Packing Trees; Gracefully Labelling Trees

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Packings

Two graph F_1, F_2 *pack* into a graph G if there exist injective homomorphisms $\phi_1 : F_1 \to G$, $\phi_2 : F_2 \to G$ such that $E(\phi_1) \cap E(\phi_2) = \emptyset$.

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Conjecture (Ringel, 1963)

Any $2n+1$ identical copies of any tree of order $n+1$ pack into K_{2n+1}.

Conjecture (Gyarfás-Lehel, 1978)

Let $T_1, \ldots, T_n$ be a family of trees, $v(T_i) = i$. Then $T_1, \ldots, T_n$ pack into K_n.

Best possible: total number of the edges in the trees equals the number of edges of the host graph

Graceful labeling of trees

Suppose that a graph G and a map $h : V(G) \to \mathbb{R}$. Then *h induces label* $|h(x) - h(y)|$ on the edge $xy \in E(G)$.

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Suppose that T is an n-vertex tree. A map $h : V(T) \to [n]$ is a *graceful labelling* if it is injective and the induced edge-labels are pairwise distinct.

Conjecture (Kotzig-Ringel-Rosa "Graceful Tree Labeling C.", 1967)

For every tree T there is a labeling which is graceful.

Graceful Tree Labeling Conjecture $\Rightarrow$ Ringel Conjecture

The results

Theorem (Böttcher, H., Piguet, Taraz; arXiv:1404.0697, Israel J Math+)

For every $\Delta \in \mathbb{N}$ and $\epsilon > 0$ there exists n_0 such that for every $n > n_0$ we have:
Let $T_1, \ldots, T_k$ be a family of trees, $v(T_i) \leq n$, $\Delta(T_i) \leq \Delta$, $\sum e(T_i) \leq \binom{n}{2}$. Then $T_1, \ldots, T_k$ pack into $K_{(1+\epsilon)n}$.

Theorem (Adamaszek, Adamaszek, Allen, Grosu, H.)

For every $\Delta \in \mathbb{N}$ and $\epsilon > 0$ there exists n_0 such that for every $n > n_0$ we have:
Let T be an n-vertex tree with maximum degree $\leq \Delta$. Then there exists a graceful labeling (i.e., vertex labels distinct, induced edge labels distinct) with $(1+\epsilon)n$ labels.

Packing trees: The proof

Theorem (Böttcher, H., Piguet, Taraz)

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Proof attempt: randomized embedding algorithm

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Proof:

- glue the trees $\Rightarrow$ family $T'_1, \dots, T'_\ell$, $n/2 \le v(T'_i) \le n$
- cut each T'_i into almost equi-sized layers $L_i^{(1)}, \dots, L_i^{(R)}$
- inductively, in steps $s = 1, 2, \dots, R$:
  - embed all the trees $\{L_i^{(s)}\}_{i=1}^{\ell}$ (no updates)
  - preserve quasirandomness of the graph $K^{(s)}$ formed by unused edges

"limping random walks", analysis for true random walks would probably be possible (Barber–Long'14)