C} \\ \varphi(r_C) \in Y_i}} |V(C) \cap T_\mathrm{e}| < (1 - 2\gamma/3)k/2 + 2(\gamma k/(12\vartheta) + 1) \leq \delta(G[X_i, Y_i]) \, .$$

□

The next three statements (Lemma 8.5, Lemma 8.6, and Proposition 8.7) deal with the Deficient case. In this case, it may happen that none of the clusters are suitable for embedding of the tree $T \in \mathscr{T}_{k+1}$. For this reason, we must find connecting structures that allow us to distribute parts of the tree to different clusters. Each of the following three statements is used for a different type of trees.

If the configuration of the graph is Deficient, we show that $\tilde{V} = \emptyset$. First we bound the sizes of the sets L and S: $|L| < \lambda(1+\gamma)k/2 + (1-\sigma)|\tilde{V}|$, $|S| > \lambda(1-\gamma)k/2 + (1+\sigma)|\tilde{V}|$. Since $|L| \geq |S|$, we infer, that $|\tilde{V}| < \sigma k/2$. This in turn implies that $\tilde{V} = \emptyset$. Thus, $\lambda = \vartheta$. Observe also that

$$\vartheta(k+1) > n\,. \tag{8.5}$$

Lemma 8.5. *Suppose that G admits a (β,σ)-Extremal Deficient partition $V_1,\ldots,V_\vartheta,\tilde{V}$, ($\tilde{V}=\emptyset$), such that $\{L^i\}_{i=1}^{\vartheta}$ is a partition of L. For $i \in [\vartheta]$ define $S^i_\sharp = \{u \in S : \deg(u,L^i) > (1/2-\gamma)k\}$.*

Then there exist distinct indices $i_1, i_2 \in [\vartheta]$ such that there exists an $L^{i_1} \leftrightarrow L^{i_2}$-edge, or a $L^{i_1} \leftrightarrow S^{i_2}_\sharp$-edge, or there exists a vertex $x_0 \in S$ such that $\deg(x_0,L) \geq (1/2-\gamma)k$, $\min\{\deg(x_0,L^{i_1}),\deg(x_0,L^{i_2})\} \geq 1$.

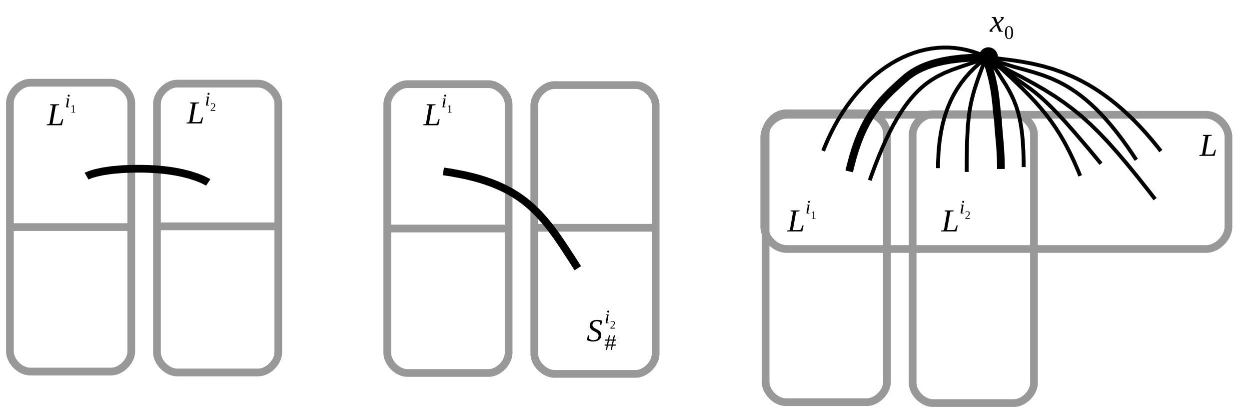

Figure 2: Three possible connecting structures guaranteed by Lemma 8.5.

Proof. We may assume that the sets $S^i_\sharp$ are mutually disjoint, otherwise there exists a $L^{i_1} \leftrightarrow S^{i_2}_\sharp$-edge ($i_1 \neq i_2$). Also, we are done if there exists an $L^{i_1} \leftrightarrow L^{i_2}$-edge, or there exists an $L^{i_1} \leftrightarrow S^{i_2}_\sharp$-edge ($i_1 \neq i_2$). We suppose that this is not the case in the following.

We write $Y = S \setminus \bigcup_i S^i_\sharp$. For any $i \in [\vartheta]$ and any vertex $u \in L^i$ there are at least $\max\{k+1-$

$|L^i| - |S^i_\sharp|, 0\}$ edges emanating from u to Y. Thus,

$$\begin{aligned}
e(L,Y) &\geq \sum_i |L^i| \max\{k+1-|L^i|-|S^i_\sharp|, 0\} \\
&\geq \sum_i (1/2-\gamma)k(k+1-|L^i|-|S^i_\sharp|) \\
&= (1/2-\gamma)k(\vartheta(k+1)-|L|-|S|+|Y|) \\
&\overset{(8.5)}{>} (1/2-\gamma)k|Y|
\end{aligned}$$

By averaging, there is a vertex $x_0 \in Y$ such that $\deg(x_0, L) > (1/2-\gamma)k$. From the definition of Y, $\deg(x_0, L^i) < (1/2-\gamma)k$, for any $i \in [\vartheta]$. Hence, x_0 is adjacent to at least two sets from $\{L^j\}_j$, as required. $\square$

Lemma 8.6. *Suppose that G admits a (β, σ)-Extremal Deficient partition $V_1, \ldots, V_\vartheta, \tilde{V}$ ($\tilde{V} = \emptyset$), such that $\{L^i\}_{i=1}^{\vartheta}$ is a partition of L. There exist $i_0 \in [\vartheta]$ and a vertex $v \in L^{i_0}$ such that* $\deg(v, L^{i_0}) + \deg(v, \bigcup_{j \neq i_0}(L^j \cup S^j)) \geq k/2$, *where $S^j = \{v \in S : \deg(v, L^j) \geq k/(3\vartheta)\}$.*

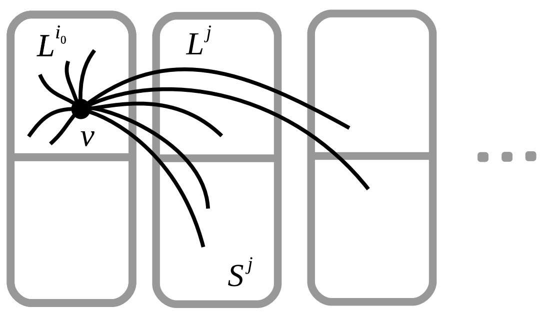

Figure 3: Connecting structure guaranteed by Lemma 8.6.

Proof. Partition $\bigcup_j S^j$ into sets $\tilde{S}^j$, $j \in [\vartheta]$ such that $\tilde{S}^j \subseteq S^j$. As $|L| \geq |S|$, there exists an index $i \in [\vartheta]$ such that $|\tilde{S}^i| \leq |L^i| \leq k/2$. Without loss of generality, assume that $k/2 - |\tilde{S}^1|$ is the maximum value among all values $k/2 - |\tilde{S}^i|$ ($i \in [\vartheta]$). Then $k/2 - |\tilde{S}^1|$ is non-negative.

Suppose that Lemma 8.6 is not true. Then for all vertices $v \in L^1$ it holds

$$\deg(v, S \setminus \bigcup_{j \neq 1} \tilde{S}^j) \geq \deg(v, S \setminus \bigcup_{j \neq 1} S^j) > k/2.$$

Thus $\deg(v, S^-) > k/2 - |\tilde{S}^1|$, where $S^- = \{u \in S : \deg(u, L^i) < k/(3\vartheta), \forall i = 1, \ldots, \vartheta\}$. A double counting argument on the edges between L^1 and S^- gives

$$|S^-| \frac{k}{3\vartheta} > e(L^1, S^-) > |L^1| \left(\frac{k}{2} - |\tilde{S}^1|\right),$$

implying that

$$|S^-| > \frac{3\vartheta|L^1|}{k}\left(\frac{k}{2} - |\tilde{S}^1|\right). \tag{8.6}$$

On the other hand, as

$$\sum_j |L^j| = |L| \geq |S| = \sum_j |\tilde{S}^j| + |S^-|,$$

there exists an $i \in [\vartheta]$ such that $|L^i| \geq |\tilde{S}^i| + |S^-|/\vartheta$. From the maximality of $k/2 - |\tilde{S}^1|$ and from (8.6) we deduce that

$$\frac{k}{2} - |\tilde{S}^1| \geq \frac{k}{2} - |\tilde{S}^i| \geq |L^i| - |\tilde{S}^i| \geq \frac{|S^-|}{\vartheta} > \frac{3|L^1|}{k}\left(\frac{k}{2} - |\tilde{S}^1|\right),$$

implying $k > 3|L^1|$, a contradiction. ☐

Proposition 8.7. *Suppose that G admits a (β,σ)-Extremal Deficient partition $V_1,\dots,V_\vartheta,\tilde{V}$ ($\tilde{V} = \emptyset$). Furthermore, suppose that the sets $\{L^i\}_{i\in[\vartheta]}$ partition the set L. Then there exists an index $i_0 \in [\vartheta]$ and matchings $\mathscr{E}^{i_0}$, and $\mathscr{J}^{i_0}$ such that the following hold.*

- *$\mathscr{E}^{i_0}$ is a $L^{i_0} \leftrightarrow (L \setminus L^{i_0})$-matching, $\mathscr{J}^{i_0}$ is a $L^{i_0} \leftrightarrow S$-matching.*
- *Each edge $xy \in \mathscr{J}_{i_0}$, $x \in L^{i_0}, y \in S$ has the property that $\deg(y, L^j) > k/(5\vartheta)$ for some $j \neq i_0$.*
- *$V(\mathscr{E}^{i_0}) \cap V(\mathscr{J}^{i_0}) = \emptyset$.*
- *$|L^{i_0}| + |\mathscr{E}^{i_0}| + |\mathscr{J}^{i_0}| \geq \frac{k+1}{2}$.*

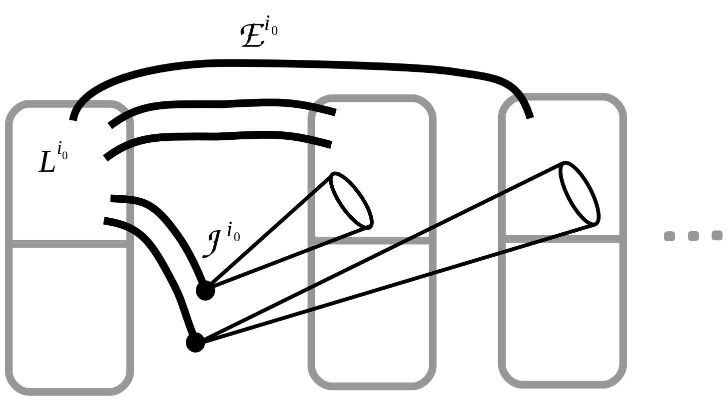

Figure 4: Connecting structure guaranteed by Proposition 8.7.

Proof. For each $i \in [\vartheta]$ let $S^i_\heartsuit = \{u \in S : \deg(u, L^i) > k/(5\vartheta)\}$. It holds by (β,σ)-Extremality that $|S^i_\heartsuit| > (1/2-\gamma)k$. We first find for each $i \in [\vartheta]$ two vertex-disjoint matchings E^i and D^i, such that E^i is a $L^i \leftrightarrow (L \setminus L^i)$-matching, D^i is a $L^i \leftrightarrow (S \setminus S^i_\heartsuit)$-matching, and such that the matchings $\{D^i\}_{i\in[\vartheta]}$ are pairwise vertex-disjoint.

For each i take E^i to be a maximum $L^i \leftrightarrow (L \setminus L^i)$ matching, and if $|L^i| + |S^i_\heartsuit| + |E^i| > k+1$, truncate E^i so that $|L^i| + |S^i_\heartsuit| + |E^i| = \max\{k+1, |L^i| + |S^i_\heartsuit|\}$. In the following we assume that

$$|L^1| + |S^1_\heartsuit| + |E^1| \geq |L^2| + |S^2_\heartsuit| + |E^2| \geq \ldots \geq |L^\vartheta| + |S^\vartheta_\heartsuit| + |E^\vartheta| \,. \tag{8.7}$$

Start with $i = 1$, and increase the index i gradually. Take D^i to be a maximum $(L^i \setminus V(E^i)) \leftrightarrow (S \setminus (S^i_\heartsuit \cup \bigcup_{j<i} V(D^j)))$ matching and truncate it so that $|L^i| + |S^i_\heartsuit| + |E^i| + |D^i| = \max\{k+1, |L^i| + |S^i_\heartsuit| + |E^i|\}$. We show that such a matching D^i exists. If $|L^i| + |S^i_\heartsuit| + |E^i| \geq k+1$, then set $D^i = \emptyset$. Otherwise, we want to find D^i of size $d_i = k+1 - |L^i| - |S^i_\heartsuit| - |E^i|$. By (8.7) it holds for the set $B_i = S \cap \bigcup_{j<i} V(D^j)$ that $|B_i| < \vartheta d_i$. Each vertex $u \in L^i$ has at least d_i neighbors outside $L^i \cup S^i_\heartsuit \cup V(E^i)$. Color arbitrary d_i edges emanating from each vertex $u \in L^i$ outside $L^i \cup S^i_\heartsuit \cup V(E^i)$ by black, and the remaining edges incident to u by grey. Easy calculation gives

$$e_{\mathrm{black}}(L^i \setminus V(E^i), S \setminus (S^i_\heartsuit \cup B_i)) > d_i(1/2 - 3\gamma)k - \vartheta d_i \frac{k}{5\vartheta} > \frac{d_i k}{5} \,. \tag{8.8}$$

Since the maximum degree in the graph $G_{\mathrm{black}}[L^i \setminus V(E^i), S \setminus (S^i_\heartsuit \cup B_i)]$ is upperbounded by $\max\{k/(5\vartheta), d_i\} = k/(5\vartheta)$, we see that there is no vertex cover of $G_{\mathrm{black}}[L^i \setminus V(E^i), S \setminus (S^i_\heartsuit \cup B_i)]$ of size less than

$$\frac{d_i k/5}{k/(5\vartheta)} \geq d_i \,.$$

Hence, by König's Matching Theorem, there exists a matching D^i of size d_i with the desired properties. We set $X_i = V(D^i) \setminus L^i$.

Let us summarize the properties of the obtained structure. For any $i \in [\vartheta]$ it holds

$$|L^i| + |S^i_\heartsuit| + |E^i| + |X_i| \geq k+1, \text{ and} \tag{8.9}$$

$$X_i \cap \bigcup_{j \neq i} X_j = \emptyset \quad \text{and} \quad S^i_\heartsuit \cap X_i = \emptyset \,. \tag{8.10}$$

The aim of the following several lines is to prove that there must be an index $i \in [\vartheta]$ such that sufficiently many vertices from $S^i_\heartsuit \cup X^i$ are contained in $\bigcup_{j \neq i} S^j_\heartsuit$, thus providing with the desired bridges from the cluster V_i. It holds

$$
\begin{aligned}
n-|L| \geq \left|\bigcup_i (S^i_\heartsuit \cup X_i)\right| &\overset{(8.10)}{\geq} \sum_i |S^i_\heartsuit| + \sum_i |X_i| - \sum_i \left|(S^i_\heartsuit \cup X_i) \cap \bigcup_{j\neq i} S^j_\heartsuit\right| \\
&\overset{(8.9)}{\geq} \vartheta(k+1) - |L| - \sum_i \left|(S^i_\heartsuit \cup X_i) \cap \bigcup_{j\neq i} S^j_\heartsuit\right| - \sum_i |E^i| \,,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\sum_i \left(|L^i| + |E^i| + \left|(S^i_\heartsuit \cup X_i) \cap \bigcup_{j\neq i} S^j_\heartsuit\right|\right) &\geq |L| + \vartheta(k+1) - n \geq \vartheta(k+1) - \frac{n}{2} \\
&\overset{(8.5)}{\geq} \frac{\vartheta(k+1)}{2} \,.
\end{aligned}
$$

By averaging, there exists an index $i_0 \in [\vartheta]$ such that

$$
|L^{i_0}| + |E^{i_0}| + \left|(S^{i_0}_\heartsuit \cup X_{i_0}) \cap \bigcup_{j\neq i_0} S^j_\heartsuit\right| \geq \frac{k+1}{2} \,. \tag{8.11}
$$

Set $\mathscr{E}^{i_0} = E^{i_0}$. The matching $\mathscr{J}^{i_0}$ consists of two vertex disjoint matchings $\mathscr{J}_1$ and $\mathscr{J}_2$. The matching $\mathscr{J}_1$ is defined by $\mathscr{J}_1 = \{e \in D^{i_0} : e \cap \bigcup_{j\neq i_0} S^j_\heartsuit \neq \emptyset\}$. We take $\mathscr{J}_2$ any matching in $G[S^{i_0}_\heartsuit \cap \bigcup_{j\neq i_0} S^j_\heartsuit, L^{i_0} \setminus V(\mathscr{E}^{i_0} \cup \mathscr{J}_1)]$ that covers $Q = S^{i_0}_\heartsuit \cap \bigcup_{j\neq i_0} S^j_\heartsuit$. Since $|Q| < \gamma k$, such a matching can be found greedily. □

8.1 Proof of Proposition 8.1

Suppose the tree T and the graph G satisfying the hypothesis of Proposition 8.1 are given. Throughout the proof we write $\alpha = 60\gamma$.

For each $i \in [\lambda]$ we define $X^i = \{v \in V_i : \deg(v, L^i) > k/(5\vartheta)\}$. Vertices in

$$
\bigcup_{i\in\lambda} L^i \cup \bigcup_{i\in[\lambda]} X^i
$$

are *substantial*, vertices in

$$
\mathscr{O} = V \setminus \left(\tilde{V} \cup \bigcup_{i\in[\lambda]} L^i \cup \bigcup_{i\in[\lambda]} X^i\right)
$$

are *negligible*. Observe that there are at most $2r\gamma k$ negligible vertices. The substantial vertices are suitable for embedding: suppose we have a forest F of order at most $k/(5\vartheta)$ consisting of rooted components $(r_1,C_1),\ldots,(r_p,C_p)$. Let $v_1 \in V_{i_1},\ldots,v_p \in V_{i_p}$ be arbitrary distinct substantial vertices. Then F can be embedded in G so that every component C_x is embedded in V_{i_x}, with its root r_x mapped to the vertex v_x. If G is Abundant, we set $\Lambda \subseteq [\lambda]$ to be the set of indices i_0 such that $|L^{i_0}| \geq (k+1)/2$, and set $\mathscr{E}^{i_0} = \mathscr{J}^{i_0} = \emptyset$. If G is Deficient, we apply Proposition 8.7 to obtain an index i_0 and two matchings $\mathscr{E}^{i_0}$ and $\mathscr{J}^{i_0}$ such that $|L^{i_0}|+|\mathscr{E}^{i_0}|+|\mathscr{J}^{i_0}| \geq (k+1)/2$. We then set $\Lambda = \{i_0\}$.

For each $i_0 \in \Lambda$, we shall try to embed the tree T so that most of the vertices of T are embedded in V_{i_0}. We shall show that if all the attempts fail, then there exists a set Q satisfying the hypothesis of Proposition 8.1. The embedding plan is as follows. We try to embed most of T_o in (a subset of) L^{i_0} and the internal vertices of T_e into vertices which are well-connected to L^{i_0} (the leaves of T_e being treated in the last stage). The set L^{i_0} may be not large enough to absorb all the vertices from T_o, since we only know that $|L^{i_0}| > (1/2-\gamma)k+1$ and T_o may be as large as $k/2$. We use the edges of the matchings $\mathscr{E}^{i_0}$ and $\mathscr{J}^{i_0}$ in order to distribute the excess parts of T outside V_{i_0}. We want then to show that the set of vertices well-connected to L^{i_0} is large enough to absorb the internal vertices of T_e. However, this need not to be the case; but then we are able to exhibit the desired set Q.

The following statement provides an embedding of the tree, given a suitable embedding structure. We defer its proof to the end of the section.

Proposition 8.8. *For any tree $T \in \mathscr{T}_{k+1}$ with $\ell < \alpha k$ leaves the following holds. Let H and H_κ, $\kappa \in I$ (the index set I is arbitrary) be vertex disjoint subgraphs of G. The graph H is bipartite, $H = (A,B;E)$. Suppose that the graphs H, and H_κ ($\kappa \in I$) have the following properties.*

- $\delta(H_\kappa) > 25\alpha k$ *for each* $\kappa \in I$.
- $\delta(A) \geq k$.
- *There exists $A \leftrightarrow (\bigcup_\kappa(V(H_\kappa)))$-matching $\mathscr{E}$, and a family $\mathscr{M}$ of vertex disjoint $A \leftrightarrow (V \setminus V(H)) \leftrightarrow (\bigcup_\kappa V(H_\kappa))$ paths. Moreover, $V(\mathscr{E}) \cap V(\mathscr{M}) = \emptyset$.*
- $|\mathscr{E}|+|\mathscr{M}| < \alpha k$.
- $|A|+|\mathscr{E}| \geq |T_\mathrm{o}|$.
- $|B|+|\mathscr{E}|+|\mathscr{M}| \geq |T_\mathrm{e}|-1$.
- $\delta(A,B) \geq |B|-\alpha k$.
- *The set B has a decomposition $B = B_\mathrm{a} \cup B_\mathrm{d}$, $|B_\mathrm{d}| \leq \alpha k$, $\delta(B_\mathrm{a},A) \geq |A|-\alpha k$, and there exists a family $\mathscr{Q} = \{P_1,\ldots,P_r\}$ of $r = |B_\mathrm{d}|$ vertex-disjoint $A \leftrightarrow B_\mathrm{d} \leftrightarrow A$ paths. Moreover, $V(\mathscr{Q}) \cap (V(\mathscr{E}) \cup V(\mathscr{M})) = \emptyset$.*

Then there exists an embedding of T in G.

For each $i_0 \in \Lambda$ we try to find a structure suitable for applying Proposition 8.8. We do the following for each $i_0 \in \Lambda$.

We write $e = |\mathscr{E}^{i_0}|$ and $b = |\mathscr{J}^{i_0}|$. Fix a set $L_* \subseteq L^{i_0}$ of size $|T_{\mathrm{o}}| - b - e$ which contains $F = (V(\mathscr{E}^{i_0}) \cup V(\mathscr{J}^{i_0})) \cap L^{i_0}$. Set $W_{\mathrm{a}} = (L^{i_0} \setminus L_*) \cup S_{\diamond}^{i_0}$. Note that $|W_{\mathrm{a}}| > |T_{\mathrm{e}}| - \gamma k$. Take a maximum family $\mathscr{P} = \{P_1, \ldots, P_a\}$ of vertex-disjoint $(L_* \setminus F) \leftrightarrow (V \setminus (L_* \cup W_{\mathrm{a}})) \leftrightarrow (L_* \setminus F)$-paths, and let W_{d} be their middle vertices.

Assume that $|W_{\mathrm{a}}| + |W_{\mathrm{d}}| + |\mathscr{E}^{i_0}| \geq |T_{\mathrm{e}}| - 1$. Consider a family of paths $\mathscr{P}' \subseteq \mathscr{P}$ by truncating $\mathscr{P}$ so that $|\mathscr{P}'| = \min\{|\mathscr{P}|, \alpha k\}$, and denote W'_{d} the set of middle vertices of $\mathscr{P}'$. We apply Proposition 8.8, setting the parameters of the proposition as follows: $A = L_*, B_{\mathrm{a}} = W_{\mathrm{a}}, B_{\mathrm{d}} = W'_{\mathrm{d}}, \mathscr{Q} = \mathscr{P}', \mathscr{E} = \mathscr{E}^{i_0} \cup \mathscr{J}^{i_0}, \mathscr{M} = \emptyset, I = [\lambda] \setminus \{i_0\}$, and $H_\kappa = G[L^\kappa \cup S_\diamond^\kappa]$ (for each $\kappa \in I$). Proposition 8.8 will be used several other times. When using it later, we shall explicitly mention only those parameters of the proposition which differ from the ones above.

Now, assume that $|W_{\mathrm{a}}| + |W_{\mathrm{d}}| + |\mathscr{E}^{i_0}| < |T_{\mathrm{e}}| - 1$. Then $|\mathscr{P}| < \gamma k$. From each vertex $u \in L_* \setminus (F \cup V(\mathscr{P}))$ at least two edges $e_u^x = ux_u$ and $e_u^y = uy_u$ are emanating into $V \setminus (L_* \cup W_{\mathrm{a}} \cup W_{\mathrm{d}} \cup \mathscr{E}^{i_0})$. Set $R_{i_0} = \bigcup_{u \in L_* \setminus (F \cup V(\mathscr{P}))} \{x_u, y_u\}$. By the maximality of $\mathscr{P}$ all the vertices x_u, y_u, $(u \in L_* \setminus (F \cup V(\mathscr{P})))$ are distinct. At most $2\vartheta\gamma k$ of these are negligible vertices. Denote the set of substantial vertices of R_{i_0} by M_{i_0}, and call the set $Y_{i_0} = R_{i_0} \cap \tilde{V}$ the *shadow* of L_*. If $|M_{i_0}| \geq 2\gamma k$ then one can find a matching $\mathscr{N}_1 \subseteq \bigcup_{u \in L_* \setminus (F \cup V(\mathscr{P}))} \{e_u^1, e_u^2\}$ of size γk, and Proposition 8.8 can be applied (with $\mathscr{E} = \mathscr{E}^{i_0} \cup \mathscr{N}_1$, $B_{\mathrm{d}} = W_{\mathrm{d}}$, and $\mathscr{Q} = \mathscr{P}$) to show that $T \subseteq G$. Otherwise, $|Y_{i_0}| \geq 2|L_*| - |\mathscr{O}| - |M_{i_0}| \geq 2|L_*| - \vartheta\gamma k$. The choice of $L_* \subseteq L^{i_0}$ was arbitrary, with the only restriction $F \subseteq L_*$. Thus the above procedure can be applied for another choice of L_*. Denote by $\tilde{Y}_{i_0}$ the union of shadows corresponding to all possible choices of L_* (for a fixed vertex $u \in L^{i_0} \setminus (F \cup V(\mathscr{P}))$, the choice of x_u and y_u does not depend on the choice of L_*). Thus we get that $T \subseteq G$ by Proposition 8.8, or $|\tilde{Y}_{i_0}| \geq 2|L^{i_0}| - 3\vartheta\gamma k$.

Suppose that we were not able to use Proposition 8.8 so far for any $i_0 \in \Lambda$. If there exists $i_0 \in \Lambda$ such that $|\tilde{Y}_{i_0} \cap \bigcup_{i \in \Lambda \setminus \{i_0\}} \tilde{Y}_i| \geq 4\gamma k$, then $T \subseteq G$. Indeed, one can find a family $\mathscr{N}_2$ of at least γk vertex disjoint $L^{i_0} \leftrightarrow (\tilde{Y}_{i_0} \cap \bigcup_{i \in \Lambda \setminus \{i_0\}} \tilde{Y}_i) \leftrightarrow \left(\bigcup_{i \in \Lambda \setminus \{i_0\}} L^i\right)$-paths and apply Proposition 8.8 with $\mathscr{M} = \mathscr{N}_2$. We assume in the rest that such i_0 does not exist. Since $|\bigcup_{i \in \Lambda} \tilde{Y}_i| \geq \sum_{i \in \Lambda} (|\tilde{Y}_i| - |\tilde{Y}_i \cap \bigcup_{j \in \Lambda \setminus \{i_0\}} \tilde{Y}_j|)$, we have that

$$\left| \bigcup_{i \in \Lambda} \tilde{Y}_i \right| \geq 2 \sum_{i \in \Lambda} |L^i| - 4\vartheta^2 \gamma k \,. \tag{8.12}$$

Set $Y = \bigcup_{i \in \Lambda} \tilde{Y}_i$.

We distinguish three cases:

(♣1) *It holds* $|L \cap Y| \leq k/8$ *and* $e(Y, \tilde{V} \setminus Y) < \sigma k^2$.

Solution of (♣1): The idea is to show that the set $Q = \tilde{V} \setminus Y$ satisfies the requirements of

Proposition 8.1. To this end, it is enough to show that

$$|Q \cap L| > \frac{1}{2}|Q| \,. \tag{8.13}$$

By the hypothesis of (♣1), not many vertices in Y are large. Thus the ratio of the large vertices in the graph $G[\bigcup_{i\in\Lambda} V_i \cup Y]$ is substantially smaller than one half. Then there must be substantially more than half of the large vertices in the complementary set Q, and (8.13) follows. We make the idea rigorous by the following calculations. For any $i \in \Lambda$ set $l_i = |L^i|$.

$$\begin{aligned}\frac{1}{2}n \le |L| &\le (\lambda - |\Lambda|)k/2 + \sum_{i\in\Lambda} l_i + |L\cap Y| + |L\cap Q| + |L \setminus (\tilde{V} \cup \bigcup_{j\in[\lambda]} L^j)| \\ &< (\lambda - |\Lambda|)k/2 + \sum_{i\in\Lambda} l_i + k/8 + |L\cap Q| + \gamma n \,.\end{aligned}$$

Thus,

$$\begin{aligned}|L\cap Q| &> \frac{1}{2}n - (\lambda - |\Lambda|)k/2 - \sum_{i\in\Lambda} l_i - k/8 - \gamma n \\ &> \frac{1}{2}\left(|\tilde{V}| - 2\sum_{i\in\Lambda} l_i\right) + |\Lambda|k/2 - k/8 - 2\gamma n \\ &\overset{(8.12)}{>} \frac{1}{2}|Q| + |\Lambda|k/2 - k/7 > \frac{1}{2}|Q| \,,\end{aligned}$$

which was to be shown.

(♣2) *It holds* $|L\cap Y| > k/8$ *and* $e(Y, \tilde{V}\setminus Y) < \sigma k^2$.
Solution of (♣2): We show that $T \subseteq G$. Since the average degree in the graph $G[Y]$ is at least $qk/20$, there exists a subgraph $H_* \subseteq G[Y]$ with $\delta(H_*) \ge qk/40$. By averaging, there exists $i_0 \in \Lambda$ such that

$$|Y_{i_0} \cap V(H_*)| > qk/(40\vartheta) \,. \tag{8.14}$$

Fix such an index i_0. By (8.14) there exists a $L^{i_0} \leftrightarrow V(H_*)$-matching $\mathscr{E}$ of size $\alpha k/2$. By Proposition 8.8 (with $I = \{*\}$) it holds $T \subseteq G$.

(♣3) *It holds* $e(Y, \tilde{V}\setminus Y) \ge \sigma k^2$.
Solution of (♣3): We show that $T \subseteq G$. The average degree of the bipartite graph $G[Y, \tilde{V}\setminus Y]$ is at least $q\sigma k$. Thus there exists a graph $H_* \subseteq G[Y, \tilde{V}\setminus Y]$ with $\delta(H_*) \ge q\sigma k/2$. There must be an index $i_0 \in \Lambda$ such that $|Y_{i_0} \cap V(H_*)| > \sigma qk/(2\vartheta)$. Fix such an index i_0 and find matching $\mathscr{E}$ as in (♣2). By Proposition 8.8 (with $I = \{*\}$) it holds $T \subseteq G$.

Proof of Proposition 8.8. Root T at an arbitrary vertex $v \in T_0$. An *c-induced path* $a_1 \ldots a_{c+1} \subseteq T$ is a path whose internal vertices have degree two in T. Take a maximum family $\mathscr{F}$ of vertex disjoint

6-induced paths in T. We show that $|V(\mathscr{F})| \geq k-19\ell$.

Let $D_3 = \{u \in V(T) : \deg_T(u) \geq 3\}$ and $D_i = \{u \in V(T) : \deg_T(u) = i\}$ for $i = 1,2$. By Fact 3.4, we have $|D_3| < \ell$ (and $|D_2| \geq k-2\ell$). From

$$2k = \sum_{u\in V(T)} \deg(u) = |D_1| + 2|D_2| + \sum_{u\in D_3} \deg(u) \geq 2k - 3\ell + \sum_{u\in D_3} \deg(u) \ ,$$

we deduce that there are at most $3\ell+1$ maximal (w. r. t. inclusion) paths formed by vertices of degree 2 or 1 not containing the root v. On each such maximal path, at most 5 vertices are not covered by $\mathscr{F}$. Thus the total number of vertices uncovered by $\mathscr{F}$ is at most $5(3\ell+1)+|D_3|+|\{v\}| \leq 19\ell$. The order $\preceq_v$ naturally extends to an order of the paths of $\mathscr{F}$. For a family $\mathscr{F}' \subseteq \mathscr{F}$ we write $T(\downarrow \mathscr{F}')$ to denote all the vertices of $V(\mathscr{F}')$, and all vertices which are below some vertex of $V(\mathscr{F}')$, i.e.,

$$T(\downarrow \mathscr{F}') = \bigcup_{u\in V(\mathscr{F}')} V(T(\downarrow u)) \ .$$

One can find a family $\mathscr{R} \subseteq \mathscr{F}$ satisfying the three properties below.

(P1) $|\mathscr{R}| \leq |\mathscr{E}| + |\mathscr{M}|$.

(P2) $|T(\downarrow \mathscr{R})| < 25\alpha k$, and $3(|\mathscr{E}|+|\mathscr{M}|) \leq \min\{|T_{\mathrm{e}} \cap T(\downarrow \mathscr{R})|, |T_{\mathrm{o}} \cap T(\downarrow \mathscr{R})|\}$.

(P3) $\mathscr{R}$ is a $\preceq_v$-antichain.

We describe a procedure how to obtain such a family $\mathscr{R}$. By an inductive construction, we first find an auxiliary family $\mathscr{R}'$, starting with $\mathscr{R}' = \emptyset$. While $|\mathscr{R}'| < |\mathscr{E}| + |\mathscr{M}|$ we take a $\preceq_v$-minimal path in $\mathscr{F}$ which is not included in $\mathscr{R}'$ and add it to $\mathscr{R}'$. By the bound $|V(T) \setminus V(\mathscr{F})| < 19\ell$, in each step it holds that $|T(\downarrow \mathscr{R}')| \leq 6|\mathscr{R}'| + 19\alpha k$, and obviously $3|\mathscr{R}'| \leq \min\{|T_{\mathrm{e}} \cap T(\downarrow \mathscr{R}')|, |T_{\mathrm{o}} \cap T(\downarrow \mathscr{R}')|\}$.

Let $\mathscr{R}$ be the $\preceq_v$-maximal elements of $\mathscr{R}'$. The properties **(P1)**, **(P2)**, and **(P3)** are satisfied.

Set $d = 5\alpha k$. Take a family $\mathscr{X} = \{X_1, \ldots, X_d\}$ of d 5-induced vertex-disjoint $T_{\mathrm{e}} \leftrightarrow T_{\mathrm{o}} \leftrightarrow T_{\mathrm{e}} \leftrightarrow T_{\mathrm{o}} \leftrightarrow T_{\mathrm{e}}$ paths, such that no path intersects $\{v\} \cup T(\downarrow \mathscr{R}')$. For any path $R \in \mathscr{R}$ we write a_R to denote its $\preceq_v$-maximum vertex in T_{o}, and set $b_R = \mathrm{Ch}(a_R)$, $c_R = \mathrm{Ch}(b_R)$, and $d_R = \mathrm{Ch}(c_R)$. We set $U = A \cap (V(\mathscr{E}) \cup V(\mathscr{M}))$ and $Q = A \cap V(\mathscr{Q})$.

We now describe the embedding ψ of T. First note that we do not have to embed those leaves, whose parents are embedded in A. Indeed, having such a partial embedding, it easily extends to an embedding of T using high degrees of vertices in A. Hence we shall not embed them until the very last step. We embed the root v in an arbitrary vertex in $A \setminus (U \cup Q)$. We continue embedding T greedily, mapping vertices from T_{o} to $A \setminus (U \cup Q)$ and internal vertices of T_{e} to B_{a}. However, there are two exceptions in the greedy procedure.

(S1) If we are about to embed a vertex b_R (for some $R \in \mathscr{R}$), then we do not embed it, neither the part of the tree $T(\downarrow b_R)$.

(S2) If we are about to embed a vertex x_2 which was part of some path $x_1x_2x_3x_4x_5 \in \mathscr{X}$ we skip its embedding, as well as the embedding of the vertices x_3 and x_4. We continue with mapping x_5 to B_a.

Observe that we are able to finish the greedy part of the embedding since the two "skipping rules" guarantee that both in A and in B at least $d > \alpha k$ vertices of T remain unembedded.

In the next step, we build missing connections in the graph H caused by the skipping rules.

We construct an auxiliary bipartite graph $K_1 = (O_\mathrm{a}, O_\mathrm{b}; E_1)$. We arbitrarily pair up $2(d-r)$ vertices of $A \setminus (U \cup Q)$ unused by ψ into pairs $\mu_1 = \{a_1^1, a_1^2\}, \dots, \mu_{d-r} = \{a_{d-r}^1, a_{d-r}^2\}$. The remaining r pairs are formed by endvertices of the paths in $\mathscr{Q}$,

$$\mu_{i+d-r} = A \cap V(P_i) .$$

Vertices of the color class O_b are formed by the pairs μ_i ($i \in [d]$). Vertices of the color class O_a are formed by the paths in $\mathscr{X}$. A path $x_1x_2x_3x_4x_5 \in \mathscr{X}$ is adjacent in K_1 to a pair μ_i if and only if there exists a perfect matching in the graph $H[\{\psi(x_1), \psi(x_5)\}, \mu_i]$. Since $|O_\mathrm{a}| = |O_\mathrm{b}|$ and $\delta(K_1) \geq |O_\mathrm{a}| - 2\alpha k \geq |O_\mathrm{a}|/2$, there exists, by Proposition 3.6, a perfect matching M_1 in K_1. The matching M_1 gives us instructions where to embed the vertices x_2 and x_4 of any path $x_1x_2x_3x_4x_5 \in \mathscr{X}$. We extend ψ accordingly on the vertices $\bigcup_{x_1x_2x_3x_4x_5 \in \mathscr{X}} \{x_2, x_4\}$. If a path $x_1x_2x_3x_4x_5 \in \mathscr{X}$ was matched with μ_{i+d-r} (for some $i \in [r]$) in K_1 then we embed x_3 in the middle vertex of the path P_i. We write $\mathscr{X}'$ for those paths $x_1x_2x_3x_4x_5 \in \mathscr{X}$ whose vertex x_3 was not yet embedded. It holds $|\mathscr{X}'| \geq 4\alpha k$.

Let $\chi : \mathscr{R} \to U$ be an arbitrary injective mapping. We construct another bipartite graph $K_2 = (J_\mathrm{a}, J_\mathrm{b}; E_2)$. Vertices of the color class J_a are elements of $\mathscr{R} \cup \mathscr{X}'$ ($J_\mathrm{a} = \mathscr{R} \cup \mathscr{X}'$) and vertices of the color class J_b are vertices of B_a unused by ψ ($J_\mathrm{b} \subseteq B_\mathrm{a}$). A path $R \in \mathscr{R}$ is adjacent in K_1 with an $b \in J_\mathrm{b}$ if and only if $b\psi(a_R) \in E(H)$ and $b\chi(R) \in E(H)$. A path $x_1x_2x_3x_4x_5 \in \mathscr{X}'$ is adjacent to a vertex $b \in J_\mathrm{b}$ if and only if $b\psi(y_2) \in E(H)$ and $b\psi(y_4) \in E(H)$. There exists a matching M_2 in K_2 covering J_a. The existence of the matching M_2 in K_2 covering J_a is a direct consequence of Proposition 3.6. Indeed, $\delta(K_1) \geq |J_\mathrm{a}| - 2\gamma k > |J_\mathrm{a}|/2$, and $|J_\mathrm{a}| \leq |J_\mathrm{b}|$. Such a matching gives us instructions where to embed unembedded vertices x_3 (in the case of a path $x_1x_2x_3x_4x_5 \in \mathscr{X}'$ and vertices b_R (in the case of a path $R \in \mathscr{R}$). For a path $R \in \mathscr{R}$ we finish embedding the part of the tree $T(\downarrow c_R)$, extending the mapping ψ. If $\psi(c_R) \in V(\mathscr{E})$ we just use the corresponding connecting edge of $\mathscr{E}$ to embed d_R in H_κ (for some $\kappa \in I$) and continue embedding $T(\downarrow d_R)$ greedily in H_κ. If $\psi(c_R) \in V(\mathscr{M})$ we embed d_R in the middle vertex of the corresponding connecting path $\mathscr{M}$ and embed the rest of $T(\downarrow d_R)$ greedily in H_κ (for some $\kappa \in I$). $\square$

8.2 Proof of Proposition 8.2

In order to prove Proposition 8.2 we need the following two auxiliary lemmas.

Lemma 8.9. *Let G be in a (β,σ)-Extremal, Deficient configuration. Let $T \in \mathscr{T}_{k+1}$ be a tree with a vertex $r \in V(T)$ such that the forest $T-r$ contains a component C of order $v(C) \in [k/(3\vartheta), k-4\gamma k]$. Then $T \subseteq G$.*

Proof. By Lemmas 8.3 and 3.3 we can assume that $\max\{|T_e \setminus V(C)|, |T_o \setminus V(C)|\} < (k+1-v(C))/2 + (2\gamma k+1)/2 < k/2 - 2\gamma k$, otherwise $T \subseteq G$.

For $i \in [\vartheta]$ define $S_\sharp^i = \{u \in S : \deg(u, L^i) > (1/2-\gamma)k\}$. By (β,σ)-Extremality it holds that $|S_\sharp^i| > (1/2-\gamma)k$. By Lemma 8.5 there is at least one of the following three connecting structures in G. We show that $T \subseteq G$ in each of the cases separately.

(**A1**) There exists an edge xy, $x \in L^{i_1}$, $y \in L^{i_2}$, $i_1 \neq i_2$.

(**A2**) There exists an edge xy, $x \in L^{i_1}$, $y \in S_\sharp^{i_2}$, $i_1 \neq i_2$.

(**A3**) There exists a vertex $x_0 \in S$ such that $\deg(x_0, L) > (1/2-\gamma)k$, and x_0 is adjacent to vertices of at least two different clusters L^{i_1},L^{i_2} (i.e., $\min\{\deg(x_0, L^{i_1}), \deg(x_0, L^{i_2}\} \geq 1$).

To solve the cases (**A1**) and (**A2**) it is enough to map r to x, and use the edge xy to greedily embed C in $G[L^{i_2}, S_\sharp^{i_2}]$. The part $T-(V(C)\cup\{r\})$ can be greedily embedded in $G[L^{i_1}, S_\sharp^{i_1}]$.

It remains to solve the case (**A3**). Let $\imath$ be such an index i for which the value $\deg(x_0, L^i)$ is minimal positive. We embed r in x_0, C in $G[L^\imath, S_\diamond^\imath]$. The forest $F = T-(V(C)\cup\{r\})$ can be greedily embedded in the clusters $\{V_i\}_i$ (preserving adjacencies of r to the components of F). This is standard. □

Lemma 8.10. *Let F be a rooted forest with partition $V(F) = O_1 \cup O_2$, such that O_2 is independent. Let W be the set of leaves of F and set $P = \{u \in O_2 : |W \cap \mathrm{Ch}(u)| = 1\}$. Let H be a graph and let $A, B \subseteq V(H)$ be two disjoint sets such that $|A| \geq |O_1|$, $\min\{\delta(A,A), \delta(B,A)\} > |O_1| - f$, $\delta(A,B) > |B| - f$, $|B| \geq |O_2 \setminus W|$, and $\delta(A) \geq v(F)-1$. If $|P| \geq 2f$, then there exists an embedding φ of F in H such that $\varphi(O_1) \subseteq A$.*

Proof. Choose a subset $P' \subseteq P$ of size $|P'| = 2f$. Consider the subtree $F' = F - W'$, where $W' = W \cap (O_2 \cup \mathrm{N}(P'))$. We embed greedily the tree F' in $A \cup B$, so that $V(F') \cap O_1$ maps to A and $V(F') \cap O_2$ maps to B. Denote this embedding by φ'. Next we want to embed the leaves $W' \cap O_1$ in A. Denote by A' the set of vertices in A that are not used by φ', i. e., $A' = A \setminus \varphi(V(F'))$. We want to find a matching M in $H[A', \varphi'(P')]$ that covers $\varphi'(P')$. By Proposition 3.6, such a matching exists since $|A'| \geq 2f = |\varphi'(P')|$, and

$$\delta(\varphi(P'), A') > f = |P'|/2, \quad \delta(A', \varphi(P')) > f = |P'|/2\,. \tag{8.15}$$

We extend φ' to an embedding φ of F, by embedding $W' \cap O_1$ according to the matching M, and by embedding $W \cap O_2$ greedily (this is guaranteed by the minimal degree condition of the set A). □

A semiindependent partition (U_1, U_2) of a tree F is *ℓ-ideal* if each of the vertex sets U_1 and U_2 contains at least ℓ leaves of F.

If $\mathrm{disc}(T) \geq 2\gamma k$, then Lemma 8.3 ensures that $T \subseteq G$. We shall further assume only the case $\mathrm{disc}(T) < 2\gamma k$.

We prove Proposition 8.2 in two steps. In the first step we show that T has an $8\gamma k$-ideal semiindependent partition, or $T \subseteq G$. In the second step, we prove that if T has an $8\gamma k$-ideal semiindependent partition, then $T \subseteq G$.

First step. Denote by W_{e} and W_{o} the leaves in T_{e} and in T_{o}, respectively. Let $W = W_{\mathrm{e}} \cup W_{\mathrm{e}}$ be the set of all leaves of T. Set $w_{\mathrm{e}} = |W_{\mathrm{e}}|$ and $w_{\mathrm{o}} = |W_{\mathrm{o}}|$. Remark that $w_{\mathrm{e}} + w_{\mathrm{o}} \geq 60\gamma k$. We distinguish three cases based on the values of w_{e} and w_{e}.

1. If $w_{\mathrm{e}} \geq 8\gamma k$ and $w_{\mathrm{o}} \geq 8\gamma k$, then $(T_{\mathrm{o}}, T_{\mathrm{e}})$ is an $8\gamma k$-ideal semiindependent partition.

2. If $w_{\mathrm{e}} < 8\gamma k$ then it holds $w_{\mathrm{o}} \geq 52\gamma k$. We distinguish two subcases.

   - If $|\mathrm{Par}(W_{\mathrm{o}})| \leq 16\gamma k$ we consider sets $U_1 = T_{\mathrm{o}} \div (W_{\mathrm{o}} \cup \mathrm{Par}(W_{\mathrm{o}}))$ and $U_2 = T_{\mathrm{e}} \div (W_{\mathrm{o}} \cup \mathrm{Par}(W_{\mathrm{o}}))$. The partition (U_1, U_2) is semiindependent with $|U_2| - |U_1| \geq 72\gamma k$, a contradiction with the assumption $\mathrm{disc}(T) < 2\gamma k$.
   - If $|\mathrm{Par}(W_{\mathrm{o}})| > 16\gamma k$ then we choose an arbitrary subset $P' \subseteq \mathrm{Par}(W_{\mathrm{o}})$ with $|P'| = 8\gamma k$ and set $W'_{\mathrm{o}} = \mathrm{N}(P') \cap W_{\mathrm{o}}$. The partition (U_1, U_2) defined by $U_1 = T_{\mathrm{o}} \div (W'_{\mathrm{o}} \cup P')$, $U_1 = T_{\mathrm{e}} \div (W'_{\mathrm{o}} \cup P')$ is an $8\gamma k$-ideal semiindependent partition.

3. If $w_{\mathrm{o}} < 8\gamma k$ we use Fact 3.1 (Part 2) to find a full-subtree $\tilde{T} \subseteq T$ rooted in a vertex r with ℓ leaves, where $\ell \in [20\gamma k, 40\gamma k]$. The choice of $\tilde{T}$ has the property that

$$\min\{|W_{\mathrm{e}} \cap V(\tilde{T})|, |W_{\mathrm{e}} \cap V(T) \setminus V(\tilde{T})|\} \geq 12\gamma k \tag{8.16}$$

   Set $d = |V(\tilde{T}) \cap T_{\mathrm{e}}| - |V(\tilde{T}) \cap T_{\mathrm{o}}|$. We distinguish six subcases.

   (C1) $r \in T_{\mathrm{e}}$ and $d \leq \mathrm{gap}(T)/2$, **(C2)** $r \in T_{\mathrm{o}}$ and $d \geq \mathrm{gap}(T)/2$,
   (C3) $r \in T_{\mathrm{e}}$ and $d \geq \mathrm{gap}(T)/2 + 1$, **(C4)** $r \in T_{\mathrm{o}}$ and $d \leq \mathrm{gap}(T)/2 - 1$,
   (C5) $r \in T_{\mathrm{e}}$ and $d = (\mathrm{gap}(T) + 1)/2$, **(C6)** $r \in T_{\mathrm{o}}$ and $d = (\mathrm{gap}(T) - 1)/2$.

   In cases **(C1)-(C4)** we obtain an $8\gamma k$-ideal semiindependent partition by flipping either $V(\tilde{T})$ (in cases **(C1)** and **(C2)**) or $V(\tilde{T}) \setminus \{r\}$ (in cases **(C3)** and **(C4)**) from the original partition $(T_{\mathrm{o}}, T_{\mathrm{e}})$. Details are omitted.

   In the rest, we consider only the case **(C5)**, case **(C6)** being analogous. We find an $8\gamma k$-ideal semiindependent partition, or embed T in G. First observe that k is even. Consider the partition $V(T) = O_1 \cup O_2$, where $O_1 = T_{\mathrm{o}} \div V(\tilde{T})$ and $O_2 = T_{\mathrm{e}} \div V(\tilde{T})$. It holds $|O_1| = (k+2)/2$, $|O_2| = k/2$, and $\min\{|O_1 \cap W|, |O_2 \cap W|\} \geq 12\gamma k$.

(♣1) Suppose first that $W_o \cap V(T-\tilde{T}) \cap \mathrm{N}(r) \neq \emptyset$. Then take an arbitrary vertex $u \in W_o \cap V(T-\tilde{T}) \cap \mathrm{N}(r)$ and consider the partition (U_1,U_2), $U_1 = O_1 \div \{u\}$, $U_2 = O_2 \div \{u\}$. By (8.16), this is an $8\gamma k$-ideal semiindependent partition. Therefore we restrict ourselves to the case when $W_o \cap V(T-\tilde{T}) \cap \mathrm{N}(r) = \emptyset$.

(♣2) We claim that if there exist two distinct leaves $z_1,z_2 \in O_1$ with a common neighbor $\{x\} = \mathrm{Par}(\{z_1,z_2\})$, then there exists an $8\gamma k$-ideal semiindependent partition (U_1,U_2). By the assumption above we know that $x \in O_2$. Set $U_1 = O_1 \div \{x,z_1,z_2\}$ and $U_2 = O_2 \div \{x,z_1,z_2\}$. Then $|U_1| = |O_1|-1 = k/2$ and $|U_1| = |O_2|+1 = k/2+1$, and $|U_1 \cap W| = |O_1 \cap W|-2$, and $|U_2 \cap W| = |O_2 \cap W|+2$. From (8.16), the partition (U_1,U_2) is $8\gamma k$-ideal semiindependent. Therefore, we may assume that leaves in O_1 have pairwise distinct parents.

(♣3) We claim that there exists a vertex $y \in \mathrm{Par}(O_1) \cap W$ such that $\deg(y) = 2$. Suppose for contradiction that every vertex in $\mathrm{Par}(O_1) \cap W$ has degree at least three. We have already observed that every vertex in $\mathrm{Par}(O_1) \cap W$ has exactly one leaf-child in O_1. Set $W_* = O_1 \cap V(\tilde{T}) \cap W$ and $T_* = T[V(\tilde{T}) \setminus W_*]$. Observe that the leaves of T_* lying in O_2 coincide with the leaves of $\tilde{T}$ lying in O_2. We show that T_* contains at least $8\gamma k$ leaves from T_o, contradicting the assumption $w_o < 8\gamma k$. By Fact 3.2 it is enough to show that $|V(T_*) \cap T_o| \geq |V(T_*) \cap T_e| + 8\gamma k$.

$$\begin{aligned}
|V(T_*) \cap T_o| &= |V(T_*) \cap O_2| = |V(\tilde{T}) \cap T_o| \\
&\overset{(*)}{\geq} |V(\tilde{T}) \cap T_e| - 2\gamma k - 2 \\
&= |V(T_*) \cap T_e| + |W_*| - 2\gamma k - 2 \\
&\geq |V(T_*) \cap T_e| + 8\gamma k,
\end{aligned}$$

where $(*)$ follows from Lemma 3.3. Let $z \in O_1 \cap W$ be a leaf of T with parent y, $\deg(y) = 2$. We show that $T \subseteq G$ in two cases **(◇1)** and **(◇2)** separately, based on whether G is in the Abundant or Deficient configuration.

(◇1) If G admits an Abundant partition, then there exists an index $i \in [\lambda]$ such that $|L^i| \geq (k+1)/2$. As k is even, $|L^i| \geq (k+2)/2$. Choose $L_* \subseteq L^i$ such that $|L_*| = (k+2)/2$. Define $W^* = \{u \in W \cap O_1 \ :\ \mathrm{Par}(u) \in O_2\}$, and let $W' \subseteq W^*$ be the set of leaves in W^* with no brother/sister in W^*. We claim that

$$|(W \cap O_1) \setminus W^*| \leq \gamma k, \text{ and } |W^* \setminus W'| \leq \gamma k. \tag{8.17}$$

Assuming (8.17), we can use Lemma 8.10 with $A = L_*$, $B = S^i_\diamond \cup (L^i \setminus L_*)$, $f = \gamma k$, and the partition (O_1,O_2) of the tree T to get $T \subseteq G$.

It remains to prove (8.17). If $|(W \cap O_1) \setminus W^*| > \gamma k$, then consider the partition (U_1,U_2) with $U_1 = O_1 \setminus ((W \cap O_1) \setminus W^*)$ and $U_2 = O_2 \cup (W \cap O_1) \setminus W^*$. If $|W^* \setminus W'| > \gamma k$, then consider the partition (U_1,U_2) obtained from (O_1,O_2) by flipping $(W^*\setminus W')\cup \mathrm{Par}(W^*\setminus W')$. In both cases $|U_2|-|U_1|>2\gamma k$, a contradiction to our assumption that $\mathrm{disc}(T)\le 2\gamma k$.

($\diamondsuit$2) If G is in a Deficient configuration, then by Lemma 8.6 there exists an index $i\in[\vartheta]$ and a vertex $v\in L^i$ such that $\deg(v,L^i)+\deg(v,\bigcup_{j\neq i}(L^j\cup S^j))\ge k/2$, where $S^j=\{u\in S: \deg(u,L^j)\ge k/(3\vartheta)\}$. Set $\psi_1=\deg(v,L^i)$ and $\psi_2=\deg(v,\bigcup_{j\neq i}(L^j\cup S^j))$. All components of $T-\{r\}$ have size at most $k/(6\vartheta)$, or by Lemma 8.9 the tree T embeds in G (the components cannot be larger than $k-18\gamma k$ by the choice of r). Denote by $\mathscr{K}$ the set of components of $T-\{r\}$ of order at least 2. Since O_2 is an independent set, any component from $\mathscr{K}$ has non-empty intersection with O_1. Choose $\mathscr{K}_2\subseteq\mathscr{K}$ with a maximum number of vertices in O_1 satisfying the following.

- $|\mathscr{K}_2|\le\psi_2$.
- $\sum_{K\in\mathscr{K}_2}v(K)\le k/(3\vartheta)$.

Set $\mathscr{K}_1=\mathscr{K}\setminus\mathscr{K}_2$. Map r to v and embed the components of $\mathscr{K}_2$ greedily in $\bigcup_{j\neq i}(L^j\cup S^j)$ in such a way that the roots of the components are mapped to neighbors of v.

If $|V(\mathscr{K}_1)|\le k-6\gamma k-1$, then from Lemma 3.3 we deduce that $\max\{|T_{\mathrm{o}}\cap V(\mathscr{K}_1)|,|T_{\mathrm{e}}\cap V(\mathscr{K}_1)|\}\le k/2-2\gamma k$ and thus the components of $\mathscr{K}_1$ can be embedded in $L^i\cup S^i_\diamond$ greedily.

Hence, we suppose that $|V(\mathscr{K}_1)|>k-6\gamma k-1$. The maximality of $\mathscr{K}_2$ implies that $|\mathscr{K}_2|=\psi_2$. Set $U_1=O_1\cap V(\mathscr{K}_1)$ and $U_2=O_2\cap V(\mathscr{K}_1)$. Observe that U_2 is independent. We show that $|U_1|\le\psi_1$. If $r\in O_1$, then

$$|U_1|\le|O_1|-|\mathscr{K}_2|-|\{r\}|=\frac{k+2}{2}-\psi_2-1\le\psi_1\ .$$

It remains to analyze the case $r\in O_2$. Let $K\in\mathscr{K}$ be the component containing the vertex z. Then, by the choice of $\mathscr{K}_2$, there exists a component $K'\in\mathscr{K}_\in$ such that $|O_1\cap V(K')|\ge 2$. Again we conclude $|U_1|\le|O_1|-(|\mathscr{K}_2|+1)\le\psi_1$.

Observe that $\min\{|U_1\cap W|,|U_2\cap W|\}\ge 9\gamma k-6\gamma k-1>2\gamma k$, and by previous assumptions, any two leaves in U_1 have distinct parents that are in U_2 (the only leaves in O_1 with parents in O_1 are children of r and thus are not contained in $\mathscr{K}$).

We embed the trees from $\mathscr{K}_1$ in $L^i\cup S^i_\diamond$. We distinguish two cases.

- *$r\in T_{\mathrm{e}}$ or $r\in T_{\mathrm{o}}$ and $|\mathrm{N}(r)\cap U_2|\le(1/2-2\gamma)k$.*
  We apply Lemma 8.10 with $A=L^i\cap\mathrm{N}(v)$, $B=S^i_\diamond\cap\mathrm{N}(v)$, the partition of the forest $V(\mathscr{K}_1)$ being (U_1,U_2), and $P=\mathrm{Par}(U_1)$ (recall that leaves in U_1 have pairwise distinct parents in U_2).
- *$r\in T_{\mathrm{o}}$ and $|\mathrm{N}(r)\cap U_2|>(1/2-2\gamma)k$.*
  Set $\tilde{\mathscr{K}_1}=\{K\in\mathscr{K}_1 : v(K)=2,\mathrm{N}(r)\cap V(K)\subseteq U_2\}$. Then $v(\mathscr{K}\setminus\tilde{\mathscr{K}_1})\le 2\gamma k$. Con-

sider the partition $(\tilde{U}_1, \tilde{U}_2)$ obtained from (U_1, U_2) by flipping $\tilde{\mathscr{K}_1}$. Then $|\tilde{U}_1| \leq \psi_1$. Construct an embedding ϕ of the forest induced by $\mathscr{K}_1 \setminus \tilde{\mathscr{K}_1}$ such that $\phi(V(\mathscr{K}_1 \setminus \tilde{\mathscr{K}_1}) \cap \tilde{U}_1) \subseteq L^i$, $\phi(V(\mathscr{K}_1 \setminus \tilde{\mathscr{K}_1}) \cap \tilde{U}_2) \subseteq S^i_\diamond$ and $\phi(V(\mathscr{K}_1 \setminus \tilde{\mathscr{K}_1}) \cap \mathrm{N}(r)) \subseteq \mathrm{N}(v)$.

The embedding of $\{r\} \cup V(\mathscr{K})$ can be extended to the whole tree T, as r is mapped to L.

Second step. We assume that T has an $8\gamma k$-ideal semi-independent partition (U_1, U_2). The proof goes very similarly as in **($\diamondsuit$1)**, for the Abundant case, and as in **($\diamondsuit$2)** for the Deficient case. Details are omitted.

9 Lower bound

The condition on the hosting graph G of order n in the LKS Conjecture is parameterized by two numbers: first parameter defines when a a vertex is counted as "large", the second parameter is a requirement on the number of large vertices. As was observed in the Section Introduction, the first parameter cannot be lower than k, since otherwise it might happen that $K_{1,k} \not\subseteq G$. In this section lower bound on the second parameter is given. We recall that due to Theorem 1.5 the lower bound cannot in principle meet the value $n/2$, at least in some cases. There are also examples of small k's, where exact threshold on the number of large vertices required can be determined. The threshold in these examples is substantially smaller[2] than $n/2$.

The constructions given here generalize those of Zhao [22] and of Piguet and Stein [17].

For $a > b$, a *complete split-graph* $U_{a,b}$ is a graph constructed from K_a by removing all edges which are subset of a fixed $(a-b)$-element set of vertices. Equivalently, $U_{a,b}$ is a graph constructed from $K_{a-b,b}$ by adding all possible edges into the color-class of order b. The vertex set of $U_{a,b}$ decomposes naturally into the *clique part* and the *independent part*. A *double-star with m rays* S_m is a graph of order $2m+1$ which is constructed by attaching a distinct vertex to each leaf of $K_{1,m}$.

Suppose that k is even and for $j = 0, 1, \ldots, \lfloor\sqrt{k}\rfloor$ write $n = \ell_j(k+1-j) + a_j$, where ℓ_j and a_j are the quotient and the remainder of n after division by $(k+1-j)$, respectively. Let G_0 be a graph formed by ℓ_0 disjoint copies of $U_{k+1,k/2-1}$ and a_0 isolated vertices. For $j > 0$ define

$$h_j = \frac{k}{2} - j - \left\lceil \frac{1 - j + \sqrt{j^2 + 2j(2k-1) + 1}}{2} \right\rceil .$$

We construct G_j starting with ℓ_j disjoint copies of U_{k+1-j,h_j} and a_j isolated vertices. We label the

[2]For $k=1$ and $k=2$, one large vertex guarantees $\mathscr{T}_{k+1} \subseteq G$. For $k=3$ the tight requirement on the number of large vertices to guarantee $\mathscr{T}_{k+1} \subseteq G$ is $\lfloor n/3 \rfloor + 1$, for $k=4$ it is $\lfloor n/4 \rfloor + 1$.

copies of U_{k+1-j,h_j} by $U_1,\ldots,U_{\ell_j}$. In each U_i we fix a vertex set A_i of size

$$y = \left\lceil \frac{1-j+\sqrt{j^2+2j(2k-1)+1}}{2} \right\rceil$$

in its independent part. Connect each vertex of the clique part of U_i with exactly j vertices in A_{i-1} ($A_0 = A_{\ell_j}$, for convention) in such a way that no vertex in A_{i-1} receives more than $y+j-1$ edges from U_i.

The vertices in G_j's which have degree at least k are exactly those which are contained in a clique part of some complete split-graph. The number of large vertices in G_j is

$$\ell_j\left(\frac{k}{2}-j-\left\lceil \frac{1-j+\sqrt{j^2+2j(2k-1)+1}}{2}\right\rceil\right) = \begin{cases} \frac{\ell_0(k-2)}{2} & \text{, for } j=0, \text{ and} \\ \frac{n}{2}-\frac{a_j}{2}-\frac{\ell_j}{2}\left(1+j+2\left\lceil \frac{1-j+\sqrt{j^2+2j(2k-1)+1}}{2}\right\rceil\right) & \text{, otherwise.} \end{cases} \tag{9.1}$$

It is easy to observe that the path of length k is not a subgraph of G_0 and that the double-star $S_{k/2}$ is not a subgraph of G_j for $j>0$. Therefore $\mathscr{T}_{k+1} \not\subseteq G_j$ for any j. There is not an easy formula to determine which of the numbers in (9.1) is the largest. Note that maximum of the numbers in (9.1) may be fairly "discontinuous" as a function of k and n. This is not surprising, even if the bounds given here would turn out to be tight, as it has been known that divisibility plays an important role in similar problems[3].

When k is odd we construct the graph G_0 as ℓ_0 disjoint copies of $U_{k+1,(k-1)/2}$. For the graphs G_j (with $j>0$) the best construction we are aware of is to construct graphs very similar to those as when k was even and then to show that $S_{(k-1)/2} \not\subseteq G_j$.

We believe that the lower bounds presented here might be close to the truth. We put the question of determining the exact value of the number of large vertices needed as an open problem.

Problem 9.1. *Given $n,k \in \mathbb{N}$ determine the number ℓ such that any graph of order n which has at least ℓ vertices of degree at least k contains all trees of order $k+1$.*

More generally, one can ask which degree sequences of the hosting graph ensure that all trees of order $k+1$ will be contained. For $n \in \mathbb{N}$ let $\mathscr{A}_n$ be a family of n-tuples $D=(d_1,\ldots,d_n)$ such that there exists a graph G of order n with D as a degree sequence.

Problem 9.2. *Given $n,k \in \mathbb{N}$ determine all n-tuples $D=(d_1,\ldots,d_n)$, $D \in \mathscr{A}_n$ such that any graph G with degree sequence D contains all trees of order $k+1$.*

Problem 9.2 seems beyond our reach, and even a partial characterization would definitely require techniques most different from those presented in this thesis. Interestingly, the following

[3] A random example of this phenomenon is [11].

example shows that the family $\mathscr{D}$ of all degree sequences $D \in \mathscr{A}_n$ is not increasing in the coordinatewise ordering $\preceq$ on $\mathscr{A}_n \subseteq \mathbb{N}^n$. Define two degree sequences $D_0, D_1 \in \mathscr{A}_{16}$ by

$$D_0 = (0,0,\ldots,0,3,3,3,3)\,,$$
$$D_1 = (1,1,\ldots,1,3,3,3,3)\,,$$

where there are 12 zeros and ones in D_0 and D_1, respectively. We have $D_0 \prec D_1$. The only graph G_0 with the degree sequence D_0 is K_4 with 12 isolated vertices added. Obviously, $\mathscr{T}_4 \subseteq G_0$. Let G_1 be a disjoint union of four copies od $K_{1,3}$. G_1 has degree sequence D_1 and G_1 does not contain a path of length 3.

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