Property testing, parameter estimation, and graph limits

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The ○/□-switch correlates strongly with the *red*/blue switch

$\Rightarrow$these two genes must be close in the cell

the corresponding metric space is 1-dimensional.

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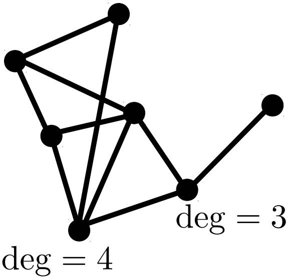
Quite often, in sciences, and in computer science alike, you want to infer properties of an object you cannot observe directly.

Here, we want to "observe" properties and parameters of graphs.



Property: YES/NO
planarity, containing a $\triangle$

Parameter: real number
chromatic number, no. of $\triangle$'s

What is the average number of "friends" in the Facebook graph?

(Graph) Parameter estimation

Setting:
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Typically, it is impossible to determine $f(G)$ before learning the entire graph G (at least in the worst case). Rather, we want to get a **high-confidence** $(= 1 - \epsilon)$ **estimate on** $f(G)$ **using few** $(= K(\epsilon))$ **randomized queries**.

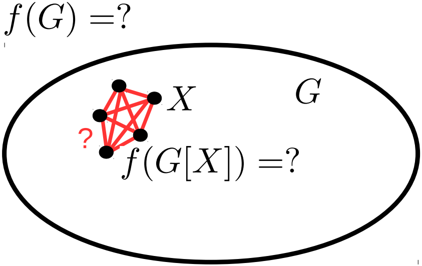
Parameter estimation in dense graphs

our universe: all graphs. . . $\mathcal{G}$

A parameter $f : \mathcal{G} \to \mathbb{R}$ is estimable if for every $\epsilon > 0$ there exists a number $K = K(\epsilon)$ such that

$$\mathbb{P}\big[f(G) - f(G[X])| > \epsilon\big] < \epsilon ,$$

where $X \subset V(G)$ is a random K-set.



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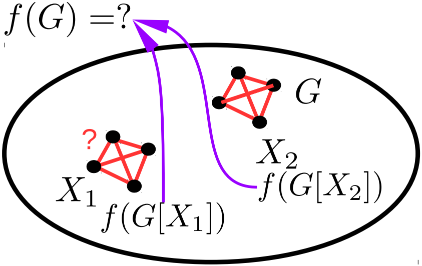

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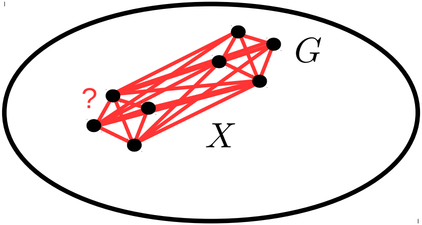
$f(G) =?$



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Why **dense**? Recall: $e(G) \leq \binom{n}{2} \approx n^2/2$.

Observe that an estimable parameter cannot change substantially after an $o(n^2)$ edge-perturbation of G.

In particular, $f(G) \approx f(\emptyset)$, whenever $e(G) = o(n^2)$.

No information about trees, planar graphs, . . .

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compactification of the space of finite graphs $\Rightarrow$
... *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0, 1]$

Why? same story as with $\mathbb{Q}$ vs $\mathbb{R}$: only the latter allows
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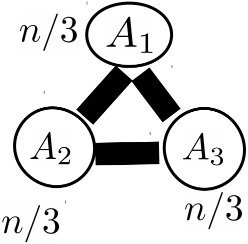
mathematical framework for parameter estimation and property testing

Limits of dense graph sequences: an abstract approach

F is a "fixed graph" of order k, G is "large" of order n
We define **subgraph density** $t(F, G)$:

$$t(F,G) := \frac{\#\text{ copies of } F \text{ in } G}{\binom{n}{k}} = \mathbb{P}\big[G[\text{random } k\text{-set}] \cong F\big]$$



$k=2:\quad t(\;, G) = \frac{1}{3} \qquad t(\;, G) = \frac{2}{3}$

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A sequence of graphs $G_1, G_2, \ldots$ **converges** if for each F, the sequence $t(F, G_1), t(F, G_2), \ldots$ converges.

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The key connection: $f : \mathcal{G} \to \mathbb{R}$ is estimable iff it is continuous.

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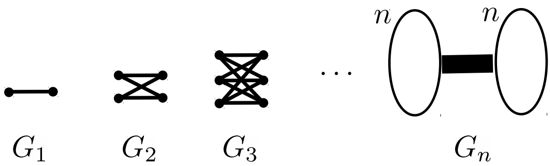

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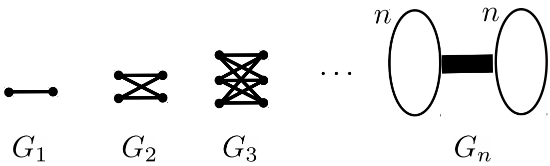
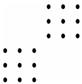
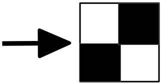


Represent these graphs by their adjacency matrices:

$$\begin{matrix}0 & 1\\ 1 & 0\end{matrix} \qquad \begin{matrix}0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\\ 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\end{matrix}$$



$\cdots$



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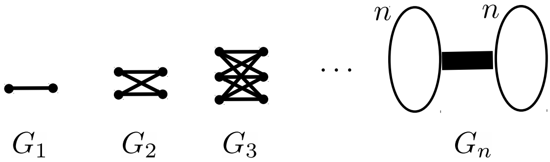

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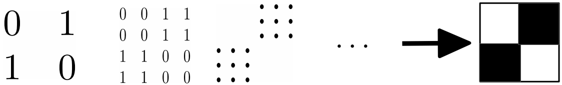
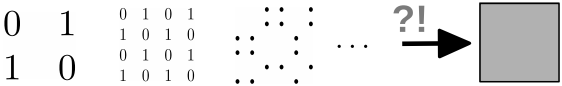


. . . works if you do things the right way. But, . . .



In general Szemerédi's Regularity Lemma can be used to determine "the right way" of ordering the vertices.

Dense model

complete picture:

characterization of testable graph properties and estimable parameters either

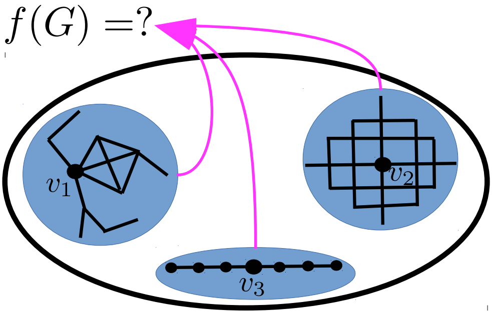
- in the language of the Szemerédi Regularity lemma ($\subseteq$Alon–Fischer–Newman–Shapira'03–'06, ...), and
- in the language of graph limits.

Parameter estimation in bounded degree graphs

our universe: graphs of degrees bounded by a constant D . . . $\mathcal{G}_D$
A parameter $f : \mathcal{G}_D \to \mathbb{R}$ is estimable if for each $\epsilon > 0$ there exists a number $K = K(\epsilon)$ and a function g such that

$$\mathbb{P}\left[|f(G) - g(B_1, B_2, \ldots, B_K)| > \epsilon\right] < \epsilon \ ,$$

where $B_1, \ldots, B_K$ are balls of radius K around K randomly selected vertices of G.



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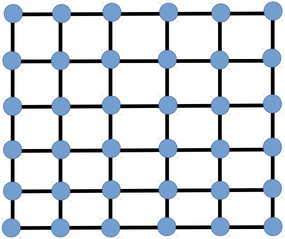

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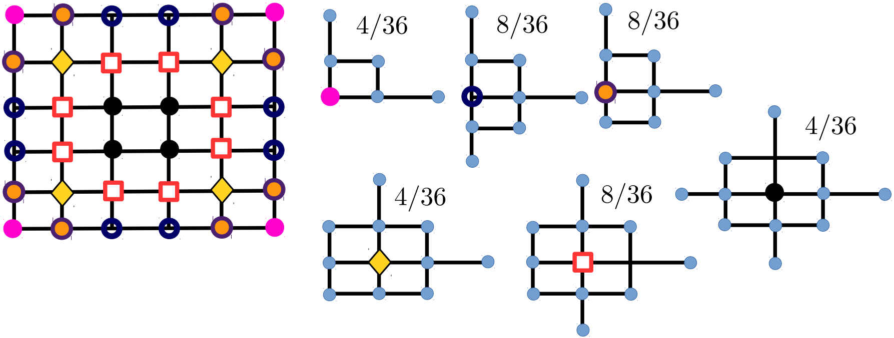

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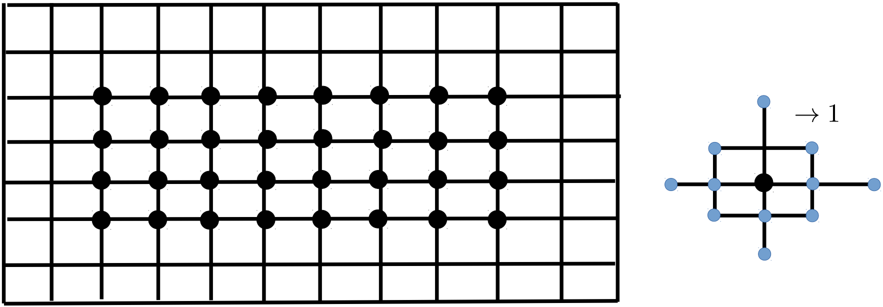

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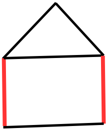

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Proof [Elek–Lippner]: (Borel oracles method)
Argue that there exists a "Borel matching" on the limit space.
Show how to make use of this structure to make estimates about matching ratio of finite graphs.
In particular, this does not give any construction of an algorithm!

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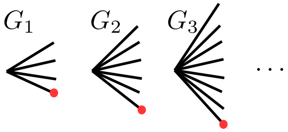

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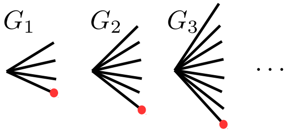

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Lyons'07: The concept of Benjamini–Schramm limit can be extended to sequences $G_1, G_2, \dots$ where for each $r \in \mathbb{N}$, the sequence $\rho_r(G_1), \rho_r(G_2), \dots$ is tight. AND NOT FURTHER

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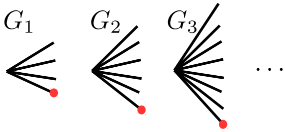

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with Lukasz Grabowski and Oleg Pikhurko, 2014+

the limit space and the soft arguments in the theory of bounded-degree graph limits make sense even for tight graph sequences

New graph classes for which the graph limit used not to be applicable:

- Erdős–Rényi $\mathbb{G}_{n,C/n}$,
- random planar graphs,...

no surprises yet.