Jan Hladký
Institute of Computer Science, Czech Academy

The cut norm and the weak* topology for graphons

Joint with
Martin Doležal, Jan Grebík,
Israel Rocha, Václav Rozhoň

Limits of dense graph sequences

Lovász–Szegedy 2006 (+Borgs, Chayes, Lovász, Sós, Vesztergombi)

Main idea: Compactify the space of finite graphs.

- underlying metric = cut distance
- limit objects = graphons

dense sequence: $\liminf_n \frac{\text{edges } G_n}{(\text{vertices } G_n)^2} > 0$

× trees
× planar graphs
× Hamming cubes $\{0,1\}^k$

(Toy) application of compactness

Mantel's Theorem (1907 / Turán 1941)

G … n-vertex graph, no triangle

Then $\text{edges}(G) \le \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil \le \frac{n^2}{4}$

Asymptotic Mantel

$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n > n_0$

G … n-vertex graph, no triangle

Then $\text{edges}(G) \le \left(\frac{1}{4} + \varepsilon\right) n^2 \quad \left(\Leftrightarrow \text{edge density} \le \frac{1}{2} + 2\varepsilon\right)$

(Toy) application of compactness

Asymptotic Mantel

$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n > n_0$

G ... n-vertex graph, no triangle

Then $\text{edges}(G) \leq \left(\frac{1}{4} + \varepsilon\right) n^2$ ($\Leftrightarrow$ edge density $\leq \frac{1}{2} + 2\varepsilon$)

Proof Suppose not. $\exists\, \varepsilon > 0$ and a sequence of triangle-free graphs of edge density $> \frac{1}{2} + 2\varepsilon$.

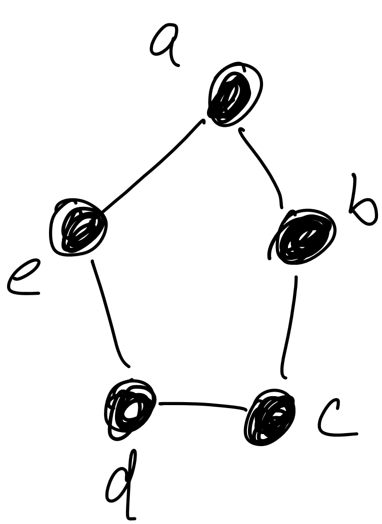
Use compactness to find a subsequential limit W.

edge density$(W) \geq \frac{1}{2} + 2\varepsilon$ triangle density$(W) = 0$ $\lightning$

Graphons from finite graphs

Graphon.... Symmetric measurable function

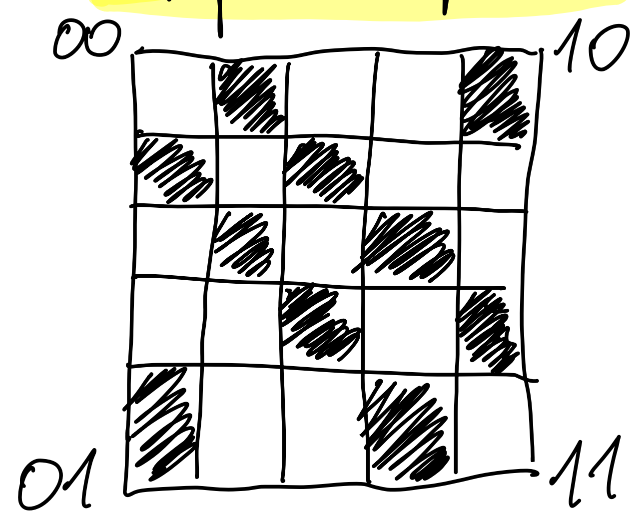
$$W: \underbrace{[0,1]^2}_{\text{just an atomless separable probability space}} \longrightarrow \underbrace{[0,1]}_{\text{intensities}}$$



Adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Graphon representation



Cut norm and cut distance

Cut norm distance

$$d_\square(U,W) = \sup_{S,T\subseteq[0,1]} \left| \int_{S\times T} U - \int_{S\times T} W \right|$$

ALL MEASURABLE!

Cut distance

$$\tilde{d}_\square(U,W) = \inf_{\pi:[0,1]\to[0,1]\ \mathrm{mpb}} d_\square(U, W^\pi)$$

where $W^\pi(x,y) = W(\pi(x),\pi(y))$

Cut distance between graphs

Examples I

Erdős-Rényi random graph $G(n, \frac{1}{2})$

vertex set $= \{1, \dots, n\}$

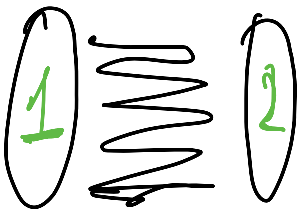
each pair of vertices forms an edge with probability $\frac{1}{2}$ (independently)

Asympotically almost surely

$$\tilde{\delta}_{\square}\left(G(n, \tfrac{1}{2}), \boxed{\tfrac{1}{2}} \right) = o(1)$$

Remark: We do not like the L^1-norm.

Examples II Complete balanced bipartite graphs



$\frac{n}{2}$ $\qquad$ $\frac{n}{2}$

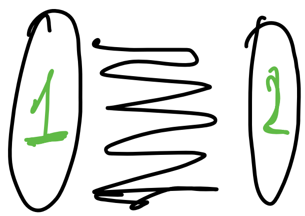
Adjacency matrix

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}$$

$\xrightarrow{\text{limit graphon}}$

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$$

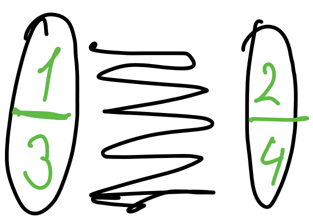
Examples II Complete balanced bipartite graphs



$\frac{n}{2}$ $\quad$ $\frac{n}{2}$

Adjacency matrix

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array} \xrightarrow{\text{limit graphon}} \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$$



$\frac{n}{2}$ $\quad$ $\frac{n}{2}$

Adjacency matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

Examples II Complete balanced bipartite graphs

$$d_\square(U,W) = \sup_{S,T\subseteq[0,1]} \left| \int_{S\times T} U - \int_{S\times T} W \right|$$

$$d_\square\left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \,,\, \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \right) \geq \left| 0 - \underbrace{\frac{1}{2}}_{\text{measure of } S} \cdot \underbrace{\frac{1}{2}}_{\text{measure of } T} \cdot \underbrace{\frac{1}{2}}_{\text{average value}} \right| = \frac{1}{8}$$

Examples II Complete balanced bipartite graphs

So, the representations of the same complete balanced bipartite graph by $2^k \times 2^k$ chessboards form a $\frac{1}{8}$ - separated system for the cut norm distance.

Noncompactness is caused by different orderings and not by complexity of graphs.

$2^{\binom{n}{2}}$ labelled graphs

$n!$ permutation

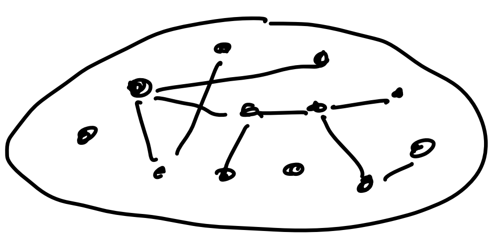
Compactness of graphons and the Regularity lemma

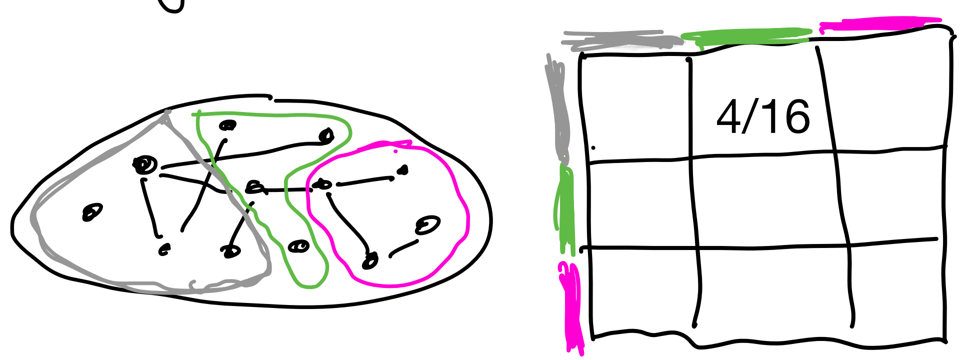
Lovász-Szegedy 2006

The space of graphons with the cut distance is compact.

Szemerédi 1978 / Frieze-Kannan 1999

$\forall \varepsilon\ \exists M$: The vertices of each graph G can be partitioned into M clusters P of almost the same sizes so that $d_\square(G, \text{densities matrix}(G, P)) < \varepsilon$

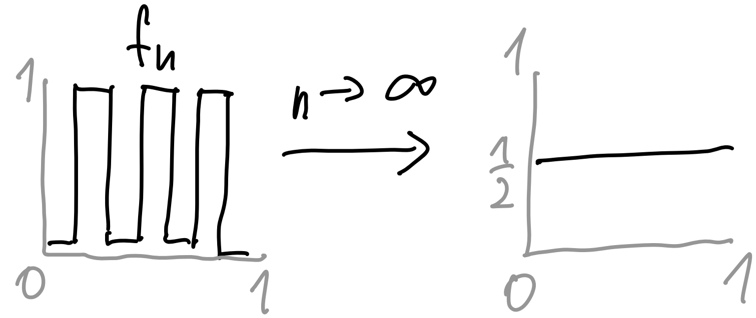




Weak* convergence (predual of L^1 functions)

One dimensional example

$$f_n : [0,1] \to [0,1]$$



Banach-Alaoglu (1930's-1940's): Weak* topology is compact.

Weak* topology for graphons

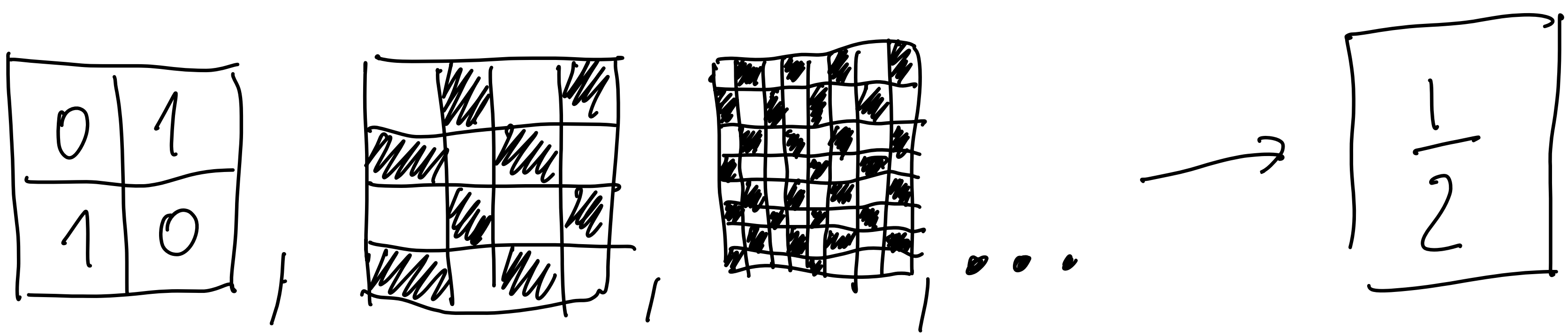
$U_n \xrightarrow{w^*} U$:

$$\sup_{S,T \subseteq [0,1]} \left| \lim_{n\to\infty} \int_{S\times T} U_n - \int_{S\times T} U \right| = 0$$

$U_n \xrightarrow{d_\square} U$:

$$\lim_{n\to\infty} \sup_{S,T \subseteq [0,1]} \left| \int_{S\times T} U_n - \int_{S\times T} U \right| = 0$$

Weak* does not preserve graph structure



Proving cut distance compactness via weak*

Setting: $W_1, W_2, \dots$ graphons

Goal: Find a subsequence $W_{n_1}, W_{n_2}, \dots$, measure preserving bijections $\pi_{n_1}, \pi_{n_2}, \dots$ and a graphon W so that

$$W_{n_i}^{\pi_i} \xrightarrow{d_\square} W$$

Approach: Define

$$ACC_{w*}(W_1, W_2, \dots) = \bigcup_{\pi_1, \pi_2, \dots} \{\text{weak}^* \text{ accu pts of } W_1^{\pi_1}, W_2^{\pi_2}, \dots\}$$

Cleverly select $W \in ACC_{w*}(W_1, W_2, \dots)$

The selection

$$\mathrm{ACC}_w\left(\boxed{\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}}, \boxed{\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}}, \boxed{\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}}, \ldots\right) \supseteq \left\{\boxed{\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}}, \boxed{\tfrac{1}{2}}\right\}$$

Theorem (Doležal – H)

Given a sequence of graphons $W_1, W_2, \ldots$

In the set $\mathrm{ACC}_{w^*}(W_1, W_2, \ldots)$ find an element W that maximizes the L^2-norm. Then W is also a cut distance accumulation point

The selection

$$\mathrm{ACC}_w\left(\begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array},\ \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array},\ \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array},\ \dots\right) \supseteq \left\{\begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array},\ \begin{array}{|c|}\hline \tfrac{1}{2}\\\hline\end{array}\right\}$$

Theorem (Doležal - H)

Given a sequence of graphons $W_1, W_2, \dots$

In the set $\mathrm{ACC}_{w^*}(W_1, W_2, \dots)$ find an element W that maximizes the L^2-norm. Then W is also a cut distance accumulation point

- You can pass to a subsequence where the maximum is attained
- "Entropy minimization"

Accumulation pts versus limits

$$ACC_{w^*}(W_1, W_2, \ldots) = \bigcup_{\pi_1, \pi_2, \ldots} \{\text{weak}^* \text{ accu pts of } W_1^{\pi_1}, W_2^{\pi_2}, \ldots\}$$

$$LIM_{w^*}(W_1, W_2, \ldots) = \bigcup_{\pi_1, \pi_2, \ldots} \{\text{weak}^* \text{ limits of } W_1^{\pi_1}, W_2^{\pi_2}, \ldots\}$$

<u>Theorem</u> Given a sequence of graphons, there exists a subsequence with

$$ACC_{w^*}(W_{n_1}, W_{n_2}, \ldots) = LIM_{w^*}(W_{n_1}, W_{n_2}, \ldots)$$

<u>Theorem</u> For a sequence of graphons, $ACC_{w^*} = LIM_{w^*}$ if and only if cut distance convergent.