# Limits of graph sequences; dense and sparse

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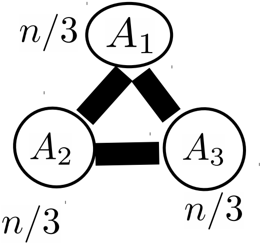
**idea:** convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
… *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0, 1]$

**Why?** same story as with $\mathbb{Q}$ vs $\mathbb{R}$: only the latter allows
reasonable e.g. variational and integral calculus
for example $argmin(x^3 - 2x)$

# Limits of dense graph sequences: an abstract approach

$F$ is a "fixed graph" of order $k$, $G$ is "large" of order $n$
We define **subgraph density** $t(F,G)$:

$$t(F,G) := \frac{\#\text{ copies of } F \text{ in } G}{\binom{n}{k}} = \mathbf{P}\big[G[\text{random } k\text{-set}] \cong F\big]$$

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$k = 4:$ $t($$, G) = 0$ $\ldots$

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A sequence of graphs $G_1, G_2, \ldots$ **converges** if for each $F$, the sequence $t(F, G_1), t(F, G_2), \ldots$ converges.
We get a **limit object** $\Psi$, $t(F, \Psi) = \lim_n t(F, G_n)$.

## Why dense graph sequences?

If the proportion of edges $\searrow 0$ ($\lim \frac{e(G_n)}{n^2} = 0$) we get a trivial limit.
That is, the theory is void for trees, planar graphs, ...

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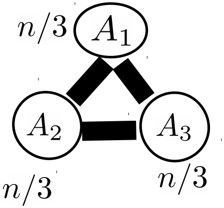
Razborov'07 flag algebras (next slide)

# Extremal graph theory and Razborov's flag algebras I

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$\qquad\quad t(P_3,G)=\frac{2}{3}\qquad t(K_3,G)=\frac{2}{9}$

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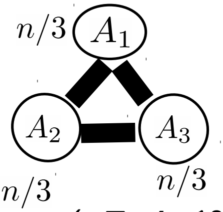


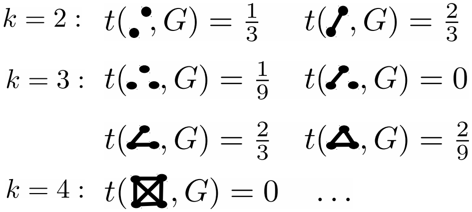


**Theorem (≈Turán 1941)** For each $\epsilon > 0$ there exists $\delta > 0$: If an $n$-vertex graph has more than $(\frac{2}{3} + \epsilon)\binom{n}{2}$ edges then it contains $> \delta$-proportion of ⊠'s.

ExGrTh studies relations between $t(F_1, G)$, $t(F_2, G)$, . . . .

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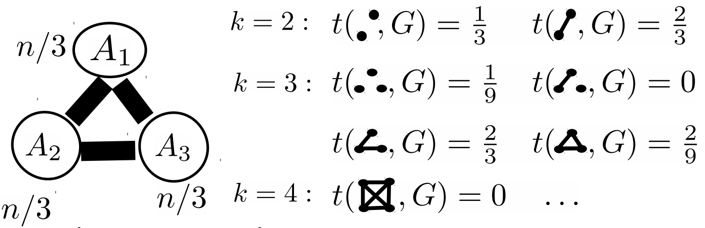


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Razborov: But lets rather study these relations on the limit space!
**Approach to proving Turán:** Suppose the theorem is false. $G_1, G_2, \ldots$ all contain $(\frac{2}{3} + \epsilon)$-proportion of edges but proportion of $\boxtimes$'s tends to 0. Pass to a subsequential limit $\Psi$. $t(|, \Psi) \geq \frac{2}{3} + \epsilon$ and $t(\boxtimes, \Psi) = 0$. Derive a contradiction.

# Extremal graph theory and Razborov's flag algebras II

Razborov provides tools for deriving relations that hold on the limit object (Cauchy-Schwarz calculus, variational calculus, . . . )

First application: Razborov'08 (AMS Robbins Prize'12) solves the triangle density problem of Lovász and Simonovits 1983:
*Suppose a graph has a given proportion of edges. What proportion of triangles can it have?*

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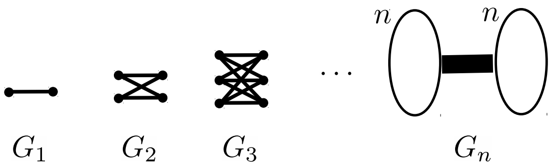
- H.-Král'-Norine'09: Caccetta-Häggkvist conj. (progress)
- H.-Hatami-Král'-Norine-Razborov'11 conjecture of Erdős 1984
- HHKNR'11 conjecture of Jagger-Štovíček-Thomason 1996
- ... and many more

**Hatami-Norine** *J.AMS'11* deciding whether an inequality between subgraph densities holds for all graph limits is undecidable

# Graphons I



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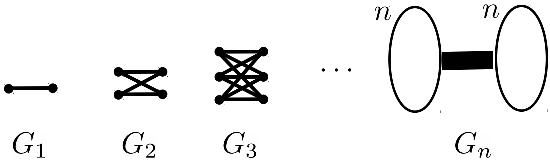
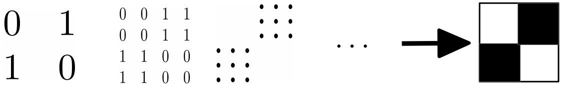



Represent these graphs by their adjacency matrices:



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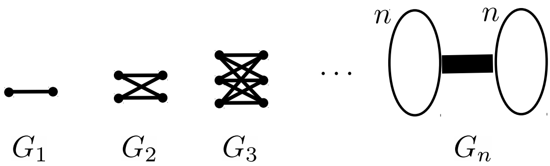


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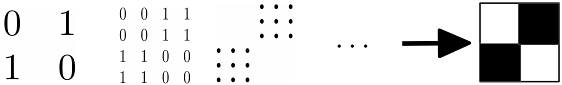
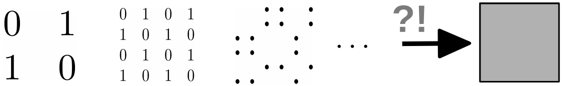



. . . works if you do things the right way. But, . . .




In general Szemerédi's Regularity Lemma can be used to determine "the right way" of ordering the vertices.

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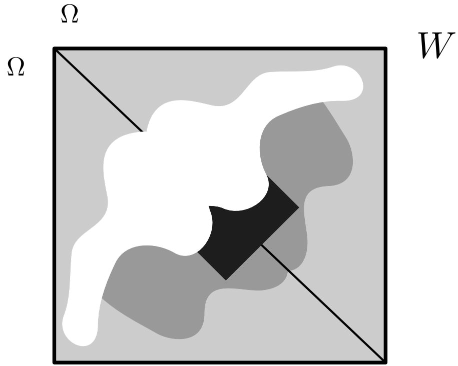
A **graphon** is a symmetric Lebesgue-m. function. $W : \Omega^2 \to [0,1]$.
**Theorem (Lovász–Szegedy)** sampling conv.$\Leftrightarrow$graphical conv.
**Theorem (L–Sz.)** Every graphon $W$ can be achieved in the limit.
**Proof:**



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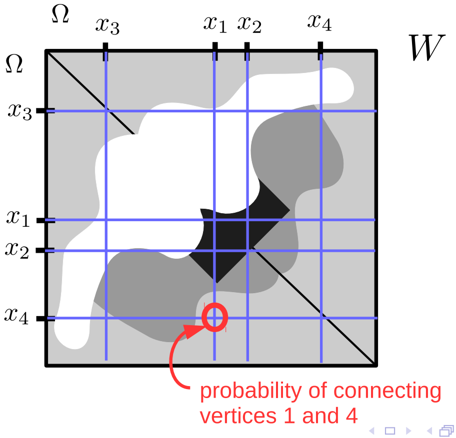

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$\mathbb{G}(n, W)$ as a generalization of the Erdős–Rényi model $\mathbb{G}(n, p)$.

Interesting model *per se*!
Bollobás–Janson–Riordan'07, H.–Mathé'15? ...

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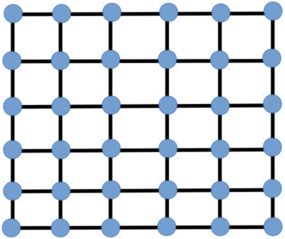

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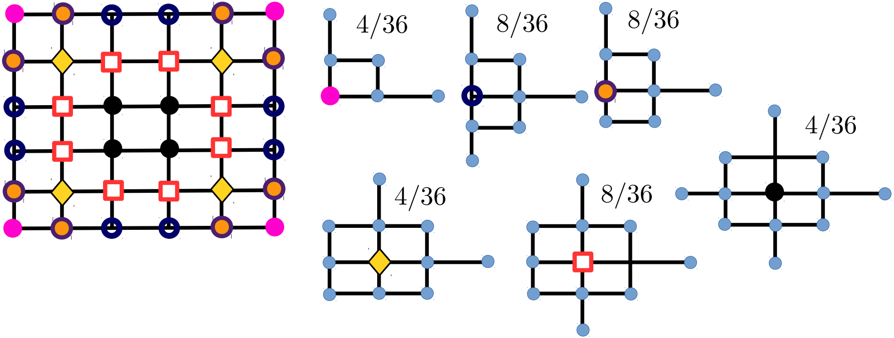

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**A soficity detour:** Notion of sofic groups (Gromov 1999). Is every group sofic?

**Conjecture (Kaplansky 1969):** For any group $G$ and commutative field $K$, the group algebra $K(G)$ is directly finite. That is $ab =_K 1$ implies $ba =_K 1$.

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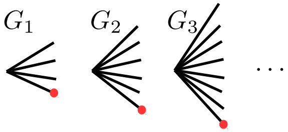


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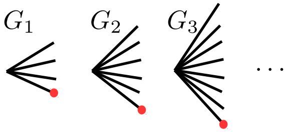


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A sequence of probability measures $\mu_1, \mu_2, \ldots$ on $\mathcal{X}$ is **tight** if for every $\epsilon > 0$ there exists a **finite** $K \subset \mathcal{X}$ such that $\mu_n(K) \geq 1 - \epsilon$ for all $n$.

**Lyons'07:** The concept of Benjamini–Schramm limit can be extended to sequences $G_1, G_2, \ldots$ where for each $r \in \mathbb{N}$, the sequence $\rho_r(G_1), \rho_r(G_2), \ldots$ is tight. AND NOT FURTHER

# Ongoing work with Lukasz Grabowski & Oleg Pikhurko

**Theorem (Hatami–Lovász–Szegedy'13)**
For every Benjamini–Schramm convergent sequence of graphs of degree $\leq D$ there is a graphing that is its local-global limit.

**Theorem (Elek'10)**
The Aldous–Lyons conjecture holds for measures supported on bounded-degree trees.

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...and hopefully we will be able to transfer more ...

Main benefit: The Erdős–Rényi random graph $\mathbb{G}(n, \frac{c}{n})$ is now within the theory.