Jan Hladký: Graphons as weak* limits

(1) “entropy minimization” with Doležal (arXiv:1705.09160)
(2) “Vietoris topology” with Doležal, Grebík, Rocha, Rozhoň (arXiv:1806.07368)
(3) “parameter minimization” with Doležal, Grebík, Rocha, Rozhoň (soon)
(4) hypergraphs with Garbe, Noel, Piguet, Rocha, Saumell

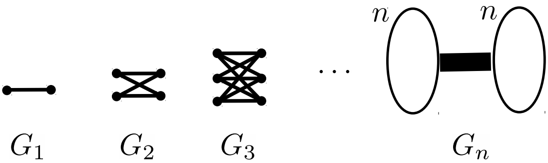
Limits of dense graph sequences

Lovász, Szegedy *JCTB'06* (Fulkerson Prize'12)
Borgs, Chayes, Lovász, Sós, Vesztergombi *Adv.Math.'06*
Borgs, Chayes, Lovász, Sós, Vesztergombi *Ann.Math.'12*

idea: convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
... *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0,1]$
Ω=separable atomless probability space $\cong [0,1]$

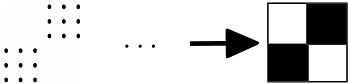
Why? same story as with $\mathbb{Q}$ vs $\mathbb{R}$: only the latter allows
reasonable e.g. variational and integral calculus
for example $argmin(x^3 - 2x)$

Graphons

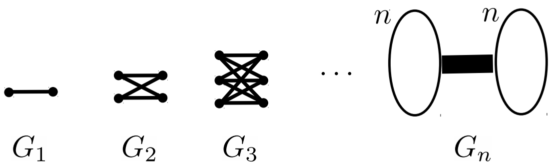


Represent these graphs by their adjacency matrices:

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix} \qquad \cdots$$

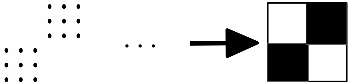


Graphons



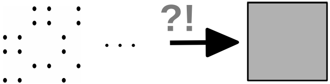
Represent these graphs by their adjacency matrices:

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$



. . . works if you do things the right way. But, . . .

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}$$



The cut-distance topology

Step 1: "Comparing the number of edges inside any vertex set"

$$d_\square(U, W) = \sup_{S \subset \Omega} \left| \int_S \int_S U(x,y) - W(x,y) \right| .$$

Step 2: "Permuting the adjacency matrix"

$$\delta_\square(U, W) = \inf_\pi d_\square(U, W^\pi) ,$$

where $\pi : \Omega \to \Omega$ runs through all measure-preserving bijections and $W^\pi(x,y) := W(\pi(x), \pi(y))$ version of W

Many important graph parameters still continuous

Lovász&Szegedy'06 $\delta_\square$ is compact (on $\Omega^2 \to [0,1]$)

Many applications in extremal graph theory, random graphs, theoretical computer science (property testing), ...

The cut-distance topology

Step 1: "Comparing the number of edges inside any vertex set"

$$d_\square(U,W) = \sup_{S\subset\Omega} \left| \int_S \int_S U(x,y) - W(x,y) \right| .$$

Step 2: "Permuting the adjacency matrix"

$$\delta_\square(U,W) = \inf_\pi d_\square(U,W^\pi) ,$$

where $\pi : \Omega \to \Omega$ runs through all measure-preserving bijections and $W^\pi(x,y) := W(\pi(x),\pi(y))$ version of W

Many important graph parameters still continuous

Lovász&Szegedy'06 $\delta_\square$ is compact (on $\Omega^2 \to [0,1]$)

Proofs of the Lovász–Szegedy Theorem

1. Lovász–Szegedy: Using Szemerédi's Regularity lemma
2. Elek–Szegedy (2012): Ultraproducts
3. Aldous–Hoover theorem on exchangeable arrays (1981)
   Persi Diaconis&Svante Janson and Tim Austin, 2008
4. **our proof(s) based on weak* convergence**

Introducing $\mathbf{ACC}_\square$ and $\mathbf{LIM}_\square$

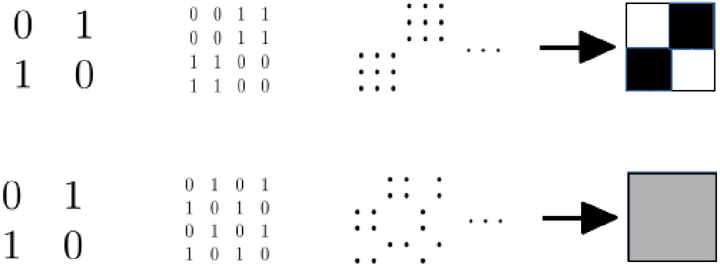
Lovász&Szegedy $\delta_\square$ is compact

$$\mathbf{ACC}_\square(\Gamma_1, \Gamma_2, \ldots) := \{\delta_\square\text{-acc pts of } \Gamma_1, \Gamma_2, \ldots\}$$
$$= \bigcup_{\pi_1, \pi_2, \ldots} \{d_\square\text{-acc pts of } \Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$$
$$\mathbf{LIM}_\square(\Gamma_1, \Gamma_2, \ldots) := \bigcup_{\pi_1, \pi_2, \ldots} \{d_\square\text{-limit of } \Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$$

Lovász&Szegedy For any sequence $\Gamma_1, \Gamma_2, ..$, $\mathbf{ACC}_\square(\Gamma_1, \Gamma_2, ..) \neq \emptyset$

Comparing the weak* and cut-distance topology

Weak* converg.: $\Gamma_1, \Gamma_2, \ldots \xrightarrow{w^*} \Gamma$ iff $\forall X \subset \Omega^2$: $\lim_n \int_X \Gamma_n = \int_X \Gamma$



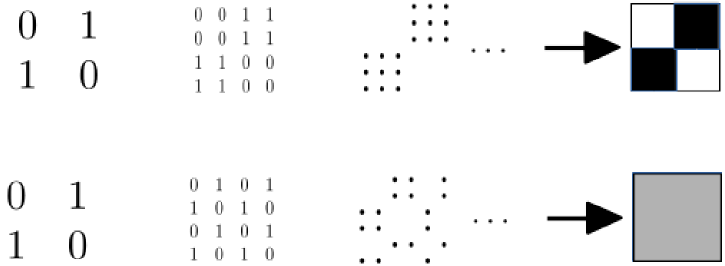
Comparing the weak* and cut-distance topology

Weak* converg.: $\Gamma_1, \Gamma_2, \ldots \xrightarrow{w^*} \Gamma$ iff $\forall X \subset \Omega^2$: $\lim_n \int_X \Gamma_n = \int_X \Gamma$

ACC$_{w^*}(\Gamma_1, \Gamma_2, \ldots) := \bigcup_{\pi_1, \pi_2, \ldots} \{$w*-acc pts of $\Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$

LIM$_{w^*}(\Gamma_1, \Gamma_2, \ldots) := \bigcup_{\pi_1, \pi_2, \ldots} \{$w*-lim pts of $\Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$

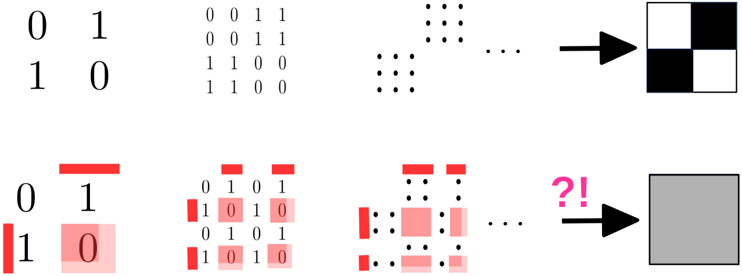
Note: **ACC**$_{w^*}(\Gamma_1, \Gamma_2, \ldots)$ is nonempty by Banach–Alaoglu Thm



Comparing the weak* and cut-distance topology

$$W_n \xrightarrow{d_\square} W \iff \limsup_n \left\{ \sup_{S\subset\Omega} \left| \int_{x\in S}\int_{y\in S} W_n(x,y) - W(x,y) \right| \right\} = 0$$

$$W_n \xrightarrow{\mathrm{w}^*} W \iff \sup_{S\subset\Omega} \left\{ \limsup_n \left| \int_{x\in S}\int_{y\in S} W_n(x,y) - W(x,y) \right| \right\} = 0$$



Graphons and the Vietoris topology (with Doležal-Grebík-Rocha-Rozhoň)

Theorem For every sequence $W_1, W_2, \ldots$ there exists a subsequence so that

$$\mathbf{ACC}_{w^*}(W_{n_1}, W_{n_2}, \ldots) = \mathbf{LIM}_{w^*}(W_{n_1}, W_{n_2}, \ldots) .$$

Theorem Any sequence of graphons $U_1, U_2, \ldots$ is cut-distance convergent if and only if $\mathbf{ACC}_{w^*}(U_1, U_2, \ldots) = \mathbf{LIM}_{w^*}(U_1, U_2, \ldots)$

Graphons and the Vietoris topology (with Doležal-Grebík-Rocha-Rozhoň)

Theorem For every sequence $W_1, W_2, \ldots$ there exists a subsequence so that

$$\mathbf{ACC}_{w^*}(W_{n_1}, W_{n_2}, \ldots) = \mathbf{LIM}_{w^*}(W_{n_1}, W_{n_2}, \ldots) .$$

Theorem Any sequence of graphons $U_1, U_2, \ldots$ is cut-distance convergent if and only if $\mathbf{ACC}_{w^*}(U_1, U_2, \ldots) = \mathbf{LIM}_{w^*}(U_1, U_2, \ldots)$ ... and converges to the most structured element in $\mathbf{LIM}_{w^*}$.

Envelopes, the structurdness order and the Vietoris topology
$\langle W \rangle := \mathbf{LIM}_{w^*}(W, W, \ldots)$
$U \preceq W \quad \text{iff} \quad \langle U \rangle \subseteq \langle W \rangle$

Cut-distance identifying graphon parameters (with Doležal-Grebík-Rocha-Rozhoň)

Motivation: The Chung-Graham-Wilson Theorem:
Among all graphons with edge density p, the constant-p graphon is the only graphon U satisfying any of the following:

- $t(C_4, U) \leq p^4$, Sidorenko's Conj: $t(B, U) \leq p^{e(B)}$
- $|\lambda_1(U)| \leq p$ and $|\lambda_2(U)| \leq 0$.
- $INT_f(U) = \int_x \int_y f(U(x, y)) \leq f(p)$ for a fixed convex function f.

Cut-distance identifying graphon parameters (with Doležal-Grebík-Rocha-Rozhoň)

Motivation: The Chung-Graham-Wilson Theorem:

Among all graphons with edge density p, the constant-p graphon is the only graphon U satisfying any of the following:

- $t(C_4, U) \leq p^4$, Sidorenko's Conj: $t(B, U) \leq p^{e(B)}$
- $|\lambda_1(U)| \leq p$ and $|\lambda_2(U)| \leq 0$.
- $INT_f(U) = \int_x \int_y f(U(x,y)) \leq f(p)$ for a fixed convex function f.

Definition $F : \mathcal{W} \to \mathbb{R}$ is a cut-distance identifying graphon parameter (CDIGP) if for each $U \prec W$ we have $F(U) < F(W)$.

Results:

- $t(C_4, \cdot)$, $t(C_6, \cdot)$, $t(C_8, \cdot), \ldots$ are CDIGPs
- generalized Sidorenko conjecture not true, i.e., $t(P_3, \cdot)$ is not CDIGP
- each kth eigenvalue is CDIGP (not precise)
- graph norms, ...