Uniform spanning tree and limits of dense graphs

Jan Hladký (TU Dresden)
Asaf Nachmias, Wojciech Samotij (Tel Aviv University)
Tuan Tran (ICS Czech Academy of Sciences)

Limits of dense graph sequences

Lovász, Szegedy *JCTB'06* (Fulkerson Prize'12)
Borgs, Chayes, Lovász, Sós, Vesztergombi *Adv.Math.'06*
Borgs, Chayes, Lovász, Sós, Vesztergombi *Ann.Math.'12*

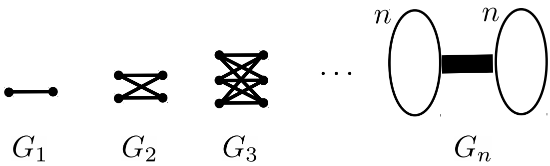
Limits of dense graph sequences

Lovász, Szegedy *JCTB'06* (Fulkerson Prize'12)
Borgs, Chayes, Lovász, Sós, Vesztergombi *Adv.Math.'06*
Borgs, Chayes, Lovász, Sós, Vesztergombi *Ann.Math.'12*

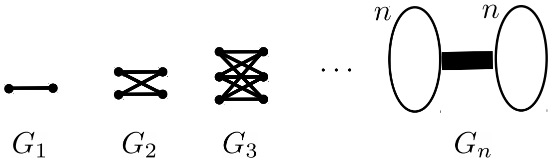
idea: convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
... *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0,1]$

Why? same story as with $\mathbb{Q}$ vs $\mathbb{R}$: only the latter allows
reasonable e.g. variational and integral calculus
for example $argmin(x^3 - 2x)$

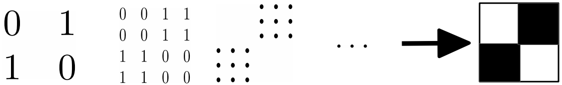
Graphons



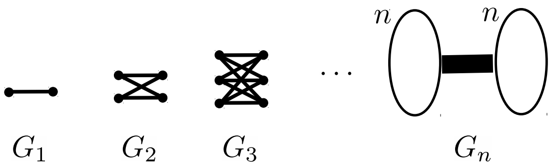
Graphons



Represent these graphs by their adjacency matrices:

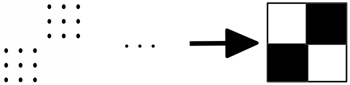


Graphons



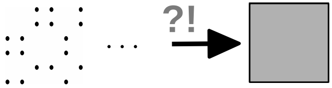
Represent these graphs by their adjacency matrices:

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$



. . . works if you do things the right way. But, . . .

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}$$



In general Szemerédi's Regularity Lemma can be used to determine "the right way" of ordering the vertices.

Parameters in graphs and graphons (examples)

G ...graph on n vertices
A ...adjacency matrix of G
What is number of triangles in G?

Parameters in graphs and graphons (examples)

G ... graph on n vertices
A ... adjacency matrix of G
What is number of triangles in G?

$$= \frac{1}{6} \sum_i \sum_j \sum_k A_{i,j} A_{j,k} A_{k,i}$$

Parameters in graphs and graphons (examples)

G ... graph on n vertices
A ... adjacency matrix of G
What is number of triangles in G?

$$= \frac{1}{6} \sum_i \sum_j \sum_k A_{i,j} A_{j,k} A_{k,i}$$

if $G \approx W$ then

$$\frac{1}{6} \sum_i \sum_j \sum_k A_{i,j} A_{j,k} A_{k,i} \approx \frac{n^3}{6} \cdot \int_x \int_y \int_z W(x,y) W(y,z) W(z,x)$$

Parameters in graphs and graphons (examples)

G ... graph on n vertices
A ... adjacency matrix of G
MAX-CUT in G?

Parameters in graphs and graphons (examples)

G ... graph on n vertices
A ... adjacency matrix of G
MAX-CUT in G?

$$= \max_{V(G)=X \sqcup Y} \sum_{x \in X} \sum_{y \in Y} A(x, y)$$

Parameters in graphs and graphons (examples)

G ...graph on n vertices
A ...adjacency matrix of G
MAX-CUT in G?

$$= \max_{V(G)=X\sqcup Y} \sum_{x\in X} \sum_{y\in Y} A(x,y)$$

if $G \approx W$ then

$$MAXCUT(G) \approx n^2 \cdot \sup_{\Omega=X\sqcup Y} \int_{x\in X} \int_{y\in Y} W(x,y)$$

If G is connected then it has at least one **spanning tree**, $\mathcal{T}(G) \neq \emptyset$
Let T be a UST of G (T is a random tree!)

If G is connected then it has at least one **spanning tree**, $\mathcal{T}(G) \neq \emptyset$
Let T be a UST of G (T is a random tree!)

Case study: UST in the complete graph K_n
$|\mathcal{T}(K_n)| = n^{n-2}$ (Cayley's formula)

If G is connected then it has at least one **spanning tree**, $\mathcal{T}(G) \neq \emptyset$
Let T be a UST of G (T is a random tree!)

Case study: UST in the complete graph K_n
$|\mathcal{T}(K_n)| = n^{n-2}$ (Cayley's formula)

Quiz question: $T \sim UST(K_n)$. How many leaves does T have?
(a) $\log_2 n$
(b) $\sqrt{\frac{2}{\pi}} \cdot \sqrt{n}$
(c) $e^{-1} n$
(d) $\frac{n}{2}$

If G is connected then it has at least one **spanning tree**, $\mathcal{T}(G) \neq \emptyset$
Let T be a UST of G (T is a random tree!)

Case study: UST in the complete graph K_n
$|\mathcal{T}(K_n)| = n^{n-2}$ (Cayley's formula)

Quiz question: $T \sim UST(K_n)$. How many leaves does T have?
(a) $\log_2 n$
(b) $\sqrt{\frac{2}{\pi}} \cdot \sqrt{n}$
(c) $e^{-1} n$
(d) $\frac{n}{2}$

And what about vertices of degree $2, 3, \ldots$?
Frequencies of larger balls? (linear scaling! / local statistics)

And what about if $G \neq K_n$? Continuity with respect to the cut-distance?

Main Theorem

If G is connected then it has at least one **spanning tree**, $\mathcal{T}(G) \neq \emptyset$
Let T be a UST of G (T is a random tree!)

Case study: UST in the complete graph K_n
$|\mathcal{T}(K_n)| = n^{n-2}$ (Cayley's formula)

Quiz question: $T \sim UST(K_n)$. How many leaves does T have?
(a) $\log_2 n$
(b) $\sqrt{\frac{2}{\pi}} \cdot \sqrt{n}$
(c) $e^{-1} n$
(d) $\frac{n}{2}$

And what about vertices of degree $2, 3, \ldots$?
Frequencies of larger balls? (linear scaling! / local statistics)

And what about if $G \neq K_n$? Continuity with respect to the cut-distance?

Main Theorem Suppose that G is a connected graph, $G \approx W$, where W is a nondegenerate graphon. Then we know stuff.

Main Theorem $G \approx W$, local structure of UST.

In this sketch of proof: number of leaves of $UST(G)$.

Given $v \in V(G)$, what is the probability that v is a leaf in UST?
(depends on v)

Main Theorem $G \approx W$, local structure of UST.

In this sketch of proof: number of leaves of $UST(G)$.

Given $v \in V(G)$, what is the probability that v is a leaf in UST?
(depends on v)

Wilson's algorithm STOC'96, loop-erased random walk (LERW)
LERW→Schramm–Loewner evolution, conformal invariance in 2D
2006 Fields medal to Wendelin Werner

Main Theorem $G \approx W$, local structure of UST.

In this sketch of proof: number of leaves of $UST(G)$.

Given $v \in V(G)$, what is the probability that v is a leaf in UST?
(depends on v)

Wilson's algorithm STOC'96, loop-erased random walk (LERW)
LERW→Schramm–Loewner evolution, conformal invariance in 2D
2006 Fields medal to Wendelin Werner

$$\mathbb{P}_T\left[v \text{ leaf in } T\right] \approx \mathbb{P}\left[0 = \sum_{u\sim v} Bernoulli\left(\frac{1}{\deg_G(u)}\right)\right]$$
$$\approx \mathbb{P}\left[0 = Poisson\left(\sum_u \frac{A_{u,v}}{\deg_G(u)}\right)\right] = \exp\left(-\sum_u \frac{A_{u,v}}{\deg_G(u)}\right)$$

Main Theorem $G \approx W$, local structure of UST.

In this sketch of proof: number of leaves of $UST(G)$.

Given $v \in V(G)$, what is the probability that v is a leaf in UST?
(depends on v)

Wilson's algorithm STOC'96, loop-erased random walk (LERW)
LERW→Schramm–Loewner evolution, conformal invariance in 2D
2006 Fields medal to Wendelin Werner

$$\mathbb{P}_T\,[v \text{ leaf in } T] \approx \mathbb{P}\left[0 = \textstyle\sum_{u\sim v} Bernoulli\left(\frac{1}{\deg_G(u)}\right)\right]$$

$$\approx \mathbb{P}\left[0 = Poisson\left(\textstyle\sum_u \frac{A_{u,v}}{\deg_G(u)}\right)\right] = \exp\left(-\textstyle\sum_u \frac{A_{u,v}}{\deg_G(u)}\right)$$

$$\text{ratio of leaves in UST(G)} \approx \frac{1}{n}\sum_v \exp\left(-\textstyle\sum_u \frac{A_{u,v}}{\deg_G(u)}\right)$$

$$\approx \int_x \exp\left(-\int_y \frac{W(x,y)}{\deg_W(y)}\right)$$