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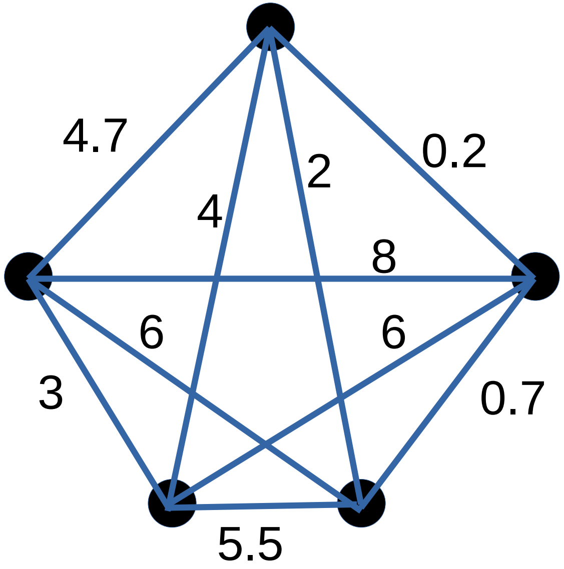
Random minimum spanning tree and dense graph limits

with Gopal Viswanathan
arXiv: 2310.11705

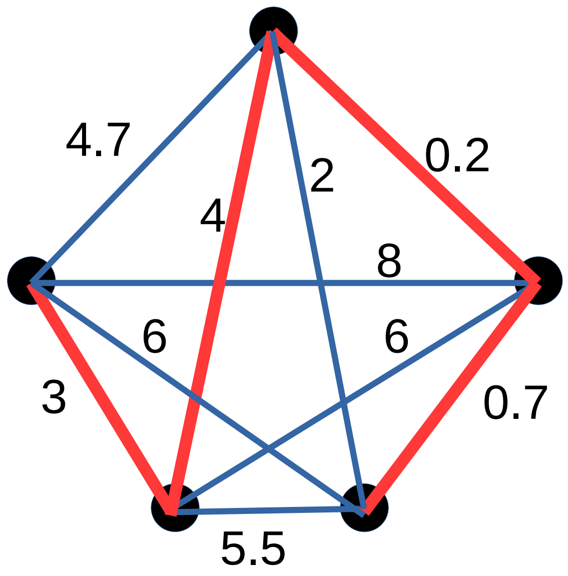
Outline

- **Minimum spanning tree & Kruskal’s algorithm**
- **Frieze’s Theorem**
- **Our result**
- **Dense graph limits**
- **Inhomogeneous branching processes**

- Minimum spanning tree



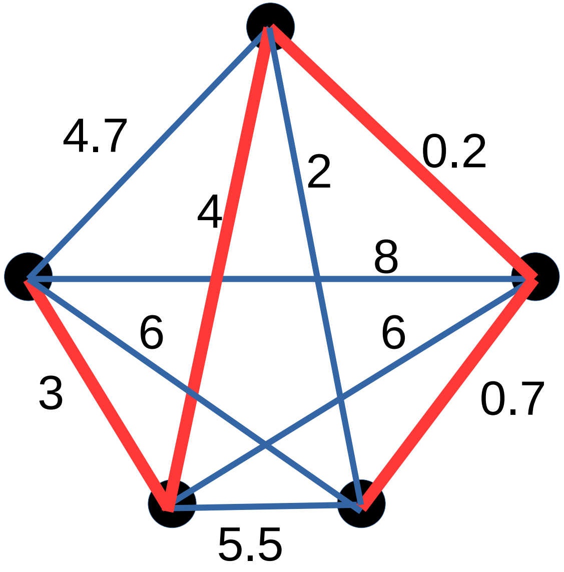
- Minimum spanning tree



- Minimum spanning tree
- Kruskal's algorithm (1956)

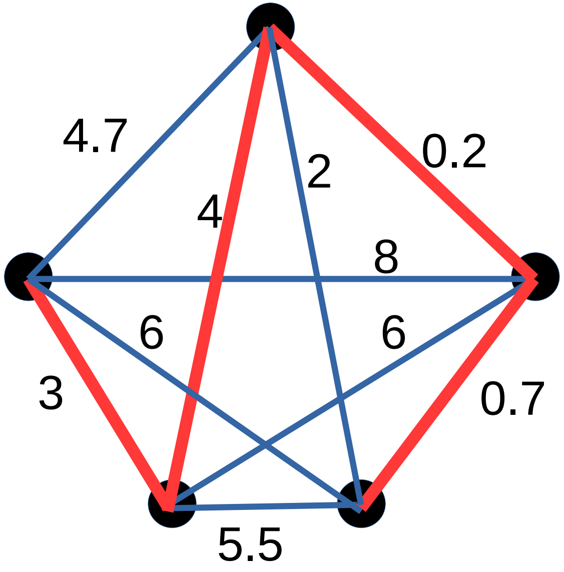
Start with T=∅. Order the edges from the lightest to the heaviest. Sequentially, include to T each edge which decreases number of components (⟺does not create a cycle).

Output: minimum spanning tree T



✓ Greedy strategy works
✗ e.g. perfect matchings

- Minimum spanning tree
- Kruskal's algorithm (1956)
- Unweighted graph $G^{\leq x}$



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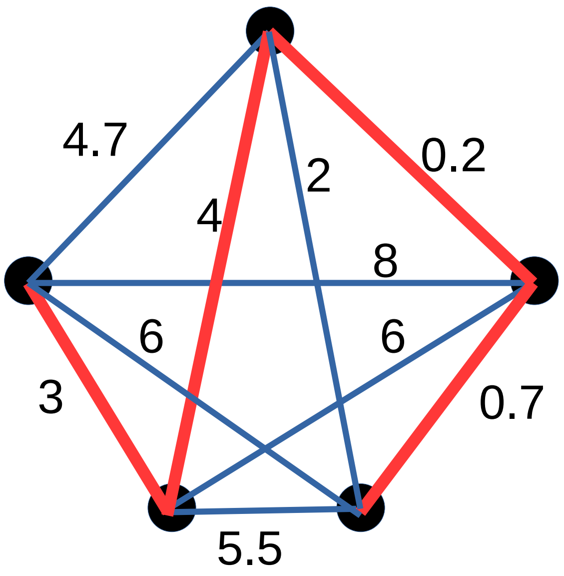

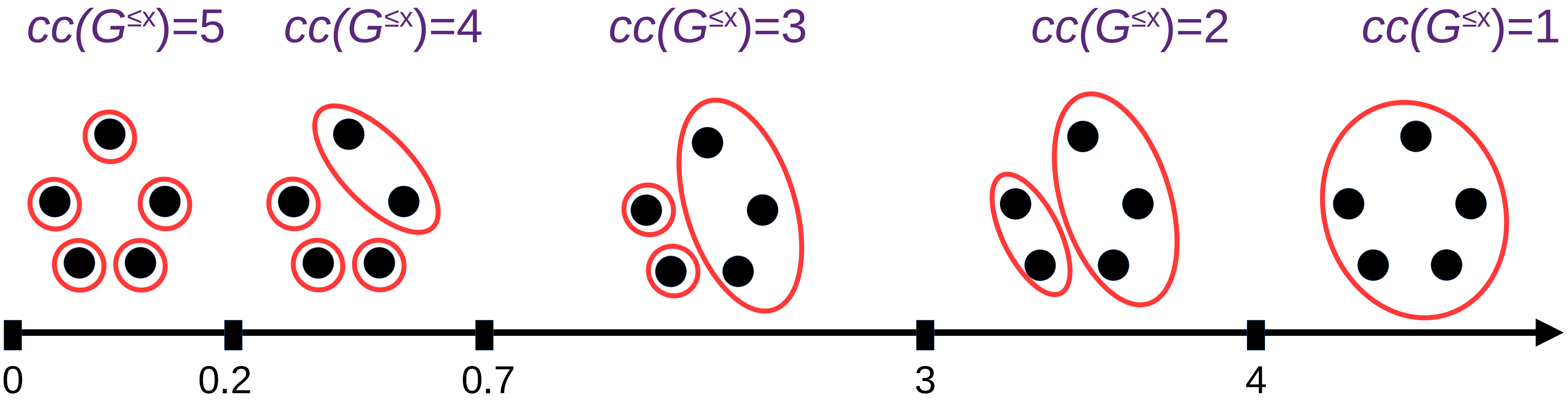

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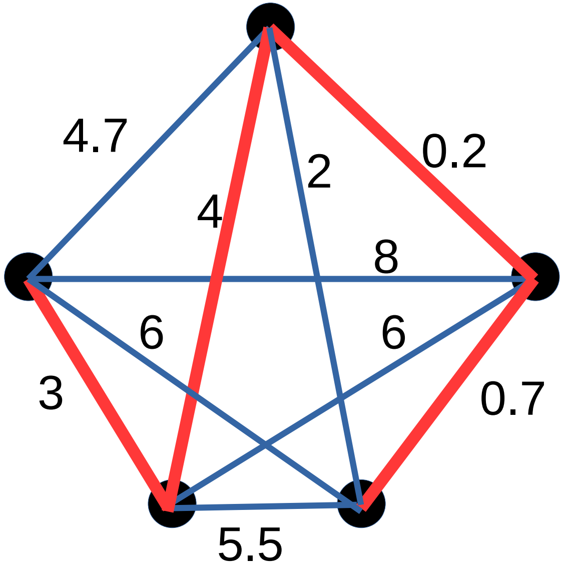

$$MST(G) = \int_{x=0}^{\infty} \left(cc\left(G^{\leq x}\right) - 1 \right)$$

Where does the bulk of the contribution come from?

Theorem (Frieze, 1985)

Put UNIFORM[0,1] weights on the edges on K_n.

Then MST converges to $\zeta(3)=1.202$ in probability as n tends to infinity.

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Then MST converges to ζ(3)=1.202 in probability as *n* tends to infinity.

Proof:

- $$MST(K_n) = \int_{x=0}^{\infty} \left(cc\left(K_n^{\cdot \le x}\right) - 1\right) = \int_{x=0}^{1} \left(cc\left(K_n^{\cdot \le x}\right) - 1\right)$$
- $K_n^{\le x}$ is the Erdős-Rényi random graph **G**(n,x)
- $$MST(K_n) = \int_{p=0}^{1} \left(cc\left(\mathbf{G}(n,p)\right) - 1\right) \approx \int_{p=0}^{999/n} \left(cc\left(\mathbf{G}(n,p)\right) - 1\right)$$

Theorem (Frieze, 1985)

Suppose that *D* is a probability distribution on $[0,\infty)$. Let *f* be its cumulative distribution function and suppose $C:=f'(0)>0$.

Use *D* for the weights of the edges of K_n.

Then MST converges to $\zeta(3)/C$ in probability as *n* tends to infinity.

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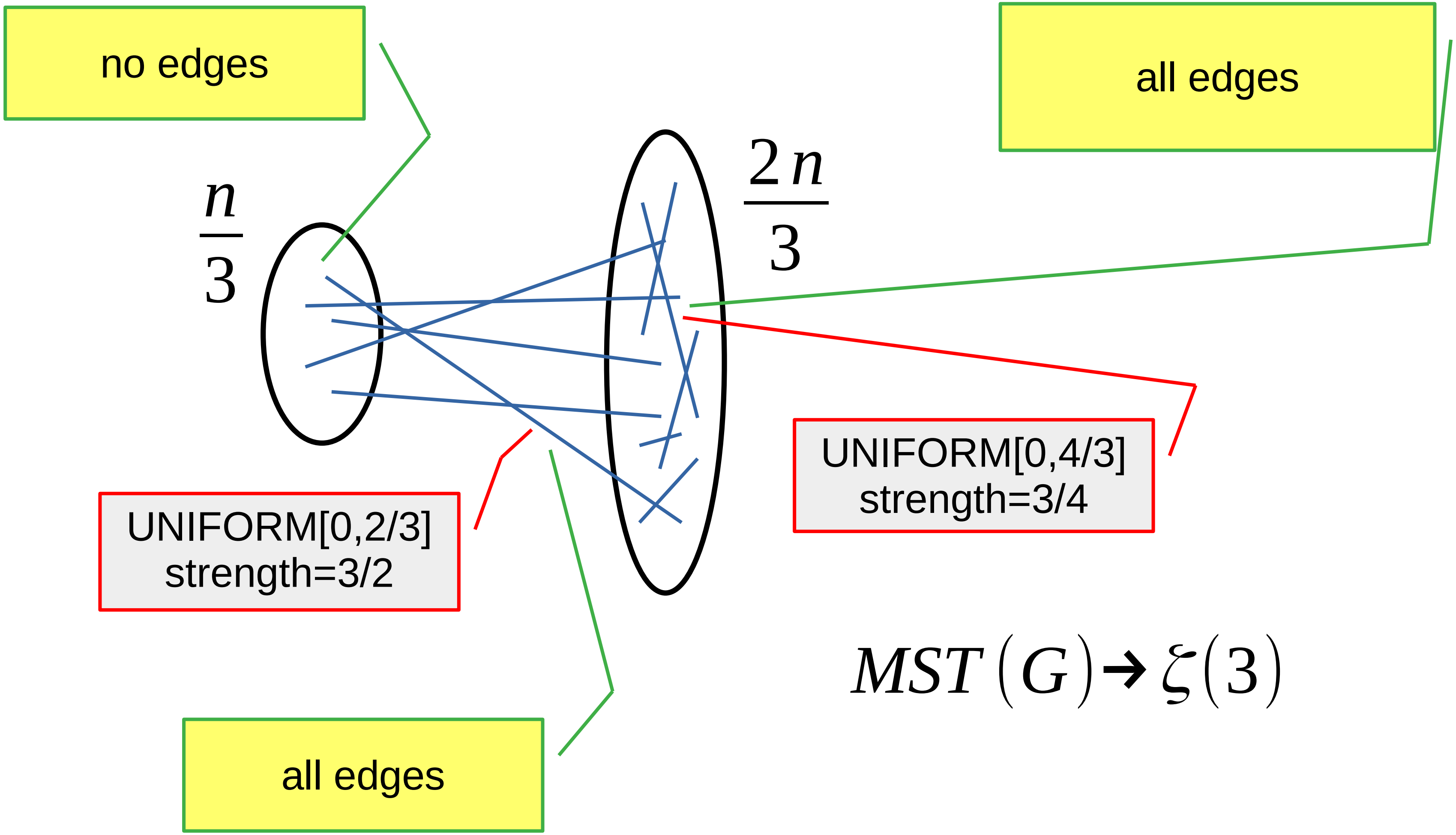
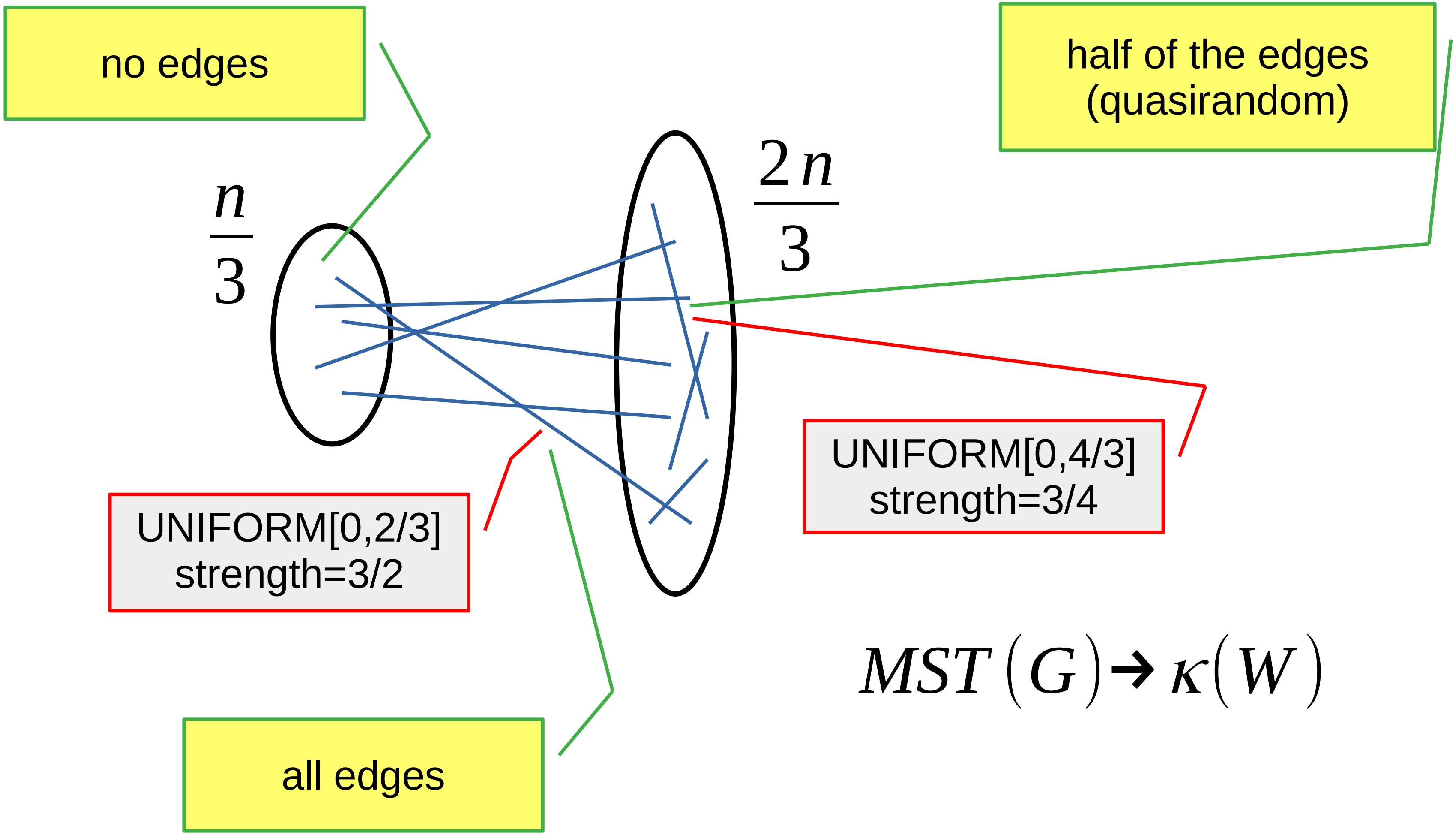
“**strength** of a distribution”: derivative of the distribution function at 0

Theorem (Frieze-McDiarmid, 1989)



$$MST(G) \rightarrow \zeta(3)$$

Theorem (H.-Viswanathan, 2023+)



$$MST(G) \rightarrow \kappa(W)$$

Theorem (H.-Viswanathan, 2023+)

Suppose that (G_n) is a sequence of graphs.

Each edge of each graph is equipped with a probability distribution (+conditions).

(G_n) converge (including strengths) to a graphon/kernel W.

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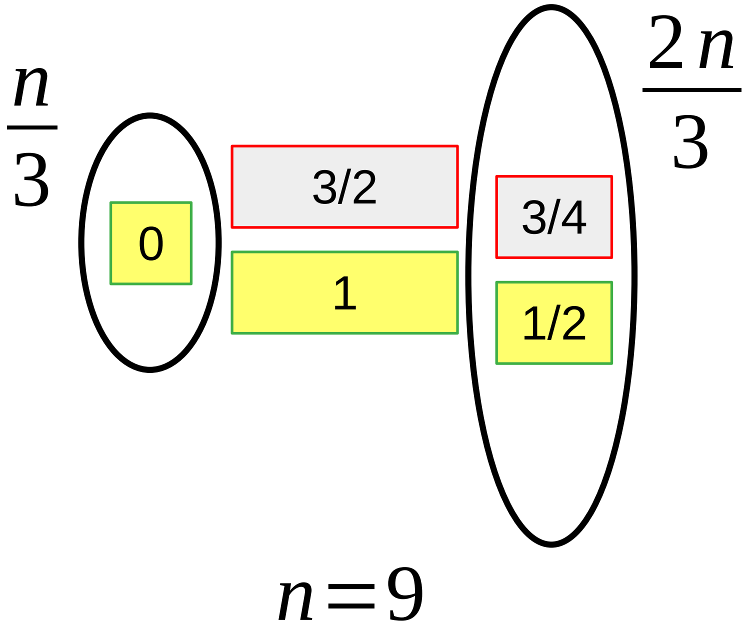
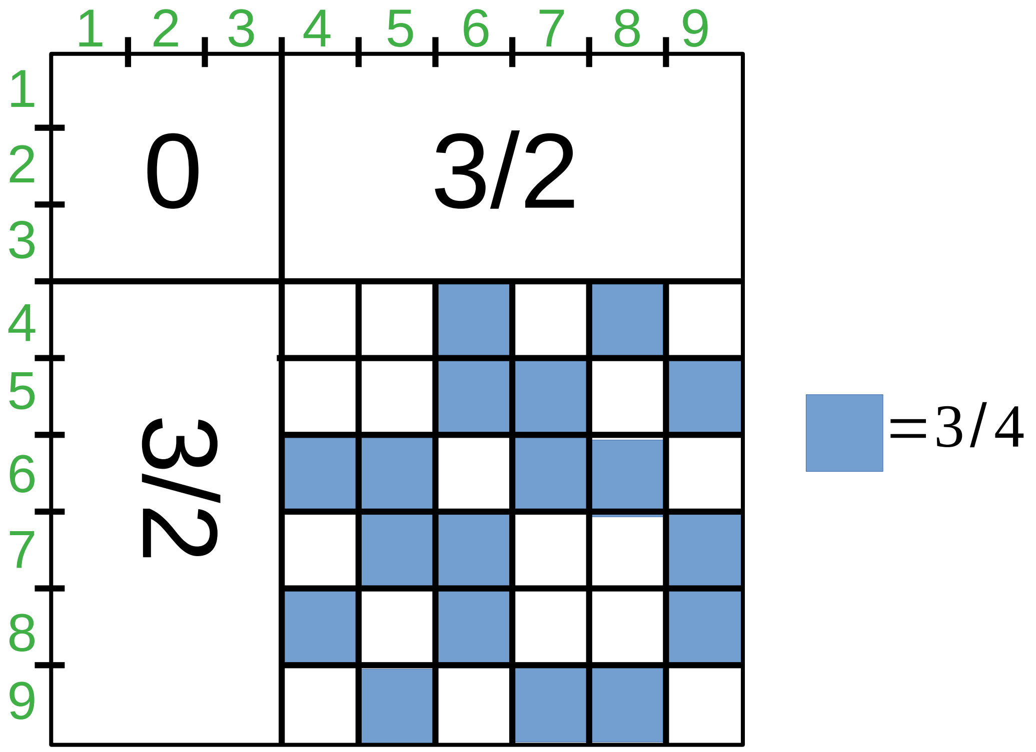
Dense graph limits, starting around 2004
Lovász, Szegedy, Borgs, Chayes, Sós, Vesztergombi

Dense graph limits // Cut distance convergence

Four steps: **(1) finite graph**, **(2) adjacency matrix**

(3) function on $[0,1]^2$ = graphon/kernel representation, **(4)** limit step





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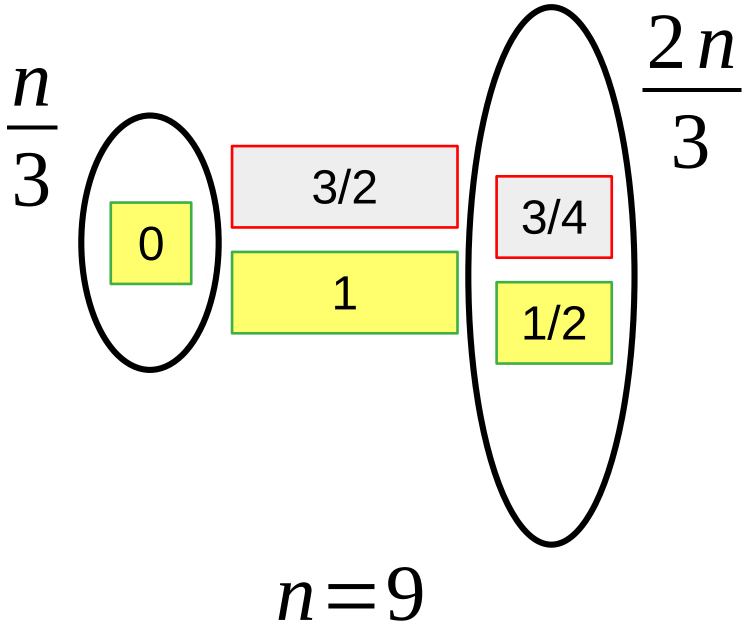

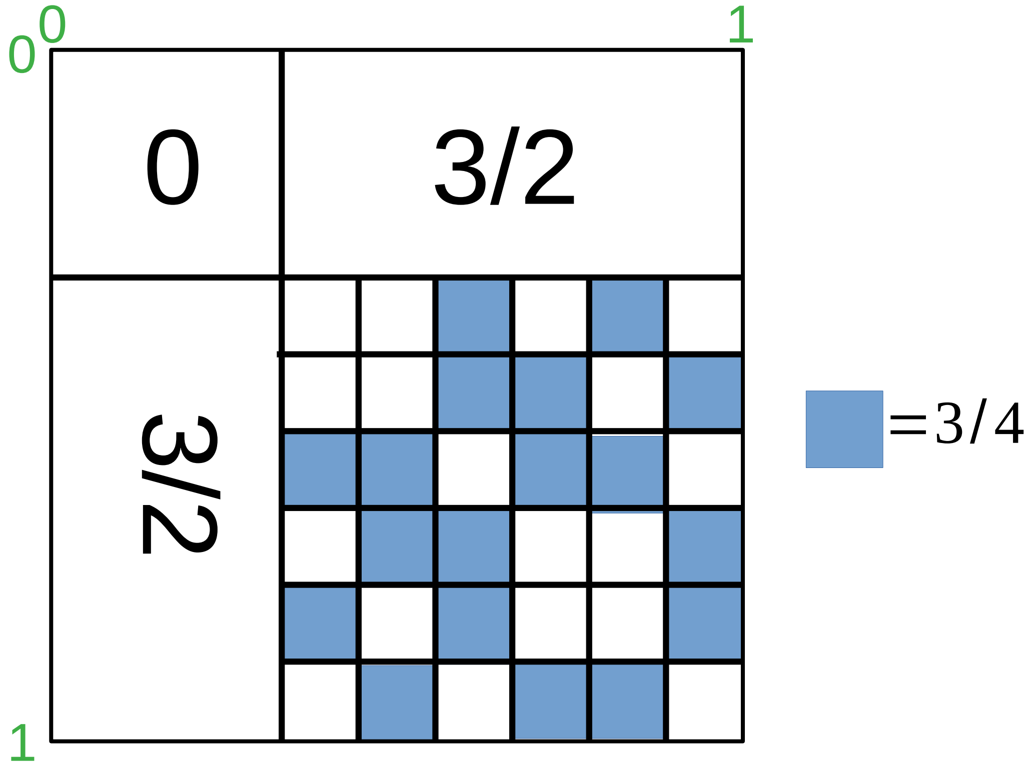

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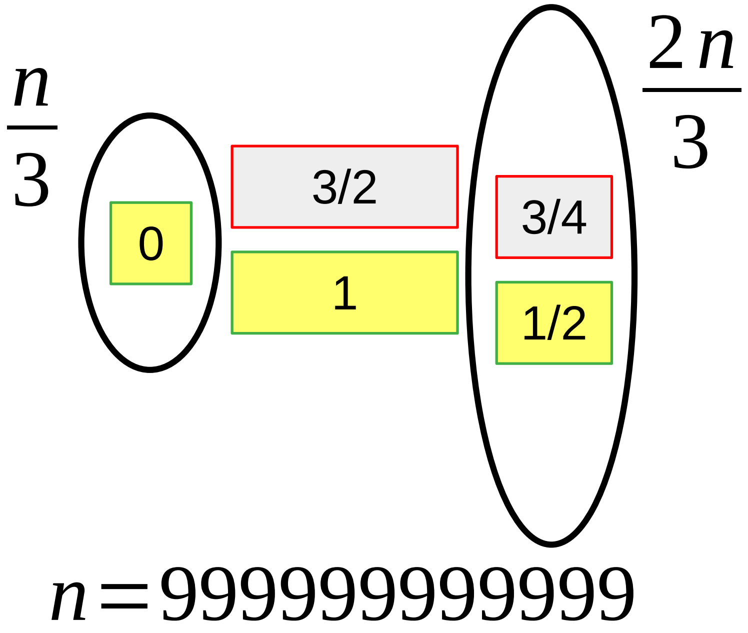

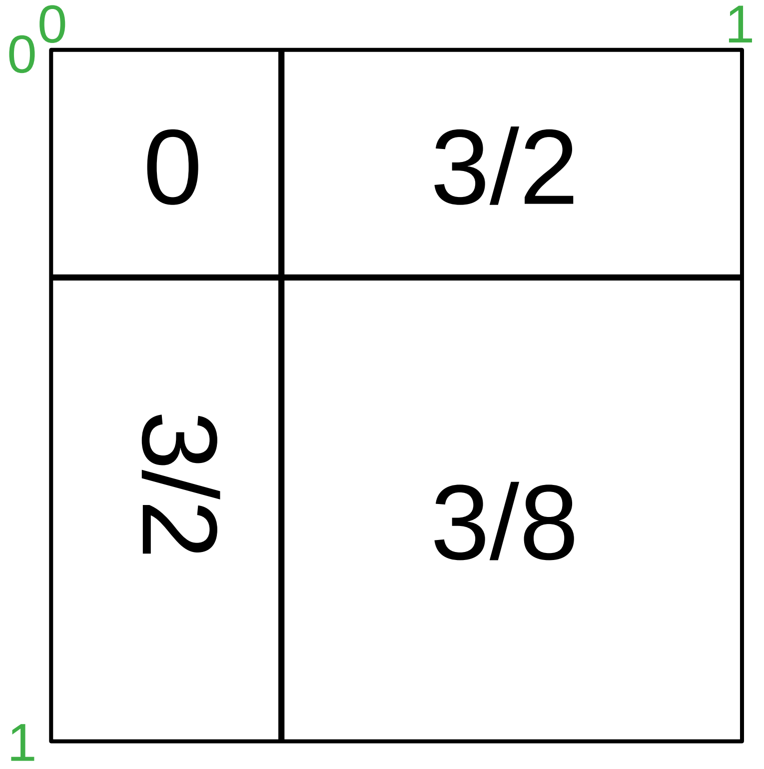

Theorem (Frieze, 1985)

MST(K_n , UNI[0,1]) $\to \zeta(3)$

Proof:

- $$MST(K_n) = \int_{p=0}^{1} (cc(\mathbf{G}(n,p)) - 1) \approx \int_{p=0}^{999/n} cc(\mathbf{G}(n,p))$$

- $$cc(H) = \sum_{k=1}^{\infty} comps\ of\ order\ k$$

$$\ldots = n \times \sum_{k=1}^{\infty} \frac{proportion\ of\ vertices\ within\ comps\ of\ order\ k}{k}$$

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- What is the component order distribution in ***G***(*n*, *17/n*) seen from a uniform vertex?

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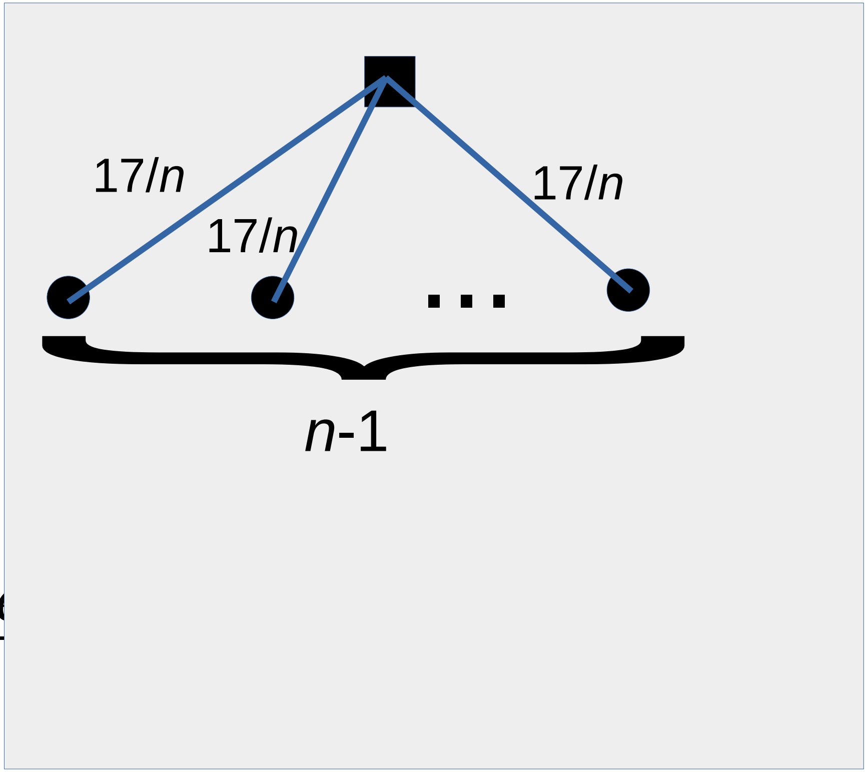

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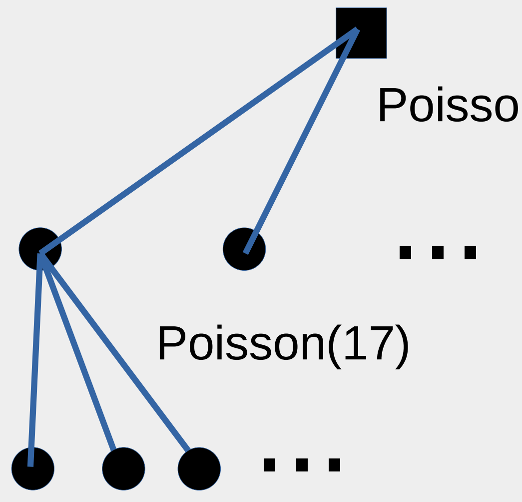

Galton-Watson branching process with offspring distribution Poisson(17)

- What is the component order distribution in $\boldsymbol{G}(n, 17/n)$ seen from a uniform vertex?

Theorem (H.-Viswanathan,

$\mathrm{MST}(G_n) \to \kappa(W)$

Proof:

- $MST(G_n) \approx \int_{p=0}^{999/n} cc(\textbf{percolate}(G_n, p$

- $cc(H) = n \times \sum_{k=1}^{\infty} \underline{proportion\ of\ vertic}$

- What is the component order distribution in ***percolate***(G_n,17/*n*) seen from a uniform vertex?

Bollobás-Borgs-Chayes-Riordan 2010:

Multitype branching process based on W

17×