Extremal Graph Theory

Jan Hladký

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An alternative definition: substructures in graphs

In this talk:

- Turán's Theorem
- Erdős–Sós Conjecture
- Szemerédi Regularity Lemma

Mantel 1907/Turán 1941 G has n vertices

If G has more than $n^2/4$ edges then it contains a triangle.

- optimal $\Rightarrow$ extremal graph
- starting point of extremal graph theory
- Aigner 1995: *Turán's graph theorem*, 6 proofs.

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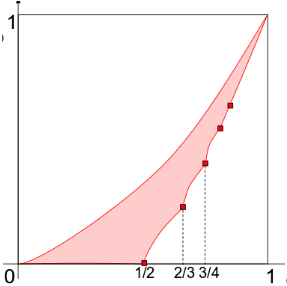
Extensions:

- other graphs than the triangle (Turán, Erdős-Stone 1964)
- 3-uniform hypergraphs (still open!!!)
- "triangle density problem"
  Alexander Razborov, 2013 Robbins Prize (AMS)



(taken from a presentation by Lovász)

Erdős–Sós Conjecture

Setting
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$\mathcal{T}_\ell$ . . . all trees on ℓ vertices

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Embedding trees: motivation $\delta(G) \geq k$, then $\mathcal{T}_{k+1} \subset G$.

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Loebl-Komlós-Sós Conjecture '95 If at least $n/2$ of the vertices of G have degrees at least k, then $\mathcal{T}_{k+1} \subset G$.
approximate solution by H., Piguet, Komlós, Simonovits, Stein, Szemerédi

Szemerédi Regularity Lemma

Szemerédi 1975: Dense subsets of integers contain arithmetic progressions of arbitrary length

If $A \subset \mathbb{N}$ such that $\limsup_n \frac{|A \cap \{1,\ldots,n\}|}{n} > 0$ then

$\forall k$ there exists $a_0, d \in \mathbb{N}$ such that

$a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (k-1)d \in A$.

History: 1953 Roth $k = 3$; 1977 Furstenberg (ergodic theory)

Szemerédi 1978: Regularity lemma Every graph can be decomposed into a bounded number of quasirandom pieces

Ruzsa and Szemerédi 1976: Removal lemma

easy consequence of the Regularity lemma (next slide)

2012: Abel Prize to Szemerédi

2002-2007: Hypergraph regularity lemma

Rödl, Schacht, Skokan, . . . ; Gowers

2012 Pólya Prize to Rödl and Schacht

Removal Lemma

Ruzsa and Szemerédi 1976: (Triangle) Removal lemma:

If a graph contains few triangles then it can be made triangle-free by removing few edges.

For every $\epsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds.

If G is an n-vertex graph ($n > n_0$) which has at most δn^3 triangles then there is a set of at most ϵn^2 edges deletion of which makes G triangle-free.

Regularity-lemma free proof: Fox (Annals Math 2012)

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Application I: Property testing

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Application I: Property testing

Application II: Roth's Theorem: Dense sets contain 3-AP's

(Version I) If $A \subset \mathbb{N}$ such that $\limsup_n \frac{|A \cap \{1,\ldots,n\}|}{n} > 0$ then there exists $a_0, d \in \mathbb{N}$ such that $a_0, a_0 + d, a_0 + 2d \in A$.

(Version II) For every $\alpha > 0$ there exists n_0 such that the following holds. $A \subset \{1, \ldots, n\}$ (for some $n > n_0$) $|A| > \alpha n$ then there exists $a_0, d \in \mathbb{N}$ such that $a_0, a_0 + d, a_0 + 2d \in A$.