Graph limits

Jan Hladký

Institute of Computer Science, Czech Academy of Sciences

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Introduction to dense graph limits

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Dense graph limits (graphons)

Lovász-Szegedy: *Limits of dense graph sequences*, JCTB 2006

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Convergent sequences of dense graphs I, Adv Math 2008

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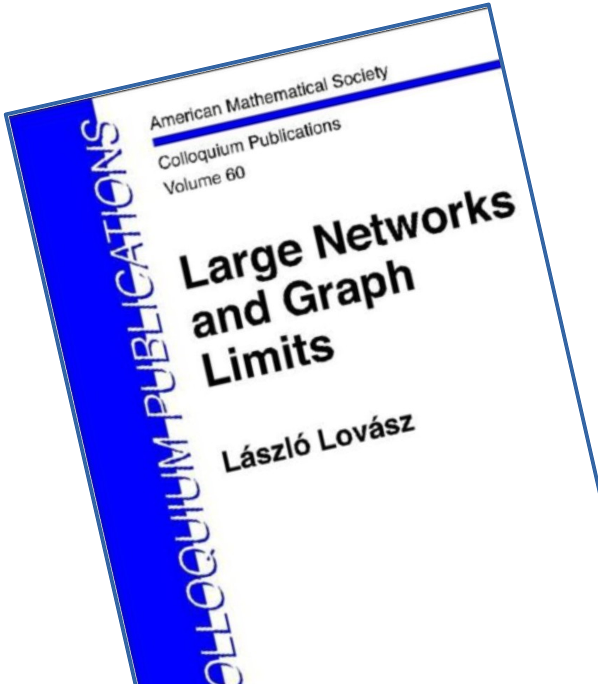
2012 Fulkerson Prize, 2013 Coxeter–James Prize, …



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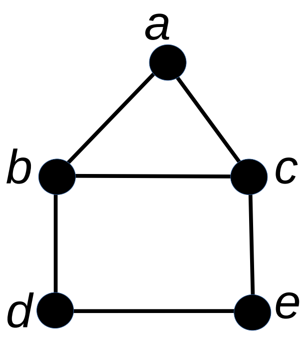
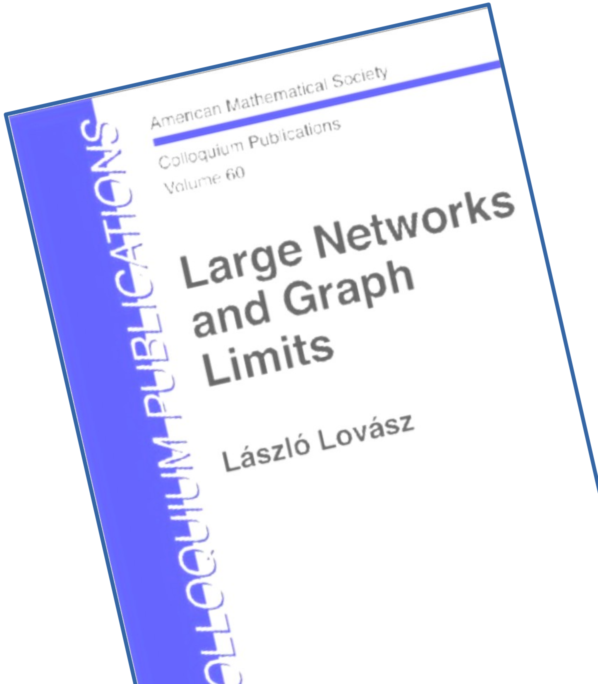
Main idea: Compactify finite graphs $\rightsquigarrow$ "graphons"

Lovász-Szegedy compactness theorem





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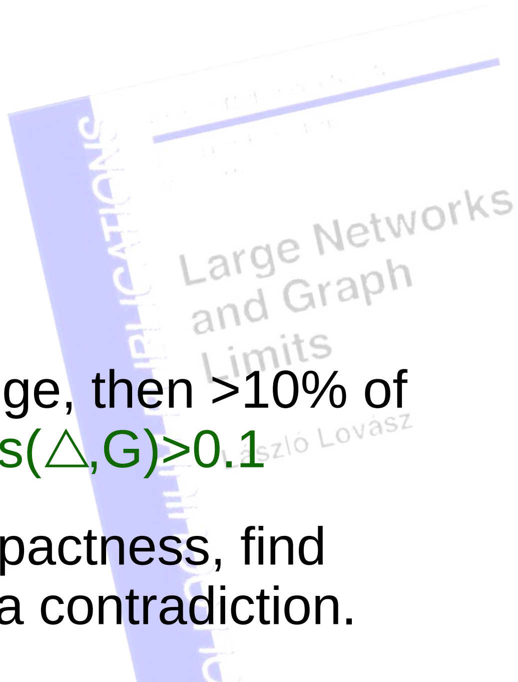


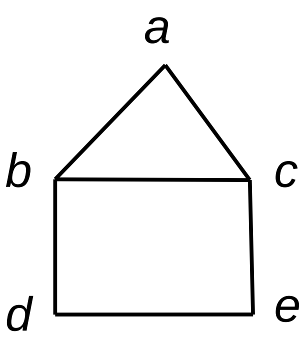
Applications in extremal graph theory:

- Sample Conjecture: If >70% pairs of vertices form an edge, then >10% of triples form a triangle homdens(/ ,G)>0.7 ⟹ homdens(△,G)>0.1
- Suppose for contradiction ∞ counterexamples. Use compactness, find a graphon counterexample. Use analytic tools to derive a contradiction.

Graphons

Main idea: Compactify finite graphs ⇝ graphons // cut norm distance // cut distance

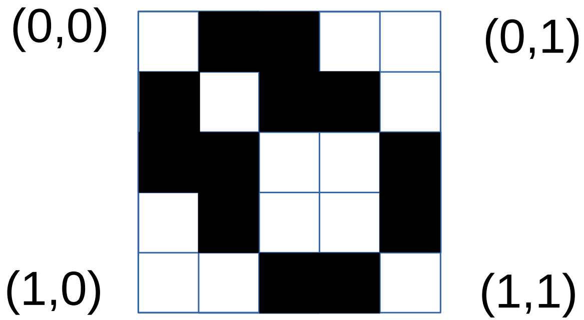
Graph



Adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Graphon representation



Graphon = symmetric Lebesgue measurable function $[0,1]^2 \to [0,1]$

The cut norm topology on graphons

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Cut norm convergence of graphons Graphons $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ converge to W if

$$\lim_{n\to\infty} \sup_{S,T\subset[0,1]} \left\{ \int_{S\times T} \Gamma_n - \int_{S\times T} W \right\} = 0$$

Lovász-Szegedy Compactness Theorem 2006

Graphons equipped with the cut distance (=cut norm distance + an additional step) are a compact space.

Proof: Based on Szemerédi's Regularity lemma

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Doležal-H., 2×(Doležal, Grebík, H., Rocha, Rozhoň)

Use weak* convergence instead

$$\sup_{S,T\subset[0,1]} \left\{ \limsup_{n\to\infty} \int_{S\times T} \Gamma_n - \int_{S\times T} W \right\} = 0$$

Compactness by the Banach-Alaoglu Theorem

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Original graph parameters for graphons

1a. Graph homomorphism counts

Example: triangles

homomorphism of a △ into a graph G

$\varphi: \{a,b,c\} \to V(G)$ such that

$\varphi(a)\varphi(b),\ \varphi(b)\varphi(c),\ \varphi(c)\varphi(a) \in E(G)$

counting homomorphisms via the adjacency matrix

$\hom(\triangle, G) = \sum_a \sum_b \sum_c A_{a,b} \cdot A_{b,c} \cdot A_{c,a}$

2a. Cuts in graphs

Example: MAXCUT

$\mathrm{MAXCUT}(G) = \max_{X \subseteq V(G)} e(X, V(G) \setminus X)$

$= \max_{X \subseteq V(G)} \sum_{a \in X} \sum_{b \in V(G) \setminus X} A_{a,b}$

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1b. Graph homomorphism densities

homomorphism density of triangles in a graphon W

$t(\triangle,W)=\int_a\int_b\int_c W(a,b)\cdot W(b,c)\cdot W(\mathrm{c},a)$

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$\mathrm{MAXCUT}(W)=\sup_{X\subseteq[0,1]}\int_{a\in X}\int_{b\in[0,1]\setminus X} W(a,b)$

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hom(△,G)=$\sum_a \sum_b \sum_c A_{a,}$

1b. Graph homomorphism densities

homomorphism density of triangles in a graphon W

Theorem (2006-2008)
Lovász-Szegedy, Borgs-Chayes-Lovász-Sós-Vesztergombi:

In the cut distance topology

- *Homomorphism densities are continuous,*
- *MAXCUT is continuous.*

2a. Cuts in graphs

Example: MAXCUT

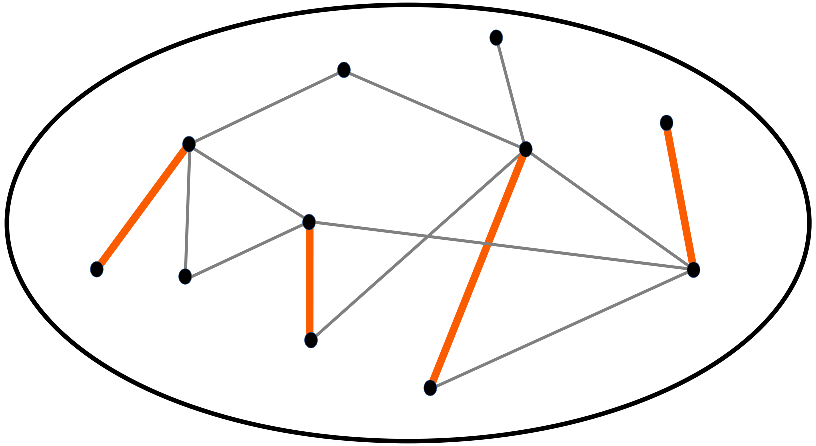
MAXCUT(G)=$\max_{X \subseteq V(G)} e(X, V(G) \setminus X)$

$=\max_{X \subseteq V(G)} \sum_{a \in X} \sum_{b \in V(G) \setminus X} A_{a,b}$

MAXCUT(W)=$\sup_{X \subseteq [0,1]} \int_{a \in X} \int_{b \in [0,1] \setminus X} W(a,b)$

Matchings in graphs and graphons

Matchings in a graph



- **Maximum size matching ➜ matching number**
- Fractional matchings
- LP duality
- Dual notions: (fractional) vertex cover

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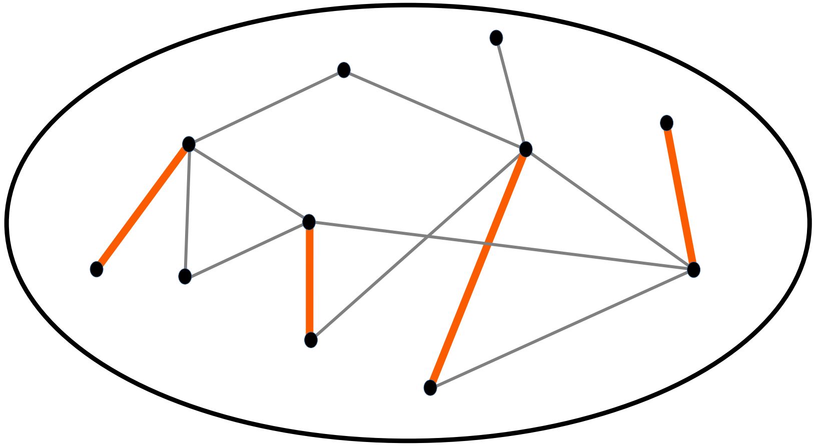

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Matchings in graphons

H.-Hu-Piguet, H. Doležal:

Continuous LP duality➜**matching number** of graphons

Theorem:
The matching number of graphons is lower-semicontinuous with respect to the cut distance topology.

Developed more generally for **tilings**.
Example: △-tiling=vertex-disjoint copies of triangles.

A new proof of **Komlós tiling theorem**.

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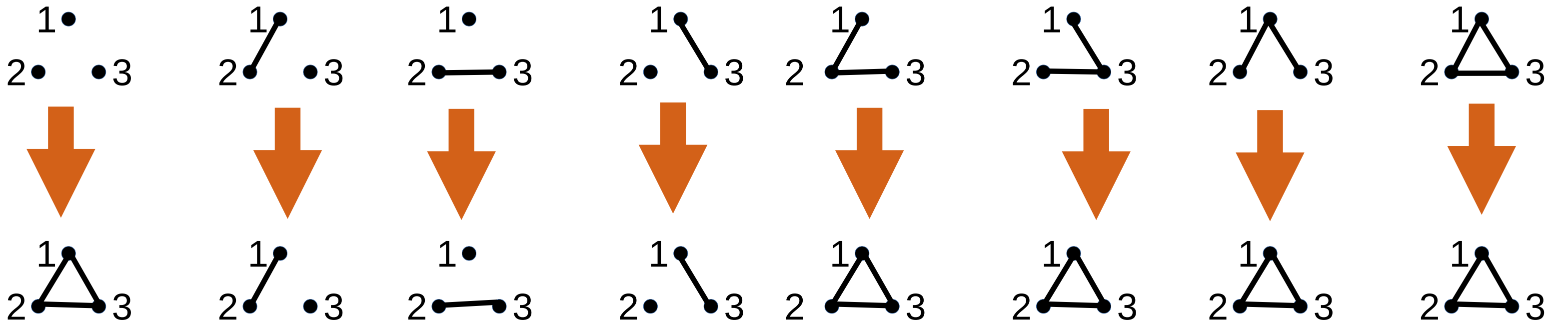
"Flip process" on a graph

A flip process is given by a rule (=transformation table on graphs of a given order k).

Example:

k=3



Flip process (for a fixed replacement rule):

1) Input graph G_0 (quite large)
2) Step i:
   (a) Pick k vertices of G_{i-1} at random
   (b) Apply the replacement rule on the induced subgraph, $G_{i-1} \rightsquigarrow G_i$

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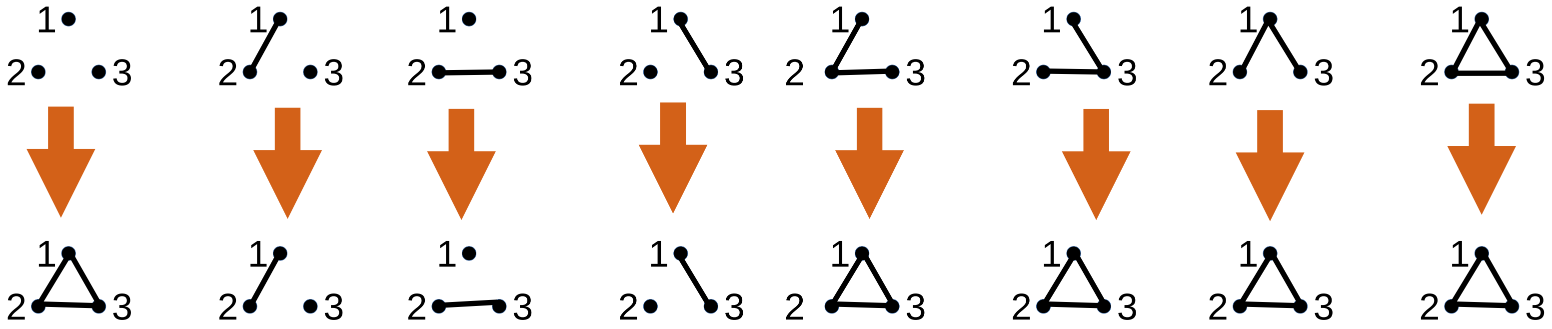
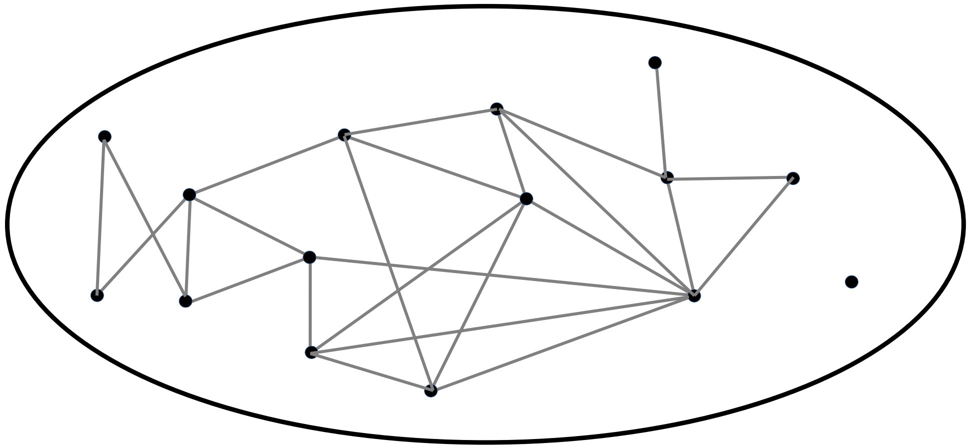


G_0



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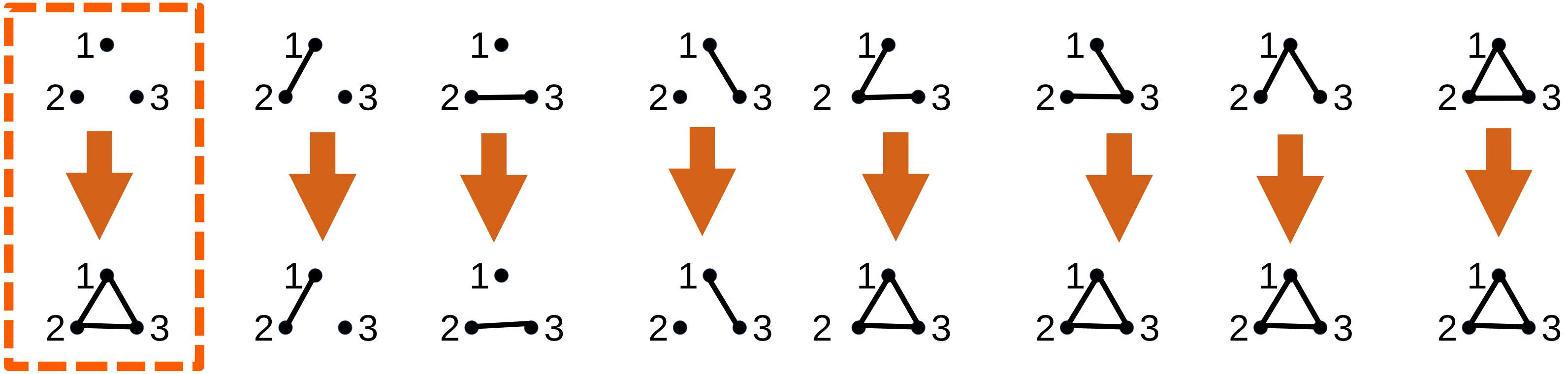

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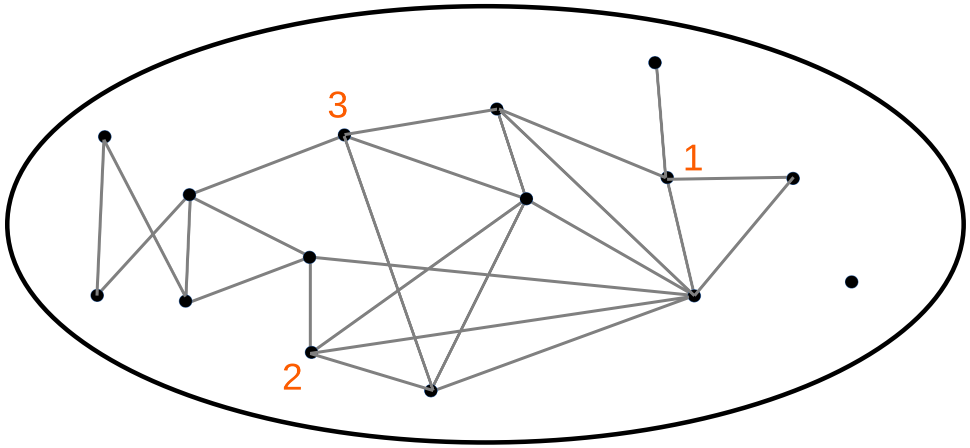

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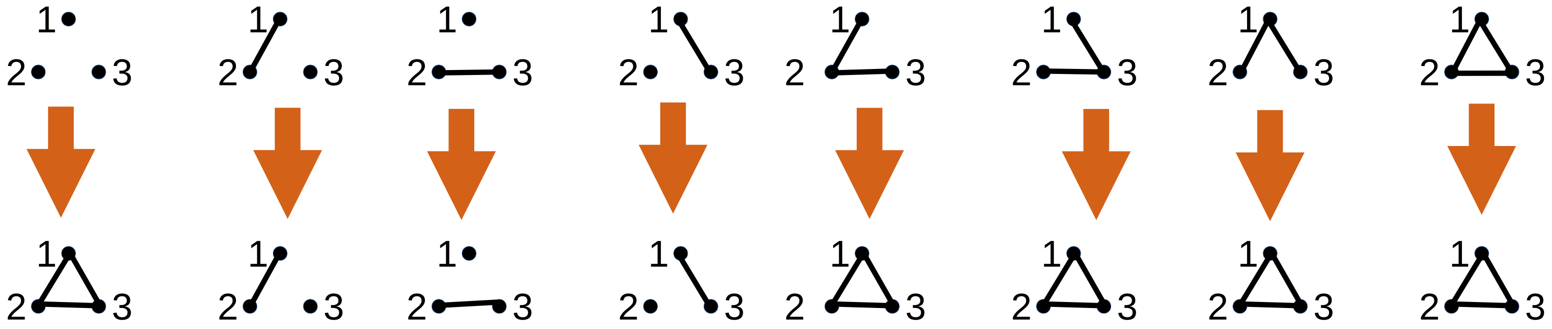
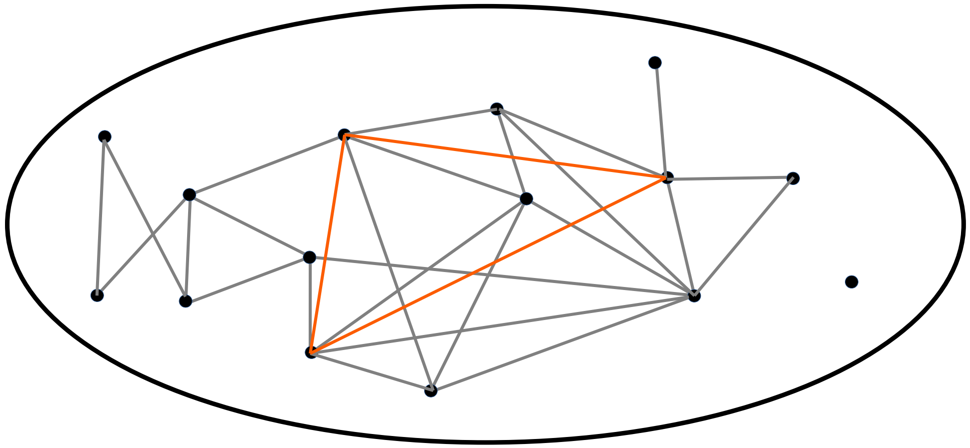


G_1



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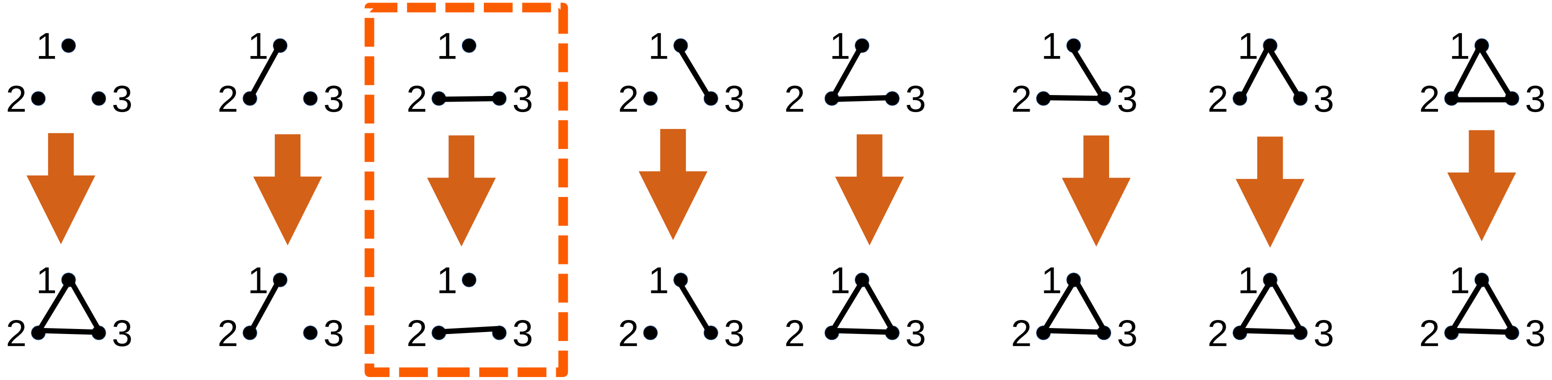

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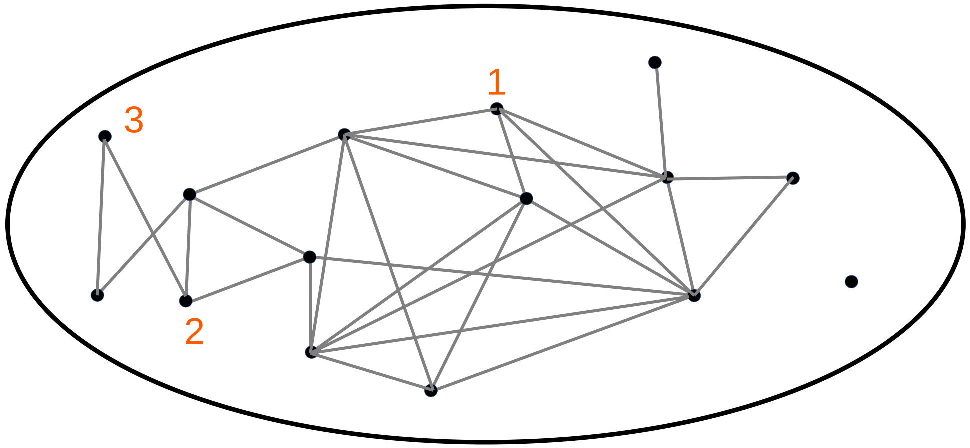

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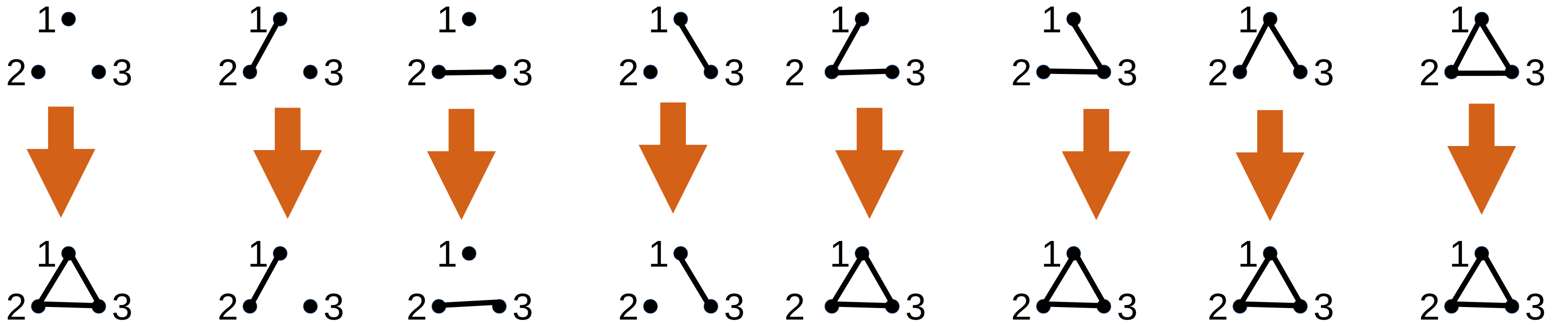
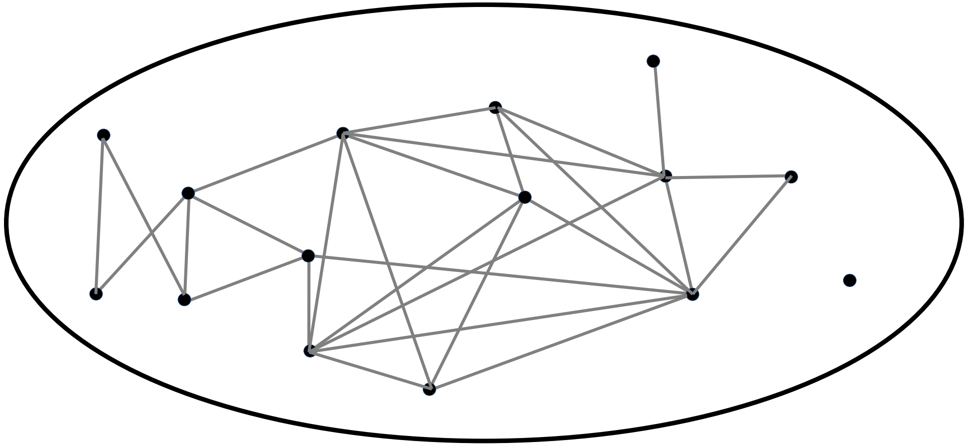


G_2



Flip processes on graphs and graphons

"Triangle removal process" studied by Bollobás-Erdős (1990), …, Bohman-Frieze-Lubotzky (2015)

Theorem [Garbe-H.-Šileikis-Skerman], informal:

For a flip process with a given rule **R**, there exist trajectories, Φ: graphons × [0,∞) → graphons,

so that if G_0 is a large n vertex graph close to graphon W, then G_{cn^2} (i.e., cn^2 random steps) is close to $\Phi(W,c)$

with high probability.

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Limits of discrete structures

	Limit object	Cut distance	Hom densities	Compactness	Minimality of the limit space
Graphs *Borgs-Chayes-Lovász-Sós-Szegedy-Vesztergombi*	graphon	✓	✓	✓	✓ By random sampling
Hypergraphs *Elek-Szegedy* *Zhao*	hypergraphon	?	✓	✓	✓ By random sampling
Permutations *Hoppen-Kohayakawa-Moreira-Ráth-Sampaio*	permuton	✓	✓	✓	✓ By random sampling
Latin squares Garbe-Hancock-H.-Sharifzadeh	Latinon	✓	✓	✓	✓ Semirandom construction