Graphons

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Graphons

- 2004 Lovász, Szegedy // Borgs, Chayes, Lovász, Sós, Vesztergombi
- **Main idea:** Compactify the space of finite graphs.
- **Definition:** A graphon is a symmetric measurable function

$$W:[0,1]^2 \to [0,1]$$

- **Graph as graphon** via adjacency matrix
- What do **values between 0 and 1** represent?
- **Cut norm distance** (simplified!)

$$d_\square(U,W) := max_{S,T\subset[0,1]} \left|\int_{S\times T} U - \int_{S\times T} W\right|$$

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- **Idea I:** *For each sequence of graphs there is a subsequence and a limit graphon to that subsequence.*

- **Idea II:** *Many important parameters are continuous.*

Flip processes

- *Garbe-Hladký-Šileikis-Skerman* fundaments of the theory
- *Araújo-Hladký-Hng-Šileikis* specific flip processes
- *Hng* uniqueness of trajectories

$\binom{n}{2}$

- *Hladký-Řada, in preparation* permutations

Erdos-Renyi random graph process

- G(n,p) *binomial Erdos-Renyi random graph*
  - n vertices, insert each potential edge with probability p
  - For this talk, $p \in (0,1)$ fixed
- G(n,m) uniform *Erdos-Renyi random graph*
  - Uniformly random graph with m edges.
  - For $m=pn^2/2$; $G(n,p) \approx G(n,m)$
- *Erdos-Renyi random graph process* (n vertices) $G_0, G_1, \ldots, G_{\binom{n}{2}}$
  - G_0 is edgeless, G_{r+1} is obtained from by turning a randomly selected nonedge into an edge
- With high probability, everything on this slide is *quasirandom*

Triangle removal process

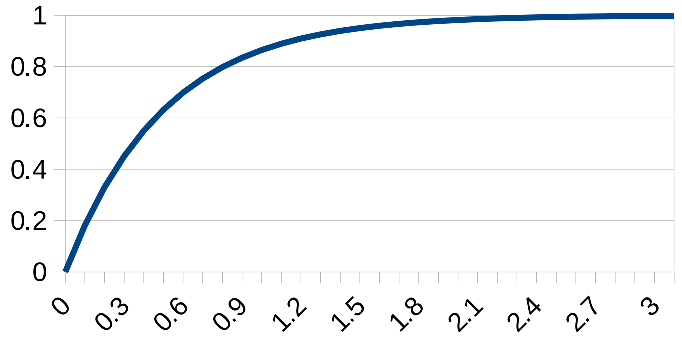
- Introduced by Bollobas-Erdos'90
- Start with G_0=clique
- In step r, pick a random triangle of G_r and delete it
- Bohman-Frieze-Lubetzky'15: *Triangle removal process typically terminates when there are $n^{3/2+o(1)}$ edges left.*
  - Key in the proof: quasirandomness during the evolution

Erdos-Renyi *flip* process

- Start with a graph G_0 (for now the edgeless graph)
- In each step, “replace” a uniformly chosen **pair** with an edge
- Density computation for G_r, $r = \alpha n^2$:

$$P[uv \text{ is an edge}] = 1 - P[uv \text{ is not an edge}]$$

$$\ldots = 1 - \left(1 - \frac{1}{\binom{n}{2}}\right)^r \approx 1 - \exp(-2r/n^2) = 1 - \exp(-2\alpha)$$



Triangle removal *flip* process

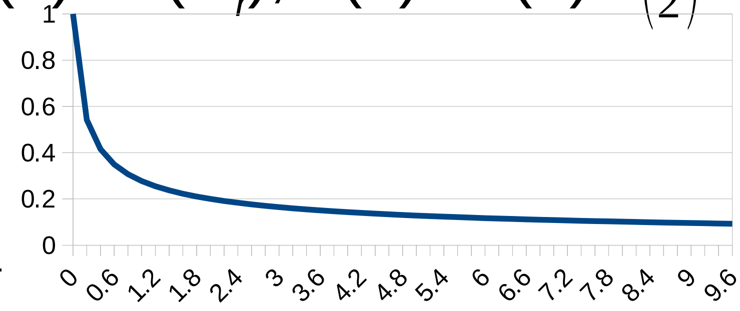
- Start with a graph G_0 (for now the complete graph)
- In each step r pick three random vertices u_1, u_2, u_3,
- If $G_r[u_1, u_2, u_3]$ induces a triangle then remove it…

  ...otherwise $G_{r+1} := G_r$.
- Density computation: G_r, $r=\alpha n^2$, $e(\alpha)$:=e(G_r), $d(\alpha):=e(\alpha) / \binom{n}{2}$

$$P[u_1 u_2 u_3 \, is \, a \, triangle] \approx d(\alpha)^3$$

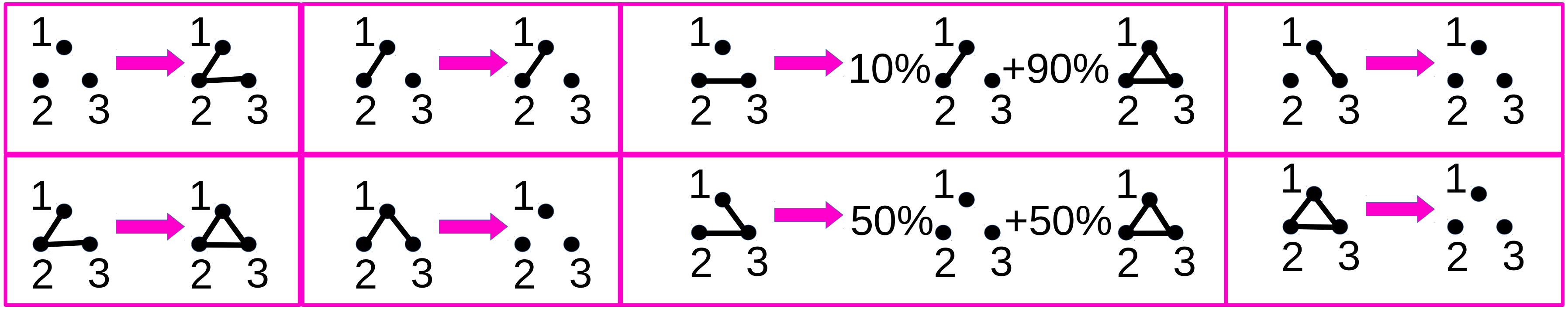
$$e(\alpha+\epsilon)-e(\alpha)\approx -3d(\alpha)^3 \cdot \epsilon n^2$$

$$\frac{d(\alpha)}{\partial \alpha} = -6d(\alpha)^3 \Rightarrow d(\alpha)=\frac{1}{\sqrt{1+12\alpha}}$$



Flip process of order k (here, k=3)

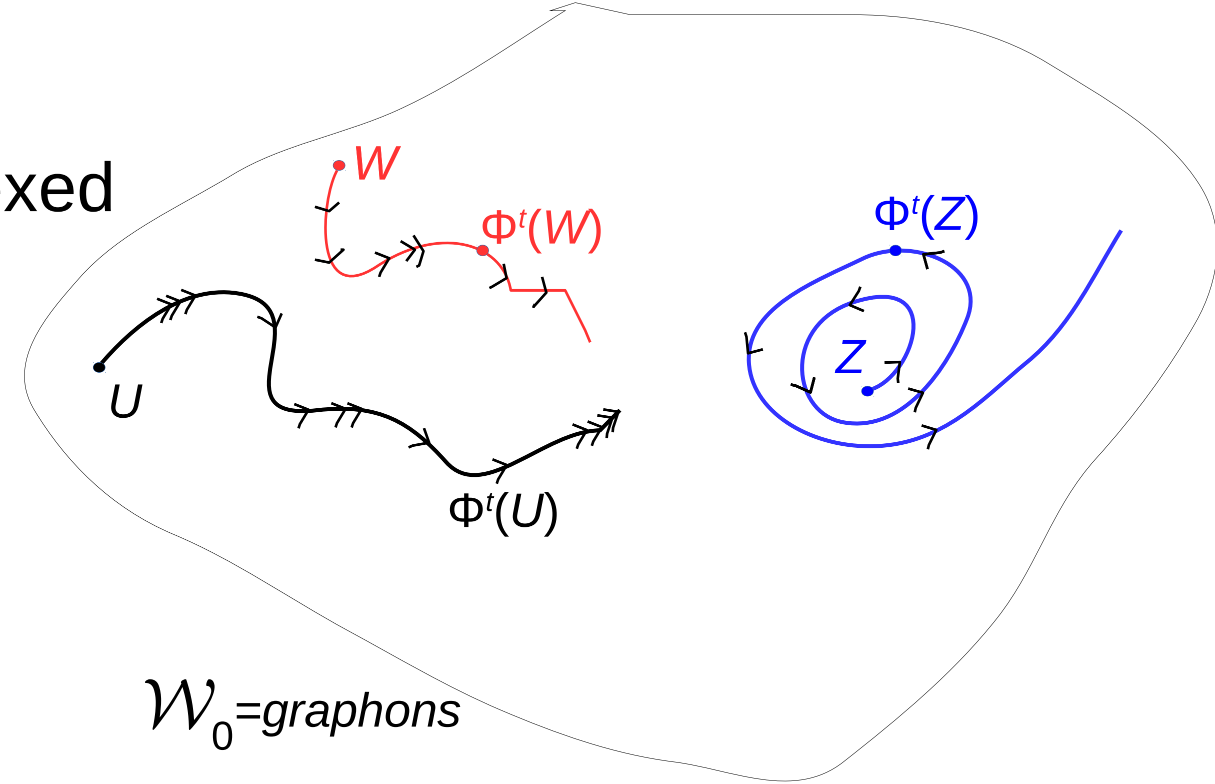
- Rule $\mathcal{R}$



- Start with a (large) graph G_0
- Step $G_r \Rightarrow G_{r+1}$: Sample k vertices and replace the induced graph according to $\mathcal{R}$

Trajectories

- Fixed rule $\mathcal{R}$ of order k
- We construct time-indexed trajectories $\Phi:\mathcal{W}_0\times[0,\infty)\rightarrow\mathcal{W}_0$



Transference theorem (law of large numbers)

Given $\mathcal{R}$ and corresponding

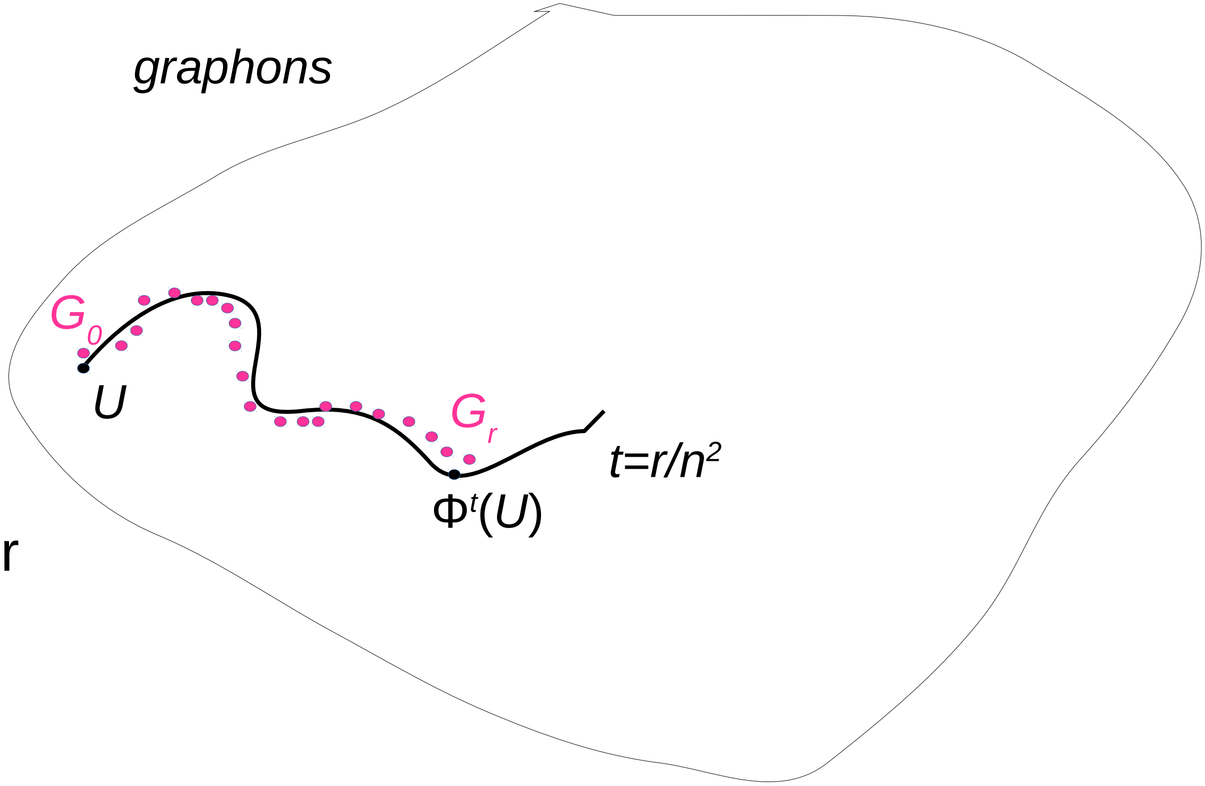
trajectories $\Phi:\mathcal{W}_0\times[0,\infty)\to\mathcal{W}_0$,

whenever a large n-vertex G_0

is close to U (in cut norm)

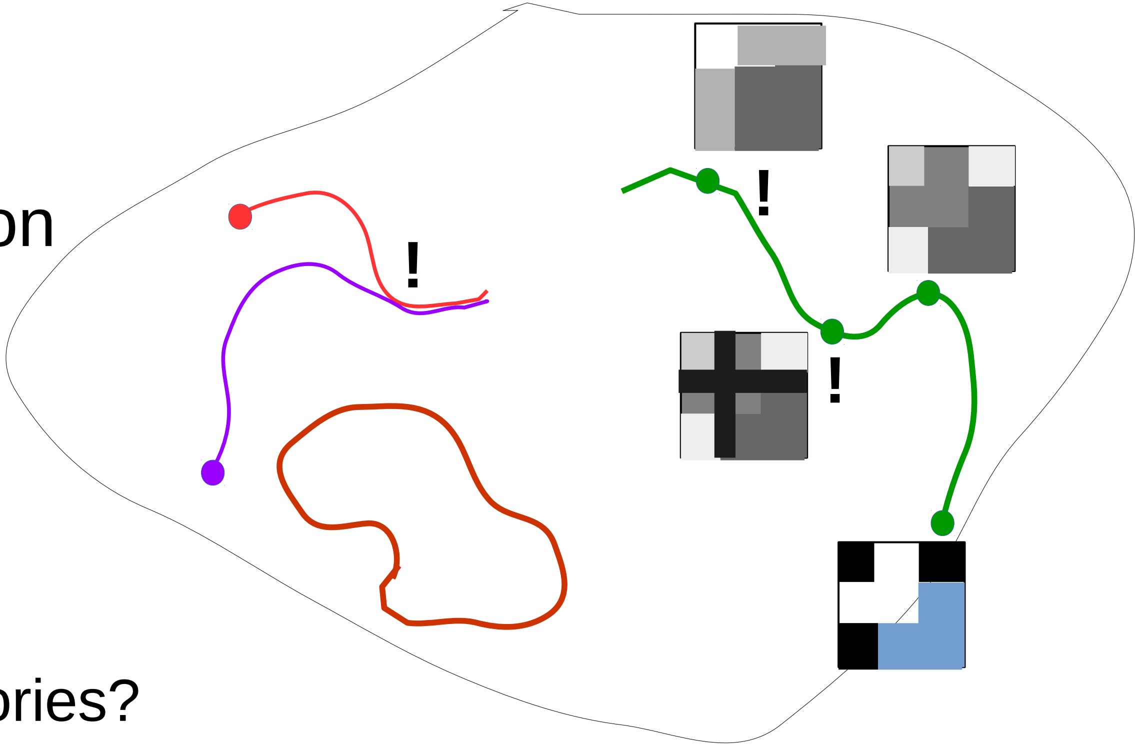
then w.h.p. G_r is close to $\Phi^t(U)$ for

$t:=r/n^2$



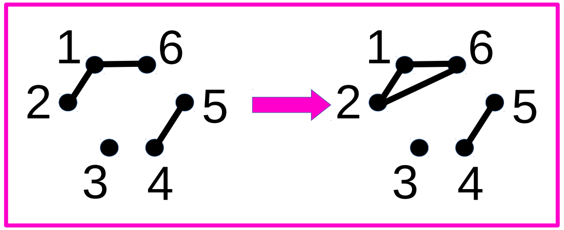
What is all this good for ?

- No confluences
- Going back in time
- Block structure preservation
- Limits $t \to \infty$:
  - Stable and unstable fixed points (often constants)
  - Periodic trajectory
  - Really complicated trajectories?
- Speed of convergence

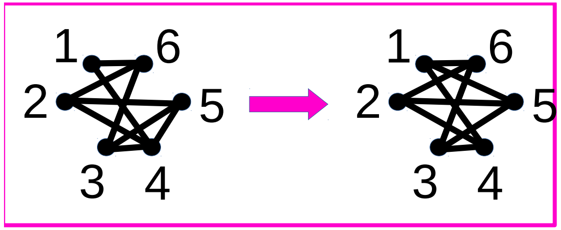


More examples of flip processes

- Ignorant flip process
- Removal flip process
- Complement flip process
- Component completion flip process
- The stirring flip process
- The extremist flip process
- The polarizing flip process



Component completion



Polarizing