Loebl–Komlós–Sós Conjecture

Jan Hladký
Institute of Mathematics
Academy of Sciences of the Czech Republic

European Commission

JH's research is supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme.

Definition (Extremal graph theory, Bollobás 1976):

Definition (Extremal graph theory, Bollobás 1976):
Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Definition (Extremal graph theory, Bollobás 1976):
Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Mantel 1907/Turán 1941 G has n vertices
If G has more than $n^2/4$ edges then it contains a triangle.

- optimal $\Rightarrow$ extremal graph
- starting point of extremal graph theory
- Aigner 1995: *Turán's graph theorem*, 6 proofs.

Definition (Extremal graph theory, Bollobás 1976):
Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Mantel 1907/Turán 1941 G has n vertices
If G has more than $n^2/4$ edges then it contains a triangle.

- optimal $\Rightarrow$ extremal graph
- starting point of extremal graph theory
- Aigner 1995: *Turán's graph theorem*, 6 proofs.

Extending in all possible ways:

If G has more than $n^2/4$ edges then it contains a triangle

Definition (Extremal graph theory, Bollobás 1976):
Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Mantel 1907/Turán 1941 G has n vertices
If G has more than $n^2/4$ edges then it contains a triangle.

- optimal $\Rightarrow$ extremal graph
- starting point of extremal graph theory
- Aigner 1995: *Turán's graph theorem*, 6 proofs.

Extending in all possible ways:

If G has more than $n^2/4$ edges then it contains at least $n/2$ triangles

Definition (Extremal graph theory, Bollobás 1976):
Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Mantel 1907/Turán 1941 G has n vertices
If G has more than $n^2/4$ edges then it contains a triangle.

- optimal $\Rightarrow$ extremal graph
- starting point of extremal graph theory
- Aigner 1995: *Turán's graph theorem*, 6 proofs.

Extending in all possible ways:

If G whose edges are colored with two colors has more than $n^2/4$ edges then it contains at least $n/2$ monochromatic triangles

Definition (Extremal graph theory, Bollobás 1976):
Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Mantel 1907/Turán 1941 G has n vertices
If G has more than $n^2/4$ edges then it contains a triangle.

- optimal $\Rightarrow$ extremal graph
- starting point of extremal graph theory
- Aigner 1995: *Turán's graph theorem*, 6 proofs.

Extending in all possible ways:

If G whose edges are colored with two colors has more than $n^2/4$ edges then it contains at least $n/2$ monochromatic triangles

So, Ramsey theory meets Extremal graph theory:
at a party of 49 (43?) there are either 5 mutual strangers or 5 mutual friends

Conjectures

Setting

G … simple, looples graph of order n

$\mathcal{T}_\ell$ … all trees of order ℓ

Conjectures

Setting

G . . . simple, looples graph of order n

$\mathcal{T}_\ell$ . . . all trees of order ℓ

Conjectures

Setting

G . . . simple, looples graph of order n

$\mathcal{T}_\ell$ . . . all trees of order ℓ

ExGrTh DENSITY CONDITION $\Rightarrow$ SUBGRAPH.

Conjectures

Setting

G . . . simple, looples graph of order n

$\mathcal{T}_\ell$ . . . all trees of order ℓ

ExGrTh DENSITY CONDITION $\Rightarrow$ SUBGRAPH.

Dirac Thm If $\delta(G) \geq n/2$ then G contains a spanning path.
Again an easy proof but later we attempt for a complicated one. . .

Conjectures

Setting

G . . . simple, looples graph of order n

$\mathcal{T}_\ell$ . . . all trees of order ℓ

ExGrTh DENSITY CONDITION $\Rightarrow$ SUBGRAPH.

Dirac Thm If $\delta(G) \geq n/2$ then G contains a spanning path.
Again an easy proof but later we attempt for a complicated one. . .

Embedding trees: motivation $\delta(G) \geq k$, then $\mathcal{T}_{k+1} \subset G$.

Conjectures

Setting
G ... simple, looples graph of order n
$\mathcal{T}_\ell$ ... all trees of order ℓ

ExGrTh DENSITY CONDITION $\Rightarrow$ SUBGRAPH.

Dirac Thm If $\delta(G) \geq n/2$ then G contains a spanning path.
Again an easy proof but later we attempt for a complicated one...

Embedding trees: motivation $\delta(G) \geq k$, then $\mathcal{T}_{k+1} \subset G$.
Can this be weakened?

Conjectures

Setting

G ... simple, looples graph of order n

$\mathcal{T}_\ell$ ... all trees of order ℓ

ExGrTh DENSITY CONDITION $\Rightarrow$ SUBGRAPH.

Dirac Thm If $\delta(G) \geq n/2$ then G contains a spanning path.
Again an easy proof but later we attempt for a complicated one...

Embedding trees: motivation $\delta(G) \geq k$, then $\mathcal{T}_{k+1} \subset G$.
Can this be weakened?

Erdős-Sós Conjecture '63 If the average degree of G is at least k, then $\mathcal{T}_{k+1} \subset G$.
proof announced by Ajtai, Komlós, Simonovits, and Szemerédi

Conjectures

Setting
G ... simple, looples graph of order n
$\mathcal{T}_\ell$ ... all trees of order ℓ

ExGrTh DENSITY CONDITION $\Rightarrow$ SUBGRAPH.

Dirac Thm If $\delta(G) \geq n/2$ then G contains a spanning path.
Again an easy proof but later we attempt for a complicated one...

Embedding trees: motivation $\delta(G) \geq k$, then $\mathcal{T}_{k+1} \subset G$.
Can this be weakened?

Erdős-Sós Conjecture '63 If the average degree of G is at least k, then $\mathcal{T}_{k+1} \subset G$.
proof announced by Ajtai, Komlós, Simonovits, and Szemerédi

Loebl-Komlós-Sós Conjecture '95 If at least $n/2$ of the vertices of G have degrees at least k, then $\mathcal{T}_{k+1} \subset G$.

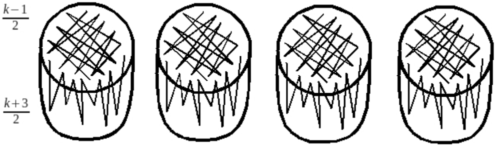
LKS Conjecture

Loebl-Komlós-Sós Conjecture '95 If at least $n/2$ of the vertices of G have degrees at least k, then $\mathcal{T}_{k+1} \subset G$.

LKS Conjecture

Loebl-Komlós-Sós Conjecture '95 If at least $n/2$ of the vertices of G have degrees at least k, then $\mathcal{T}_{k+1} \subset G$.

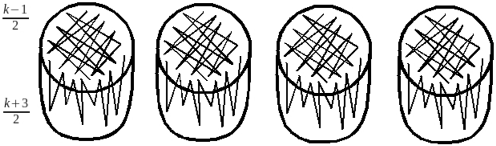
Figure : The extremal graph



LKS Conjecture

Loebl-Komlós-Sós Conjecture '95 If at least $n/2$ of the vertices of G have degrees at least k, then $\mathcal{T}_{k+1} \subset G$.

Figure : The extremal graph

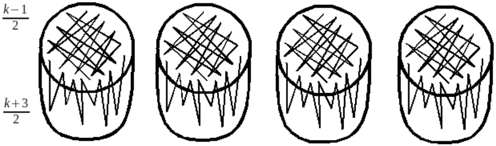


various partial results

LKS Conjecture

Loebl-Komlós-Sós Conjecture '95 If at least $n/2$ of the vertices of G have degrees at least k, then $\mathcal{T}_{k+1} \subset G$.

Figure : The extremal graph

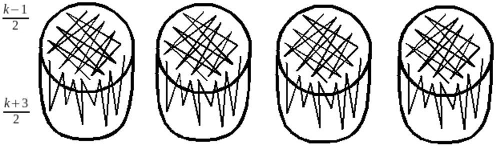


various partial results
Piguet, Stein '07 For $\epsilon, q > 0$ there exists n_0 such that the following holds. For any $n > n_0$ and $k > qn$ it holds that if G of order n has at least $(1/2 + \epsilon)n$ vertices of degree at least $(1 + \epsilon)k$, then $\mathcal{T}_{k+1} \subset G$.

LKS Conjecture

Loebl-Komlós-Sós Conjecture '95 If at least $n/2$ of the vertices of G have degrees at least k, then $\mathcal{T}_{k+1} \subset G$.

Figure : The extremal graph



various partial results

Piguet, Stein '07 For $\epsilon, q > 0$ there exists n_0 such that the following holds. For any $n > n_0$ and $k > qn$ it holds that if G of order n has at least $(1/2 + \epsilon)n$ vertices of degree at least $(1 + \epsilon)k$, then $\mathcal{T}_{k+1} \subset G$.

Hladký, Piguet '08 & Cooley '09 For $q > 0$ there exists n_0 such that the following holds. For any $n > n_0$ and $k > qn$ it holds that if G of order n has at least $n/2$ vertices of degrees at least k, then $\mathcal{T}_{k+1} \subset G$.

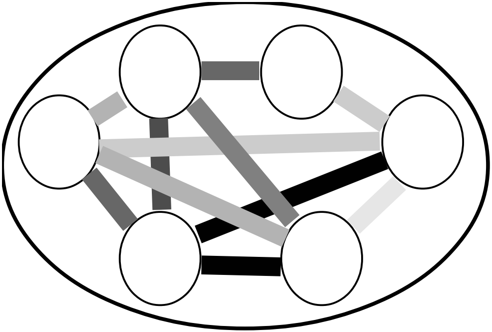
Szemerédi's Regularity Lemma and graph embedding

Szemerédi 1975: dense subsets of $\mathbb{N}$ contain a k-AP, $\forall k$
Szemerédi 1978: Regularity Lemma
Sporadic applications in ExGrTh in the '80's, boom in the '90's.
Now, various strenthenings, for graphs and other structures.

Szemerédi's Regularity Lemma and graph embedding

Szemerédi 1975: dense subsets of $\mathbb{N}$ contain a k-AP, $\forall k$
Szemerédi 1978: Regularity Lemma
Statement, informally: Vertices of each graph can be partitioned into "clusters" so that all the bipartite graphs look random-like ("regular pairs").

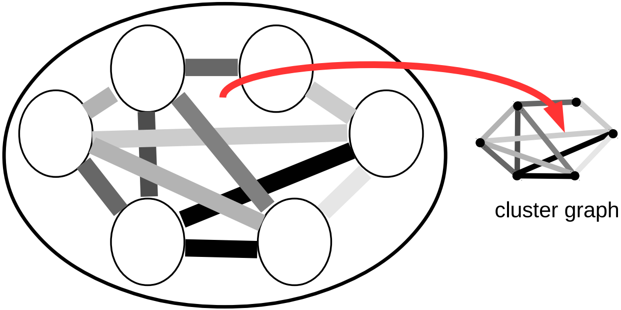
Figure : density of a pair=edges/(cluster size)^2



Szemerédi's Regularity Lemma and graph embedding

Szemerédi 1975: dense subsets of $\mathbb{N}$ contain a k-AP, $\forall k$
Szemerédi 1978: Regularity Lemma
Statement, informally: Vertices of each graph can be partitioned into "clusters" so that all the bipartite graphs look random-like ("regular pairs").

Figure : density of a pair=edges$/$(cluster size)^2



Szemerédi's Regularity Lemma and graph embedding

Szemerédi 1975: dense subsets of $\mathbb{N}$ contain a k-AP, $\forall k$
Szemerédi 1978: Regularity Lemma
Statement, informally: Vertices of each graph can be partitioned into "clusters" so that all the bipartite graphs look random-like ("regular pairs").

Embedding a spanning path in G (satisfying some density condition, e.g. Dirac's Thm) with the RL:

- regularize $G \Rightarrow$ cluster graph $\mathbf{G}$
- $\mathbf{G}$ satisfies the same density conditions
- prove that $\mathbf{G}$ is connected and contains a perfect matching (easier task!)
- this gives you directions how to embed the path (next slide)

A complicated proof of Dirac's Theorem

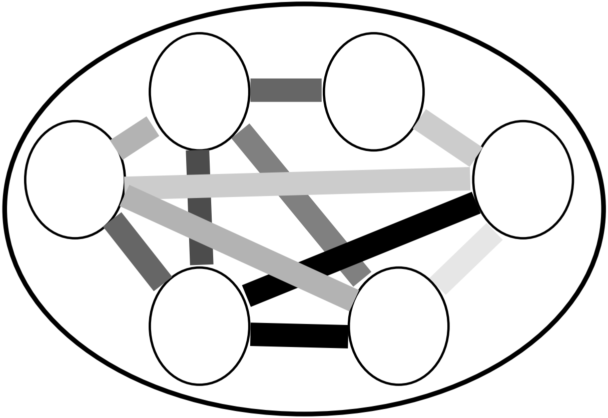
If $\delta(G) \geq n$ then G contains a spanning path.

Figure : Embedding a path

A complicated proof of Dirac's Theorem

If $\delta(G) \geq n$ then G contains a spanning path.

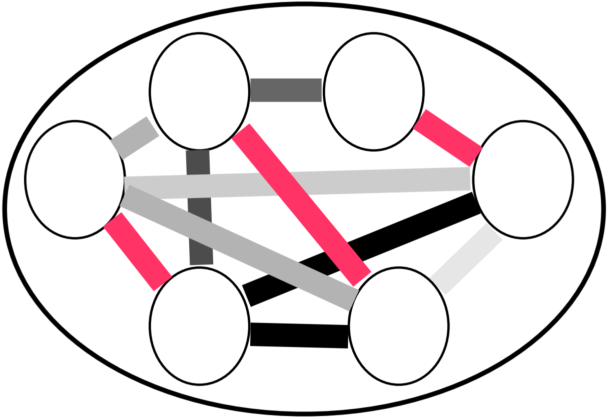
Figure : Embedding a path



A complicated proof of Dirac's Theorem

If $\delta(G) \geq n$ then G contains a spanning path.

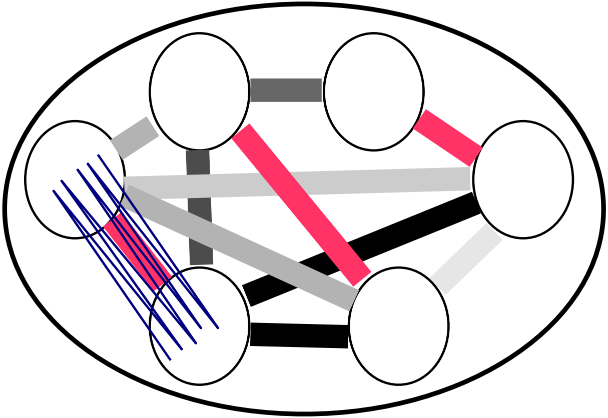
Figure : Embedding a path



A complicated proof of Dirac's Theorem

If $\delta(G) \geq n$ then G contains a spanning path.

Figure : Embedding a path



A complicated proof of Dirac's Theorem

If $\delta(G) \geq n$ then G contains a spanning path.

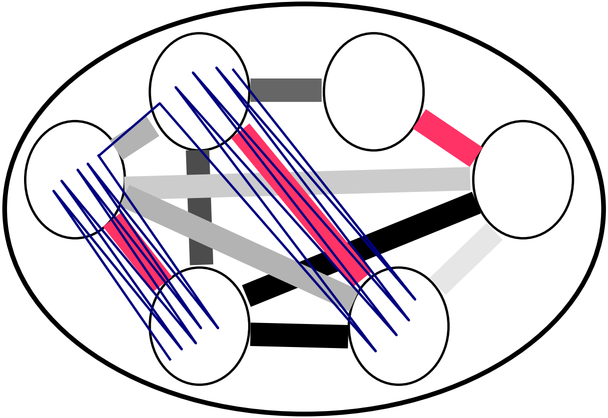


Figure : Embedding a path

A complicated proof of Dirac's Theorem

If $\delta(G) \geq n$ then G contains a spanning path.

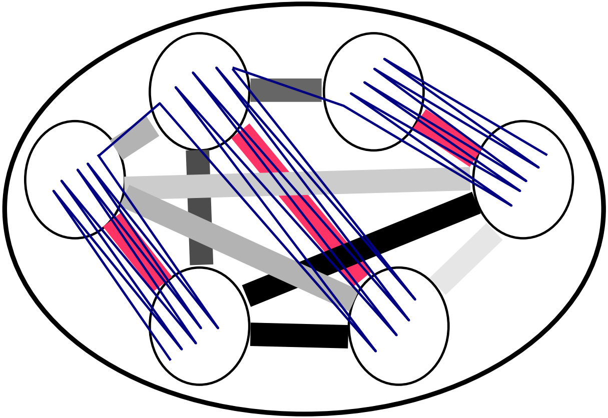


Figure : Embedding a path

Our result

H., Komlós, Piguet, Simonovits, Stein, Szemerédi

For every $\eta > 0$ there exists k_0 such that for every $k > k_0$ any graph n-vertex graph G with at least $(\frac{1}{2} + \eta)n$ with degrees at least $(1 + \eta)k$ contains any tree of order k.

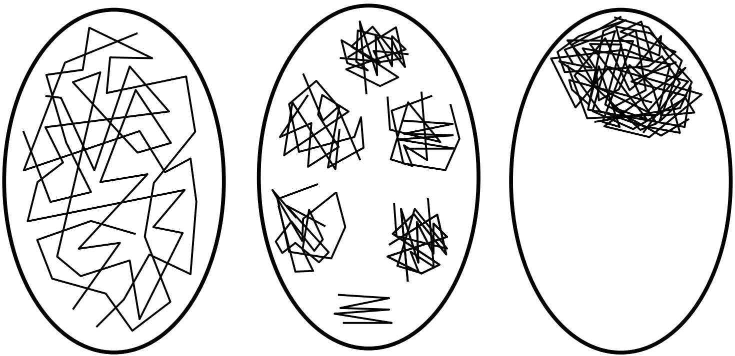
What is the structure of G?

Our result

H., Komlós, Piguet, Simonovits, Stein, Szemerédi

For every $\eta > 0$ there exists k_0 such that for every $k > k_0$ any graph n-vertex graph G with at least $(\frac{1}{2} + \eta)n$ with degrees at least $(1 + \eta)k$ contains any tree of order k.

What is the structure of G?

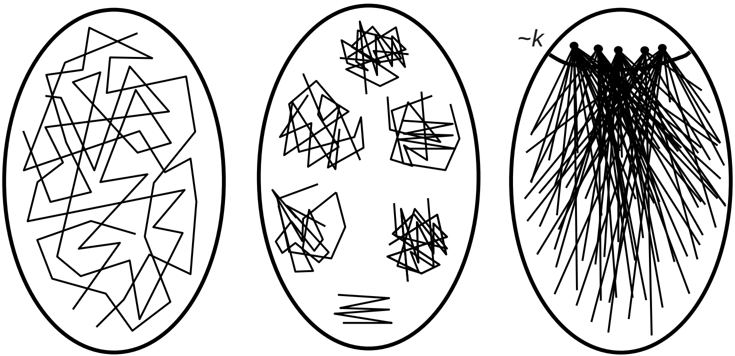


Our result

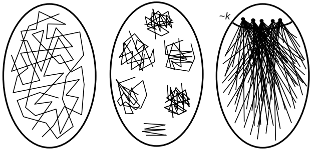
H., Komlós, Piguet, Simonovits, Stein, Szemerédi

For every $\eta > 0$ there exists k_0 such that for every $k > k_0$ any graph n-vertex graph G with at least $(\frac{1}{2} + \eta)n$ with degrees at least $(1 + \eta)k$ contains any tree of order k.

What is the structure of G?

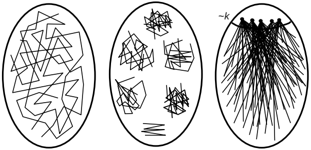


The ad-hoc Regularity Lemma



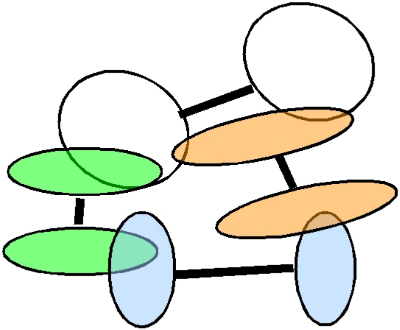
The ad-hoc Regularity Lemma

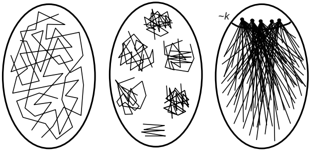
- Isolate vertices of high degree ⇒ bounded-degree graph



The ad-hoc Regularity Lemma

- Isolate vertices of high degree ⇒ bounded-degree graph
- Greedily take out dense spots ⇒ nowhere-dense graph





The ad-hoc Regularity Lemma

- Isolate vertices of high degree $\Rightarrow$ bounded-degree graph
- Greedily take out dense spots $\Rightarrow$ nowhere-dense graph
- Regularize the dense spots