Limits of dense graph sequences

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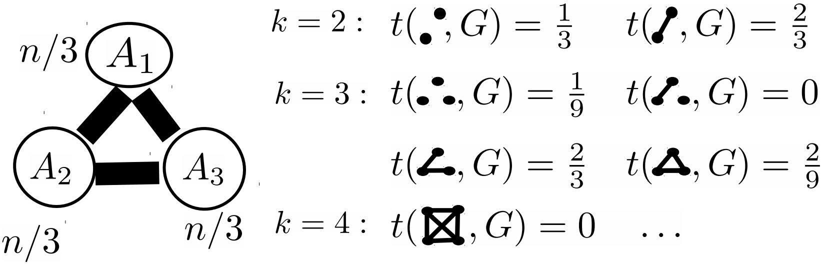
idea: convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
... *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0,1]$

Why? same story as with $\mathbb{Q}$ vs $\mathbb{R}$: only the latter allows
reasonable e.g. variational and integral calculus
for example $argmin(x^3 - 2x)$

Limits of dense graph sequences: an abstract approach

F is a "fixed graph" of order k, G is "large" of order n
We define **subgraph density** $t(F, G)$:

$$t(F,G) := \frac{\#\text{ copies of } F \text{ in } G}{\binom{n}{k}} = \mathbf{P}\big[G[\text{random } k\text{-set}] \cong F\big]$$



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A sequence of graphs $G_1, G_2, \ldots$ **converges** if for each F, the sequence $t(F, G_1), t(F, G_2), \ldots$ converges.
We get a **limit object** Ψ, $t(F, \Psi) = \lim_n t(F, G_n)$.

Why dense graph sequences?

If the proportion of edges $\searrow 0$ ($\lim \frac{e(G_n)}{n^2} = 0$) we get a trivial limit.
That is, the theory is void for trees, planar graphs, ...

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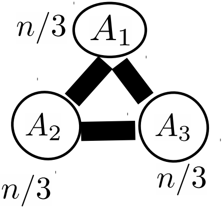
Razborov'07 flag algebras (next slide)

Extremal graph theory and Razborov's flag algebras I



$k = 2:$ $t(\overline{K_2}, G) = \frac{1}{3}$ $t(K_2, G) = \frac{2}{3}$

$k = 3:$ $t(\overline{K_3}, G) = \frac{1}{9}$ $t(K_2 \cup K_1, G) = 0$

$t(P_3, G) = \frac{2}{3}$ $t(K_3, G) = \frac{2}{9}$

$k = 4:$ $t(K_4, G) = 0$ $\dots$

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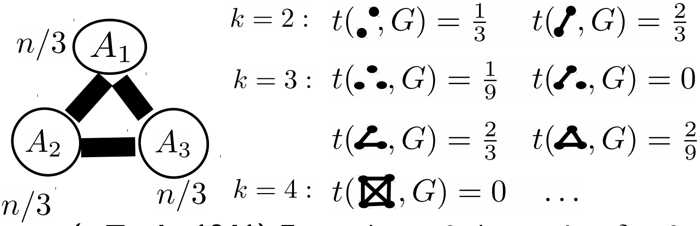

Theorem (≈Turán 1941) For each $\epsilon > 0$ there exists $\delta > 0$: If an n-vertex graph has more than $(\frac{2}{3} + \epsilon)\binom{n}{2}$ edges then it contains $> \delta$-proportion of $\boxtimes$'s.

ExGrTh studies relations between $t(F_1, G)$, $t(F_2, G)$, $\ldots$.

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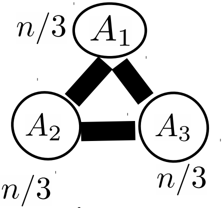

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$\qquad\quad t(P_3,G)=\frac{2}{3}\qquad t(K_3,G)=\frac{2}{9}$

$k=4:\quad t(\boxtimes,G)=0\quad\ldots$

Theorem (≈Turán 1941) For each $\epsilon>0$ there exists $\delta>0$: If an n-vertex graph has more than $(\frac{2}{3}+\epsilon)\binom{n}{2}$ edges then it contains $>\delta$-proportion of $\boxtimes$'s.

ExGrTh studies relations between $t(F_1,G)$, $t(F_2,G)$, ...

Razborov: But lets rather study these relations on the limit space!
Approach to proving Turán: Suppose the theorem is false. $G_1,G_2,\ldots$ all contain $(\frac{2}{3}+\epsilon)$-proportion of edges but proportion of $\boxtimes$'s tends to 0. Pass to a subsequential limit Ψ. $t(|,\Psi)\geq\frac{2}{3}+\epsilon$ and $t(\boxtimes,\Psi)=0$. Derive a contradiction.

Extremal graph theory and Razborov's flag algebras II

Razborov provides tools for deriving relations that hold on the limit object (Cauchy-Schwarz calculus, variational calculus, . . .)

First application: Razborov'08 (AMS Robbins Prize'12) solves the triangle density problem of Lovász and Simonovits 1983:
Suppose a graph has a given proportion of edges. What proportion of triangles can it have?

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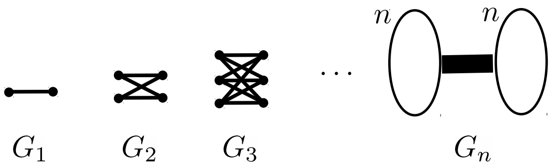
- H.-Král'-Norine'09: Caccetta-Häggkvist conj. (progress)
- H.-Hatami-Král'-Norine-Razborov'11 conjecture of Erdős 1984
- HHKNR'11 conjecture of Jagger-Štovíček-Thomason 1996
- . . . and many more

Hatami-Norine *J.AMS'11* deciding whether an inequality between subgraph densities holds for all graph limits is undecidable

Graphons I



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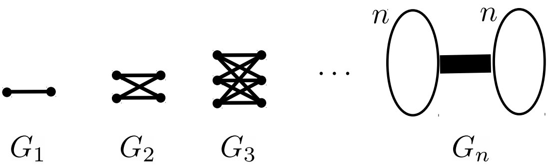
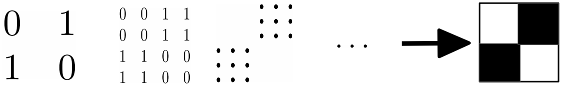


Represent these graphs by their adjacency matrices:



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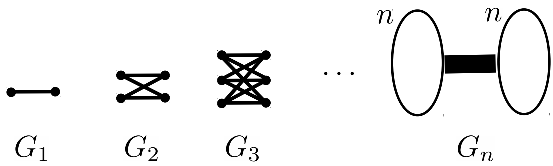

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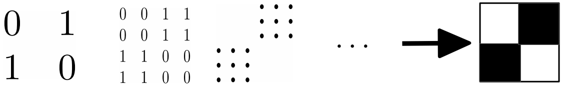
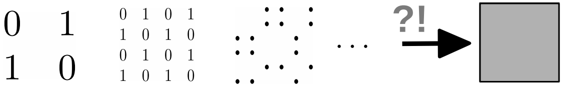


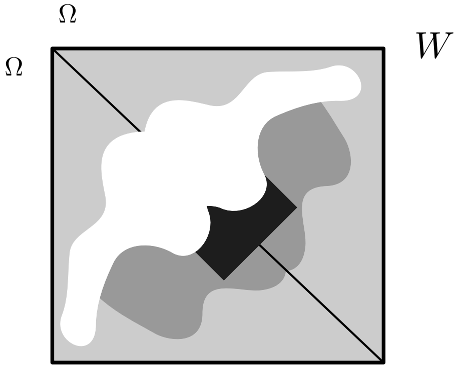
. . . works if you do things the right way. But, . . .



In general Szemerédi's Regularity Lemma can be used to determine "the right way" of ordering the vertices.

Graphons II

A **graphon** is a symmetric Lebesgue-m. function. $W : \Omega^2 \to [0,1]$.
Theorem (Lovász–Szegedy) sampling conv.$\Leftrightarrow$graphical conv.
Theorem (L–Sz.) Every graphon W can be achieved in the limit.
Proof:



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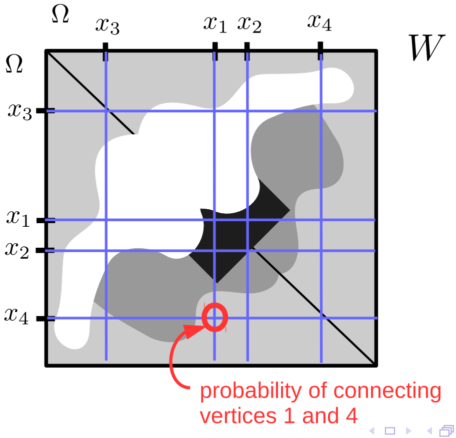

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It can be shown that almost surely, $G_1, G_2, \ldots \to W$. $\square$

$\mathbb{G}(n, W)$ as a generalization of the Erdős–Rényi model $\mathbb{G}(n, p)$.

Interesting model *per se*!
Bollobás–Janson–Riordan'07, Doležal–H.–Máthé'15