Limits of dense graph sequences

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Random graphs

introduced by Erdős and Rényi in 1959
arguably the most studied random discrete structure

$n \in \mathbb{N}$, $p \in (0,1)$
$\mathbb{G}(n,p)$ is a distribution on graphs on the vertex set $\{1,\ldots,n\}$;
connect i and j with probability p, independently for each pair ij

One can ask various questions such as:

- Does $\mathbb{G}(n,0.3)$ contain a triangle?

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One can ask various questions such as:

- Does $\mathbb{G}(n, 0.3)$ contain a triangle?
  Asymptotically almost surely, as $n \to \infty$, $\mathbb{G}(n, 0.3)$ contains a triangle.
- What is the size of the largest clique in $\mathbb{G}(n, 0.3)$?
  Asymptotically almost surely, as $n \to \infty$, the largest clique in $\mathbb{G}(n, 0.3)$ is of size $\left(\frac{2}{\log(\frac{1}{0.3})} \pm \epsilon\right) \cdot \log n$.

Limits of dense graph sequences

Lovász, Szegedy *JCTB'06* (Fulkerson Prize'12)
Borgs, Chayes, Lovász, Sós, Vesztergombi *Adv.Math.'06*
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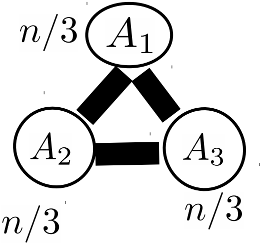
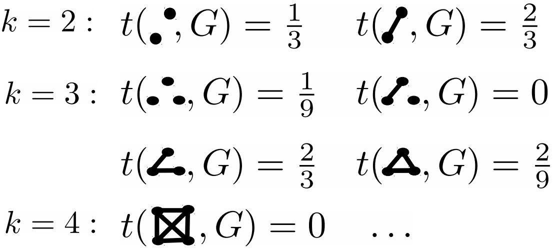
idea: convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
... *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0,1]$
Why? same story as with $\mathbb{Q}$ vs $\mathbb{R}$: only the latter allows
reasonable e.g. variational and integral calculus
for example *argmin*$(x^3 - 2x)$

Limits of dense graph sequences: an abstract approach

F is a "fixed graph" of order k, G is "large" of order n
We define **subgraph density** $t(F, G)$:

$$t(F,G) := \frac{\#\text{ copies of } F \text{ in } G}{\binom{n}{k}} = \mathbf{P}\big[G[\text{random } k\text{-set}] \cong F\big]$$





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A sequence of graphs $G_1, G_2, \dots$ **converges** if for each F, the sequence $t(F, G_1), t(F, G_2), \dots$ converges.
We get a **limit object** Ψ, $t(F, \Psi) = \lim_n t(F, G_n)$.

Why dense graph sequences?

If the proportion of edges $\searrow 0$ ($\lim \frac{e(G_n)}{n^2} = 0$) we get a trivial limit.
That is, the theory is void for trees, planar graphs, ...

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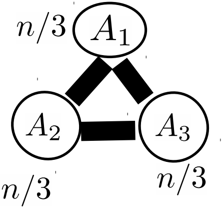
Razborov'07 flag algebras (next slide)

Extremal graph theory and Razborov's flag algebras I



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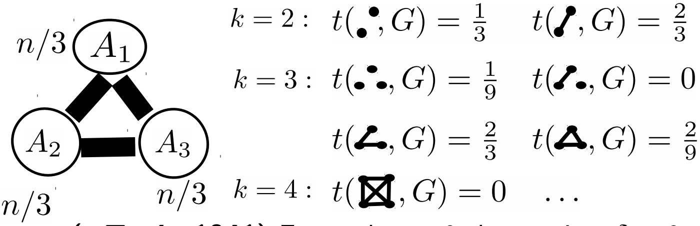

Theorem (≈Turán 1941) For each $\epsilon > 0$ there exists $\delta > 0$: If an n-vertex graph has more than $(\frac{2}{3} + \epsilon)\binom{n}{2}$ edges then it contains $> \delta$-proportion of $\boxtimes$'s.

ExGrTh studies relations between $t(F_1, G)$, $t(F_2, G)$, $\dots$.

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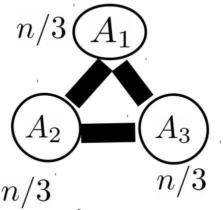

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Razborov: But lets rather study these relations on the limit space!
Approach to proving Turán: Suppose the theorem is false. $G_1, G_2, \ldots$ all contain $(\frac{2}{3} + \epsilon)$-proportion of edges but proportion of $\boxtimes$'s tends to 0. Pass to a subsequential limit Ψ. $t(|, \Psi) \geq \frac{2}{3} + \epsilon$ and $t(\boxtimes, \Psi) = 0$. Derive a contradiction.

Extremal graph theory and Razborov's flag algebras II

Razborov provides tools for deriving relations that hold on the limit object (Cauchy-Schwarz calculus, variational calculus, . . .)

First application: Razborov'08 (AMS Robbins Prize'12) solves the triangle density problem of Lovász and Simonovits 1983:
Suppose a graph has a given proportion of edges. What proportion of triangles can it have?

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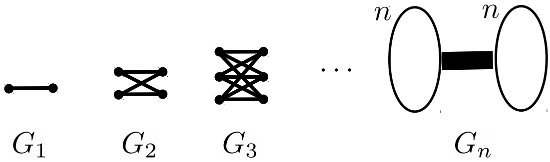
- H.-Král'-Norine'09: Caccetta-Häggkvist conj. (progress)
- H.-Hatami-Král'-Norine-Razborov'11 conjecture of Erdős 1984
- HHKNR'11 conjecture of Jagger-Štovíček-Thomason 1996
- . . . and many more

Hatami-Norine *J.AMS'11* deciding whether an inequality between subgraph densities holds for all graph limits is undecidable

Graphons I



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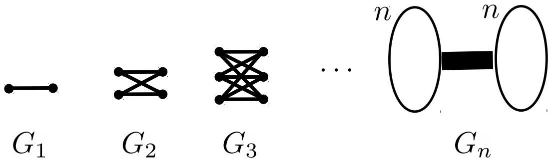
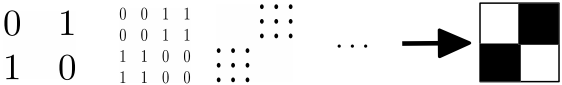


Represent these graphs by their adjacency matrices:



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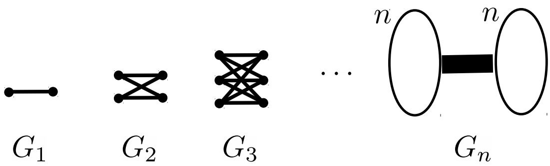

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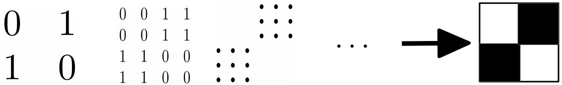
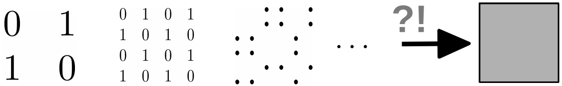


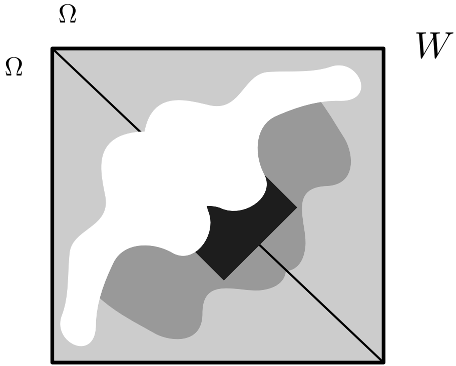
. . . works if you do things the right way. But, . . .



In general Szemerédi's Regularity Lemma can be used to determine "the right way" of ordering the vertices.

Graphons II

A **graphon** is a symmetric Lebesgue-m. function. $W : \Omega^2 \to [0,1]$.
Theorem (Lovász–Szegedy) sampling conv.$\Leftrightarrow$graphical conv.
Theorem (L–Sz.) Every graphon W can be achieved in the limit.
Proof:



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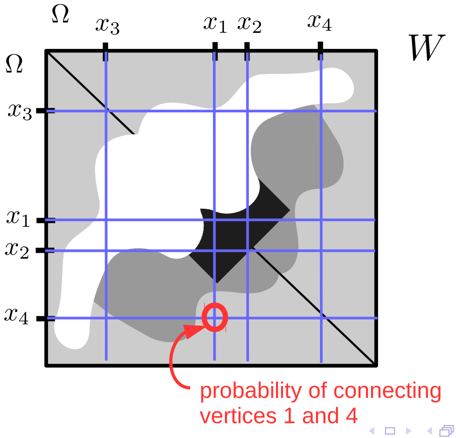

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$\mathbb{G}(n, W)$ as a generalization of the Erdős–Rényi model $\mathbb{G}(n, p)$.

Interesting model *per se*!
Bollobás–Janson–Riordan'07, Doležal–H.–Máthé'15

Cliques in inhomogeneous random graphs: motivation

(joint work with Doležal and Máthé, arXiv:1510.02335)
notation $\omega(G)$ =size of the largest clique
motivation: some parameters are continuous, e.g. triangle density

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$$\frac{1}{n^3} \cdot \sum_{i,j,k} A_{G_n}(i,j) A_{G_n}(j,k) A_{G_n}(k,i) \rightarrow \quad \int_{x,y,z} W(x,y)W(y,z)W(z,x)$$

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but the clique number is not. For example, we can find sequences

- $H_1, H_2, \ldots \to \mathbf{0}$ with $\omega(H_n) > n^{0.99}$
- $F_1, F_2, \ldots \to \mathbf{1}$ with $\omega(F_n) < \log \log n$

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In order to get rid of these pathological examples, we study samples from $\mathbb{G}(n, W)$

Cliques in inhomogeneous random graphs: the result

Matula 1976, Grimmett–McDiarmid 1975:
For any fixed $p \in (0,1)$, asymptotically almost surely,

$$\omega(\mathbb{G}(n,p)) = \left(\frac{2}{\log(1/p)} \pm \epsilon\right) \cdot \log n \ .$$

Our result:
For any "reasonable" fixed graphon W, asymptotically almost surely,

$$\omega(\mathbb{G}(n,p)) = \kappa(W) \cdot \log n \ ,$$

where

$$\kappa(W) = \sup\left\{ \frac{2\|h\|_1^2}{\int_x \int_y h(x)h(y)\log\left(1/W(x,y)\right)} \ : \ h \in L^1(\Omega), h \geq 0 \right\} \ .$$