Packing degenerate graphs

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Tree packing conjectures

Two graphs G_1, G_2 **pack** into a graph H if there exist injective homomorphisms $\phi_1 : G_1 \to H$, $\phi_2 : G_2 \to H$ such that $E(\phi_1) \cap E(\phi_2) = \emptyset$.

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Conjecture (Ringel, 1963)

Any $2n+1$ *identical copies of any tree of order* $n+1$ *pack into* K_{2n+1}*.*

Conjecture (Gyarfás–Lehel "The Tree packing conj.", 1978)

Let $T_1, \dots, T_n$ *be a family of trees,* $v(T_i) = i$*. Then* $T_1, \dots, T_n$ *pack into* K_n*.*

Best possible: total number of the edges in the trees equals the number of edges of the host graph

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Theorem (Böttcher, H., Piguet, Taraz 2014)

$\forall \epsilon > 0, \Delta \in \mathbb{N}\ \exists n_0\ \forall n > n_0$
Let $T_1, \ldots, T_k$ *be a family of trees,* $\Delta(T_i) \leq \Delta$, $v(T_i) \leq (1-\epsilon)n$, $\sum e(T_i) \leq (1-\epsilon)\binom{n}{2}$*. Then* $T_1, \ldots, T_k$ *pack into* K_n.

Packing large graphs into K_n

Böttcher, H., Piguet, Taraz 2014:

1. trees 2. bounded degree 3. $v \leq (1-\epsilon)n$ 4. $\sum e \leq (1-\epsilon)\binom{n}{2}$

Messuti, Rödl, Schacht 2015

1. non-expanding 2.-4. ———

Ferber, Lee, Mousset 2015

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Kim, Kühn, Osthus, Tyomkyn 2016

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Ringel's conjecture and the Tree packing conjecture for trees of bounded degree

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A.-B.-H.-P.

1. D-degenerate 2. $\Delta = O(n/\log n)$ 3. $\mathbf{v} \leq \mathbf{n}$ 4. ———

Theorem

For each $\epsilon > 0$ *and each* $D \in \mathbb{N}$ *there exists* $c > 0$ *and a number* n_0 *such that the following holds for each integer* $n > n_0$. *Suppose that* $(G_t)_{t\in[t^*]}$ *is a family of* D*-degenerate graphs, each of which has at most* n *vertices and maximum degree at most* $\frac{cn}{\log n}$. *Suppose further that the total number of edges of* $(G_t)_{t\in[t^*]}$ *is at most* $(1-\epsilon)\binom{n}{2}$. *Then* $(G_t)_{t\in[t^*]}$ *packs into* K_n.

Proof

$H_0 := K_n$

for $i = 1, \ldots, t^*$, take a D-degenerate ordering of G_i

embed G_i into H_{i-1} vertex-by-vertex randomly:

for $v = 1, \ldots, v(G_i)$,

$\phi_i(v) := random\left(N^{common}_{G_{i-1}}(\phi_i(N^{left}_{H_i}(v))) \setminus \phi_i([v-1])\right)$

set $H_i := H_{i-1} - \phi_i(G_i)$

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Cannot work at the end. So, run the previous for $G_i^* := G_i - U_i$, where U_i is a small independent set. Pack U_i's at the end using a matching argument