On graph norms

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- Doležal, Grebík, H, Rocha, Rozhoň
- Garbe, H, Lee
- H, Juškevičius (in progress)

Basics

- Kernel: 2-variable symmetric bounded function W: $[0,1]^2 \to \mathbb{R}$

  Graphon: ...and is nonnegative

- Subgraph density of a finite graph H: $t(H,W) = \int_{[0,1]^V} \prod_{x_i x_j \in E} W(x_i, x_j)$

- H is weakly norming if $\|W\|_{r(H)} := t(H,|W|)^{1/|E|}$ is a norm
- H is norming if $\|W\|_H := |t(H,W)|^{1/|E|}$ is a norm

Which graphs are (weakly) norming?

$\|\emptyset\| = 0$

$\|cW\| = |c|\,\|W\|$

? $\|U+W\| \leq \|U\|+\|W\|$

Sidorenko's conjecture

- Conjecture Erdős-Simonovits (1983), Sidorenko (1986):

  A graph *H* is bipartite if and only if for each graphon

$$t(H, W) \geq t(H, [\![W]\!]^{\bowtie[0,1]})$$

- Conjecture Skokan, Thoma (2004):

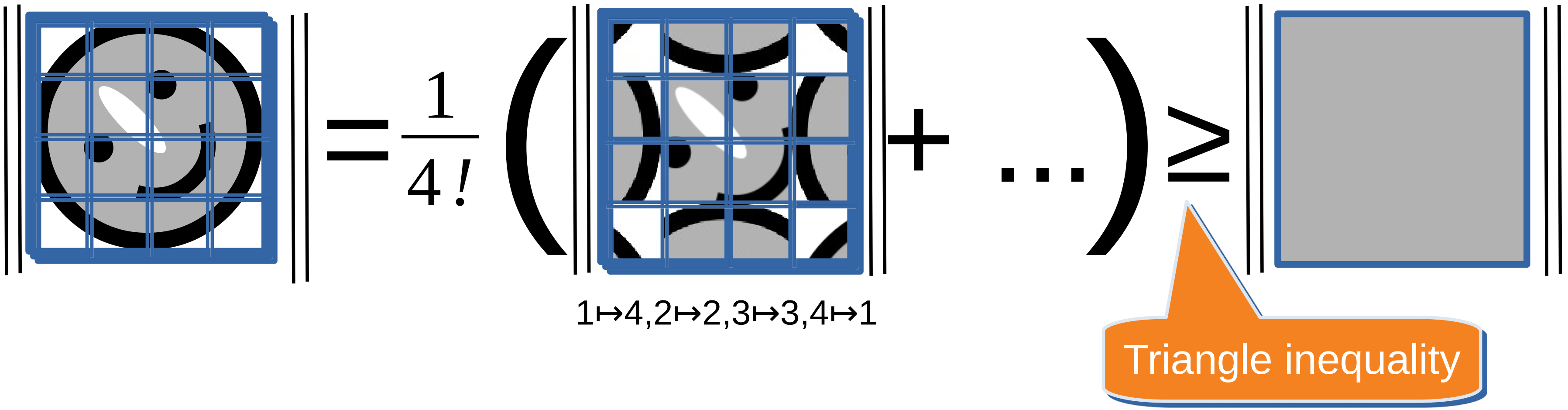
  ...and for nonconstant *W* there is never equality, unless *H* is a forest

Sidorenko’s conjecture and graph norms

- Hatami (2010):

  If *H* is weakly norming then *H* satisfies Sidorenko’s conjecture.

- **Proof:** all the (in)equalities are meant with respect to $\|\cdot\|_{r(H)}$

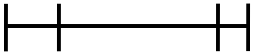


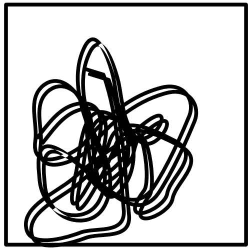
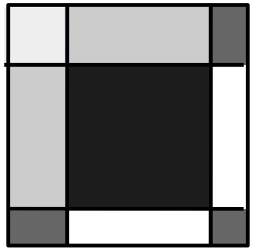
Step Sidorenko and graph norms

- Sidorenko property $t(H,W) \geq t(H,[\![W]\!]^{\bowtie[0,1]})$

- Step Sidorenko property $t(H,W) \geq t(H,[\![W]\!]^{\bowtie P})$

  for every finite partition *P* of the unit interval

P 

 W  $[\![W]\!]^{\bowtie P}$

- weakly norming⇒step Sid
  (same proof as before)

- Kral, Martins, Pach, Wrochna (2020):
  step Sidorenko≠Sidorenko

Step Sidorenko and graph norms

Doležal, Grebík, H, Rocha, Rozhoň

- For a connected graph *H* the following are equivalent
  - *H* has the step Sidorenko property
  - *H* is weakly norming

- If *H* is norming then *H* has the step forcing property

Graph norms and index-pumping

- The Frieze-Kannan weak regularity lemma (1999)

  For every ε>0 there exists *C* such that for each graphon *W* there is a partition *P* of [0,1] with at most *C* parts such that

$$\left\| W - [\![W]\!]^{\bowtie P} \right\|_{\square} < \varepsilon$$

- Proof:
  - Start with trivial *P*. Refine if there is a witness of irregularity.
  - In a refinement step, **index** goes up by *f*(ε)>0.
    - original proof **index**=L_2-norm
    - Gowers **index**=$t(C_4,\bullet)$
    - Doležal, Grebík, H, Rocha, Rozhoň **index**=$t(H,\bullet)$; *H* is norming

Gowers-like norms

- Gowers (2001): uniformity norms
- G a finite abelian group, f: $G \to \mathbb{R}$,

$$\|f\|_{U_k} := \left| \int_x \int_{y_1} \ldots \int_{y_k} \prod_{s \in \{0,1\}^k} f(x + s \cdot y) \right|^{2^{-k}}$$

- Relation to arithmetic progressions of length k+1
- Ongoing with Juškevičius: Which $S \subseteq \{0,1\}^k$ give a **norm?**

$$\|f\|_S := \left| \int_x \int_{y_1} \ldots \int_{y_k} \prod_{s \in S} f(x + s \cdot y) \right|^{1/|S|}$$