Face vectors of flag manifolds

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f-vectors of flag complexes

f-vector()=(8,12,6)

A typical question in enumerative/combinatorial geometry:
Class $\mathcal{C}$ of geometric objects. What f-vectors are attained?

Example 1: $\mathcal{C}$ =connected planar graphs
f-vectors=(v, e, f): $v - e + f = 2$ integral (Euler's formula)

Example 2: $\mathcal{C}$ =3-polytopes
f-vectors=(f_0, f_1, f_2): $f_0 - f_1 + f_2 = 2$, $f_2 \leq 2f_0 - 4$, $f_0 \leq 2f_2 - 4$
(Steinitz's theorem 1906)

Example 3: $\mathcal{C}$ =4-polytopes
f-vectors=??? not a polyhedral cone; holes

Definition A simplicial complex is *flag* if every clique induces a simplex. A *flag homology manifold* is a homology manifold that is flag.

Hopf's conjecture and the Charney–Davis conjecture

Euler characteristic: $\chi = f_0 - f_1 + f_2 - \dots$
(homology computations) $= b_0 - b_1 + b_2 - \dots$ Betti numbers

Gauss–Bonnet 1848 For a compact surface with Riemannian metric,

$$\chi = \frac{1}{2\pi} \int \text{Gauss curv } d(\text{area})$$

Corollary If the Gauss curvature is ≤ 0 everywhere then $\chi \leq 0$.

Hopf's conjecture ∼1930's For a compact Riemannian manifold of dimension $2n$ with sectional curvature ≤ 0, we have $(-1)^n \chi \geq 0$.
(dim $= 4$ Chern–Milnor'56 from Chern–Gauss–Bonnet integral)

Alexandrov's approach: non-smooth manifolds, CAT(0) spaces.
Gromov's lemma: In a non-positively curved complex links of vertices are flag.
The integrand: $\kappa(v) = 1 + \sum_i (-\frac{1}{2})^{i+1} f_i(L_v)$, where L_v is the link of v

Charney–Davis conjecture 1995
For a flag sphere of dimension $2n-1$ we have $(-1)^n \kappa \geq 0$.
(dim $= 1$ trivial, dim $= 3$ Davis–Okun'01 deep)

flag manifolds and graph theory

flag simplicial complex=all the structure encoded in the 1-skeleton (which is a graph)

we use methods of extremal graph theory

1. [Adamaszek, H., Transactions AMS 2015]
   essentially solve a conjecture of Gal 2005 by characterizing f-vectors of sufficiently large three-dimensional flag Gorenstein* complexes
2. [Adamaszek, H., arXiv:1503.05961]
   upper bound theorem for flag triangulations of manifolds of odd dimension

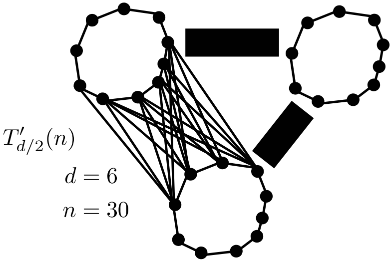
Proofs:

- geometric ingredient: close to trivial
- extremal graph theory ingredient: not much harder

Upper Bound Theorem For every even d, if M is a flag homology manifold of dimension $d-1$ on $n \geq n_0$ vertices then for every $k \leq d-1$:

$$f_k(M) \leq f_k(T'_{d/2}(n)) \; ,$$

where $T'_{d/2}(n)$ is a $d/2$-fold join of cycles of the same(± 1) length



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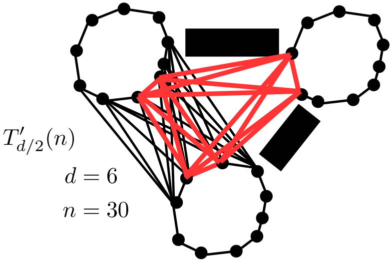
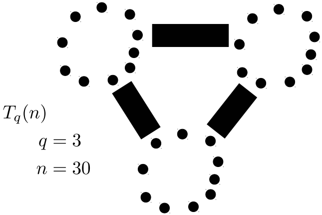


Extremal graph theory tools

Turán 1941 (Mantel 1903) If an n-vertex graph G is K_{q+1}-free then $e(G) \leq e(T_q(n))$. (Note: think of n large, q fixed.)



Stability If an n-vertex graph G has $e(G) \geq e(T_q(n)) - \delta n^2$ then it contains $\Theta(n^{q+1})$ copies of K_{q+1}, unless G is ϵ-close to $T_q(n)$.

approximate structure + ad-hoc arguments $\Rightarrow$ exact structure

Proof

Our theorem

$T'_{d/2}(n)$ maximizes face numbers (counting faces of a fixed dimension $k \leq d-1$) among flag manifolds of dim=$d-1$.

Proof

Our theorem restated

$T'_{d/2}(n)$ maximizes clique numbers (counting cliques of a fixed order $k \leq d$) among flag manifolds of dim=$d-1$.

…in particular $T'_{d/2}(n)$ maximizes the number of edges among flag manifolds of dim=$d-1$.

Turán's Theorem

$T_q(n)$ maximizes the number of edges among K_{q+1}-free graphs.