Flip processes

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Frederik Garbe, J.H., Matas Šileikis, Fiona Skerman:
From flip processes to dynamical systems on graphons,
Ann. inst. Henri Poincare (B) Probab, 2023

Pedro Araújo, J. H., Eng Keat Hng, Matas Šileikis:
Prominent examples of flip processes
arXiv: 2206.03884

Eng Keat Hng:
Characterization of flip process rules with the same trajectories
arXiv:2305.19925

Erdos-Renyi random graph process

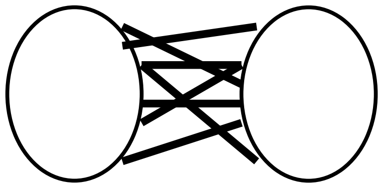
- *Erdos-Renyi random graph process* (n vertices) $G_0, G_1, \ldots, G_{\binom{n}{2}}$
  - G_0 is edgeless
  - G_{r+1} is obtained from G_r by turning a randomly selected nonedge into an edge
- For $r=\alpha n^2$, the graph G_r is a.a.s. quasirandom of density 2α.

Quasirandomness

- 1980's (Chung-Graham-Wilson, Szemeredi, ...)
- *Density* of a graph d=e(G)/$\binom{n}{2}$
- A graph is *ε-quasirandom* if for each set of vertices *U*

$$\left| e(G[U]) - d\binom{|U|}{2} \right| < \varepsilon n^2$$

- A nonquasirandom graph

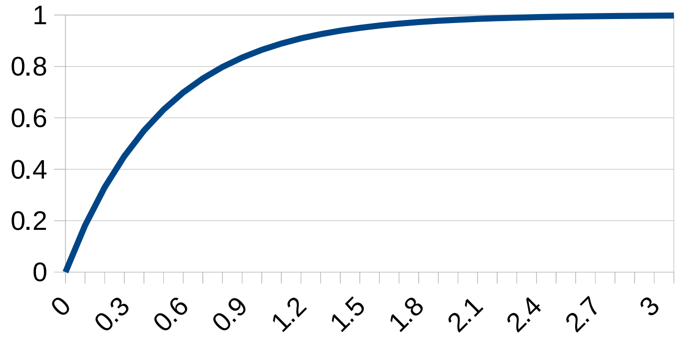


Triangle removal process

- Introduced by Bollobas-Erdos'90
- Start with G_0=clique
- In step *r*, pick a random triangle of G_r and delete it
- Bohman-Frieze-Lubetzky'15: *Triangle removal process typically terminates when there are $n^{3/2+o(1)}$ edges left.*
  - Key in the proof: quasirandomness during the evolution

Erdos-Renyi *flip* process

- Start with a graph G_0 (for now the edgeless graph)
- In each step, "replace" a uniformly chosen **pair** with an edge
- Density computation for G_r, $r=\alpha n^2$:



$$P[uv\ is\ an\ edge]=1-P[uv\ is\ not\ an\ edge]$$

$$\ldots=1-\left(1-\frac{1}{\binom{n}{2}}\right)^r\approx 1-\exp(-2r/n^2)=1-\exp(-2\alpha)$$

Erdos-Renyi 50:50 ***flip*** process

- Start with a graph G_0 on n vertices
- In each step, “replace” a uniformly chosen pair with an edge or an non-edge (50:50)
- “Converges to quasirandom graph of density 0.5”, after Cn^2 steps, $C \to \infty$

Triangle removal *flip* process

- Start with a graph G_0 (for now the complete graph)
- In each step r pick three random vertices u_1, u_2, u_3,
- If $G_r[u_1, u_2, u_3]$ induces a triangle then remove it…

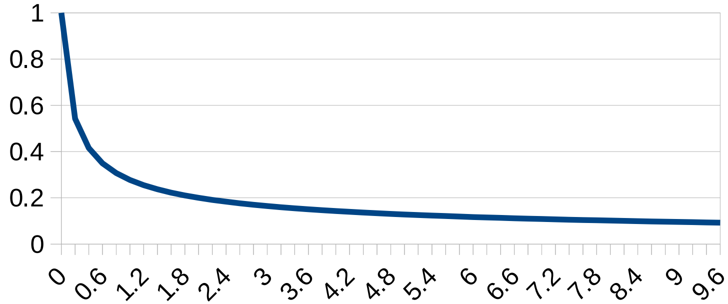
  ...otherwise $G_{r+1} := G_r$.
- Density computation: G_r, $r=\alpha n^2$, $e(\alpha):=e(G_r)$, $d(\alpha):=e(\alpha) / \binom{n}{2}$

$$P\left[u_1 u_2 u_3 \, is\, a\, triangle\right] \approx d(\alpha)^3$$

$$e(\alpha+\epsilon) - e(\alpha) \approx -3 d(\alpha)^3 \cdot \epsilon n^2$$

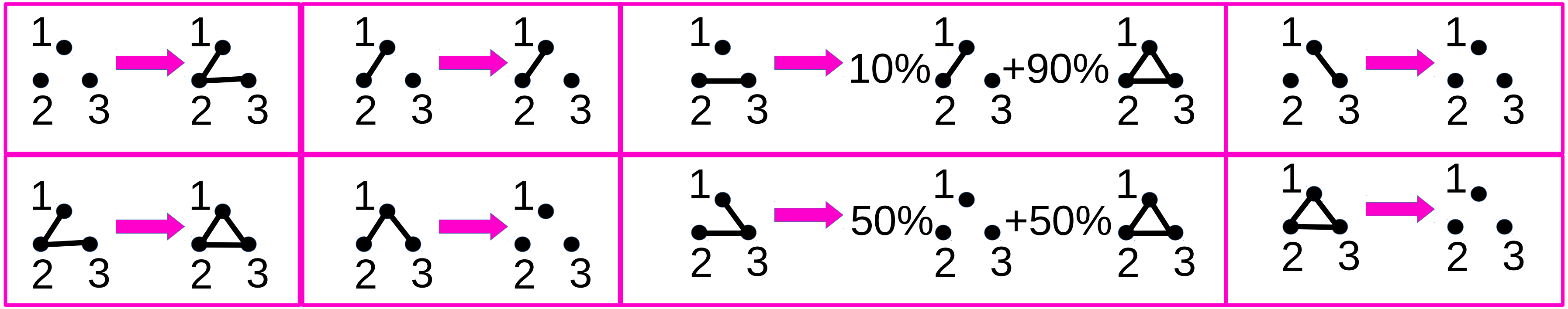
$$\frac{d(\alpha)}{\partial \alpha} = -6 d(\alpha)^3 \Longrightarrow d(\alpha) = \frac{1}{\sqrt{1+12\alpha}}$$

Separable first-order ODE



Flip process of order *k* (here, *k*=3)

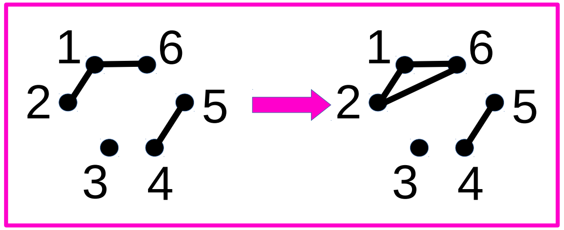
- Rule $\mathcal{R}$



- Start with a (large) graph G_0
- Step $G_r \Rightarrow G_{r+1}$: Sample *k* vertices and replace the induced graph according to $\mathcal{R}$

More examples of flip processes

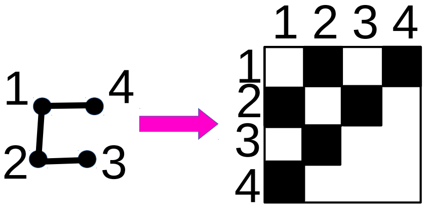
- Ignorant flip process
- *F*-Removal flip process
- Complement flip process
- Component completion flip process
- The stirring flip process
- The extremist flip process

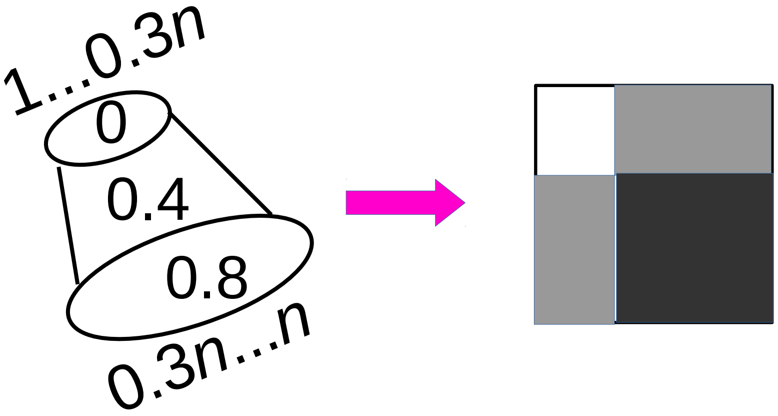


Component completion

Graphons (limits of dense graphs)

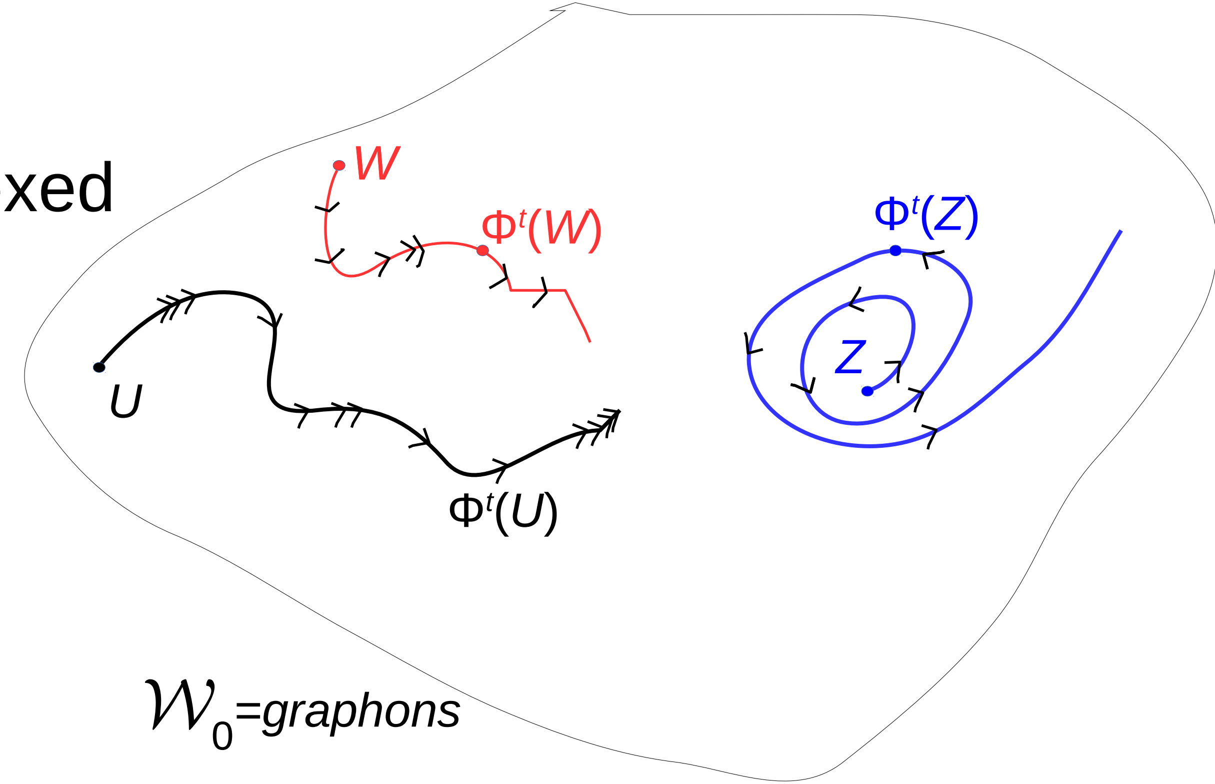
- Borgs-Chayes-Lovasz-Sos-Szegedy-Vesztergombi 2004
- Useful framework for extremal and probabilistic questions
- Graphon is a symmetric function W:$[0,1]^2 \to [0,1]$
- Cut norm measures how similar two graphons are





Trajectories

- Fixed rule $\mathcal{R}$ of order *k*
- We construct time-indexed trajectories $\Phi : \mathcal{W}_0 \times [0,\infty) \to \mathcal{W}_0$
- Construction later



Transference theorem

Theorem 5.1. *For every $k \in \mathbb{N}$ there is a constant $C > 0$ so that the following holds. Given a rule $\mathcal{R}$ of order k and a graph G on the vertex set $[n]$, let $(G_i)_{i \geq 0}$ be the flip process starting with $G_0 = G$.*

For any $T > 0$ and $\varepsilon > 0$ have

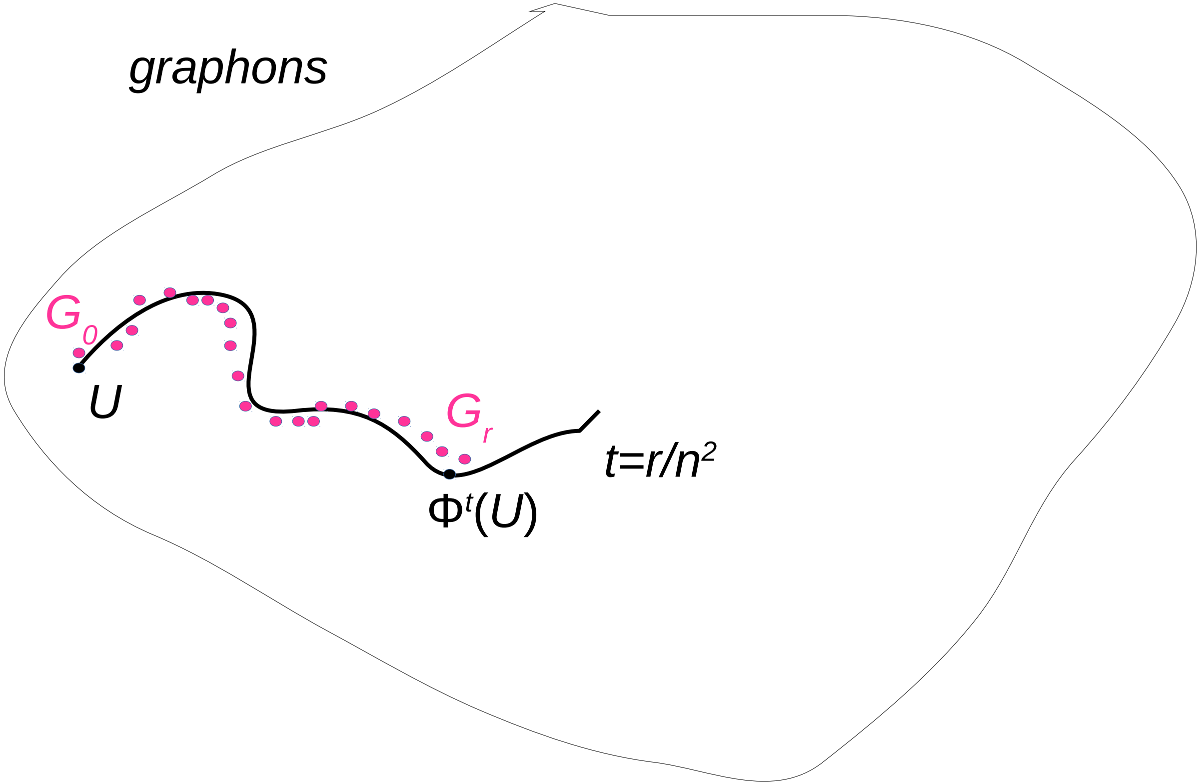
$$\max_{i \in (0, Tn^2] \cap \mathbb{Z}} \left\| W_{G_i} - \Phi^{i/n^2} W_G \right\|_\square < \varepsilon \tag{41}$$

with probability at least

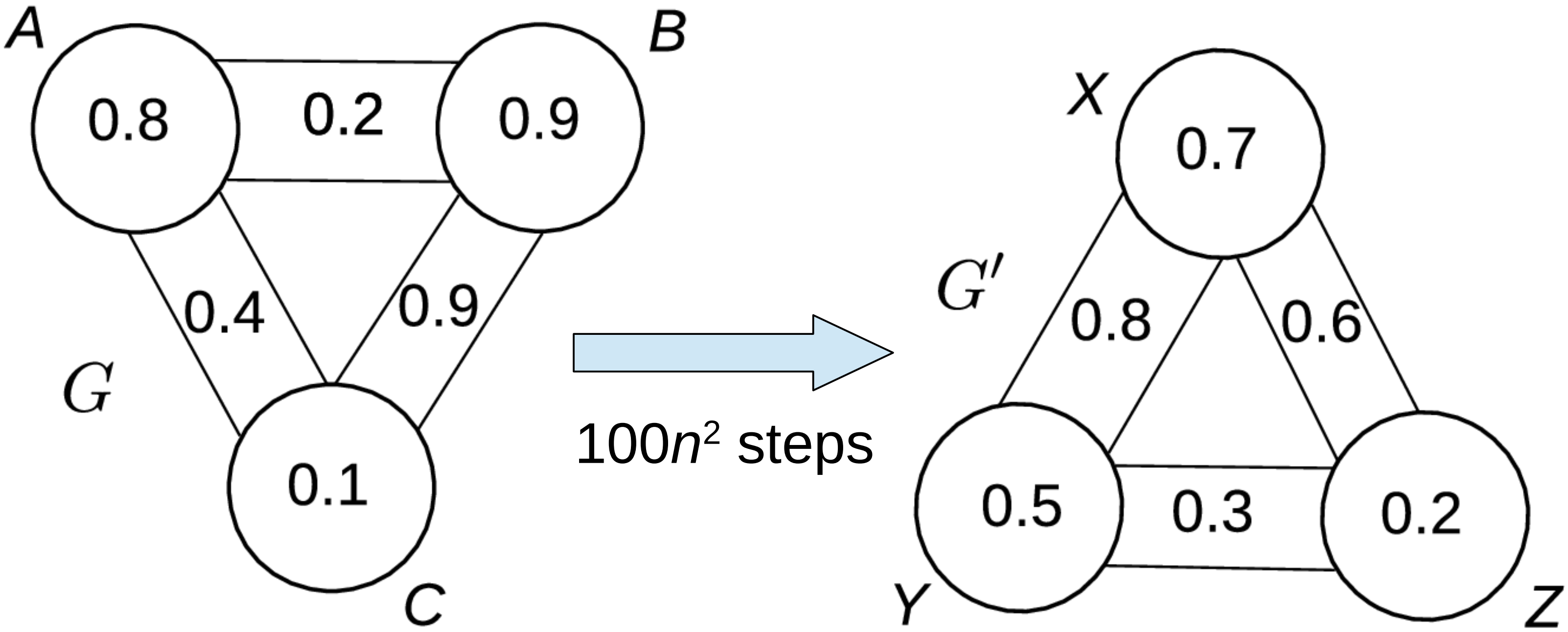
$$1 - \frac{CTe^{CT}}{\varepsilon} \exp\left((2 \ln 2)n - \frac{C\varepsilon^3 n^2}{e^{CT}} \right). \tag{42}$$

Transference theorem

Given $\mathcal{R}$ and corresponding trajectories $\Phi:\mathcal{W}_0\times[0,\infty)\rightarrow\mathcal{W}_0$, whenever a large n-vertex G_0 is close to U (in cut norm) then w.h.p. G_r is close to $\Phi^t(U)$ for $t:=r/n^2$.



Cut norm, not cut distance



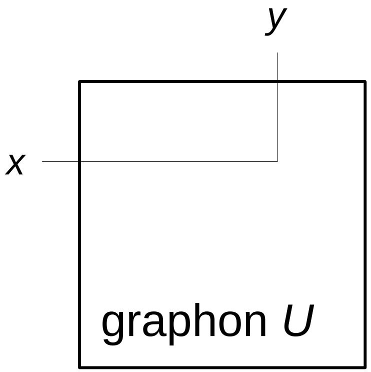
Constructing trajectories I

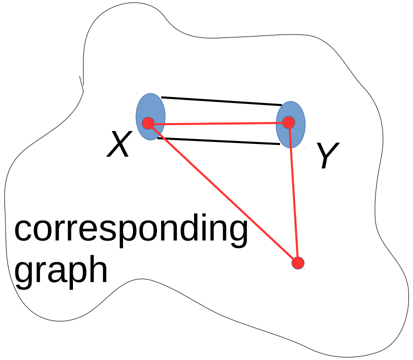
- In this example, consider the Triangle removal flip process
- $(\Phi^{\varepsilon}(U)-U)\,(x,y)$

  correspondence with a graph

  $|X|=|Y|=\gamma n$ and εn^2 steps

  $U(x,y)=e(X,Y)/(\gamma n)^2$





- Number of removed edges between X and Y in εn^2 steps:

$$\epsilon n^2 \cdot \gamma^2 \cdot t_{xy}^{\bullet\bullet}(K_3, U)$$

Density change at (x,y): $-\epsilon \cdot t_{xy}^{\bullet\bullet}(K_3, U)$

$$t_{xy}^{\bullet\bullet}(K_3, W) = \int_z W(x,y)\,W(x,z)\,W(y,z)$$

Constructing trajectories II

- Velocity field $V:\mathcal{W}_0 \to \mathcal{W}$ (signed graphons)

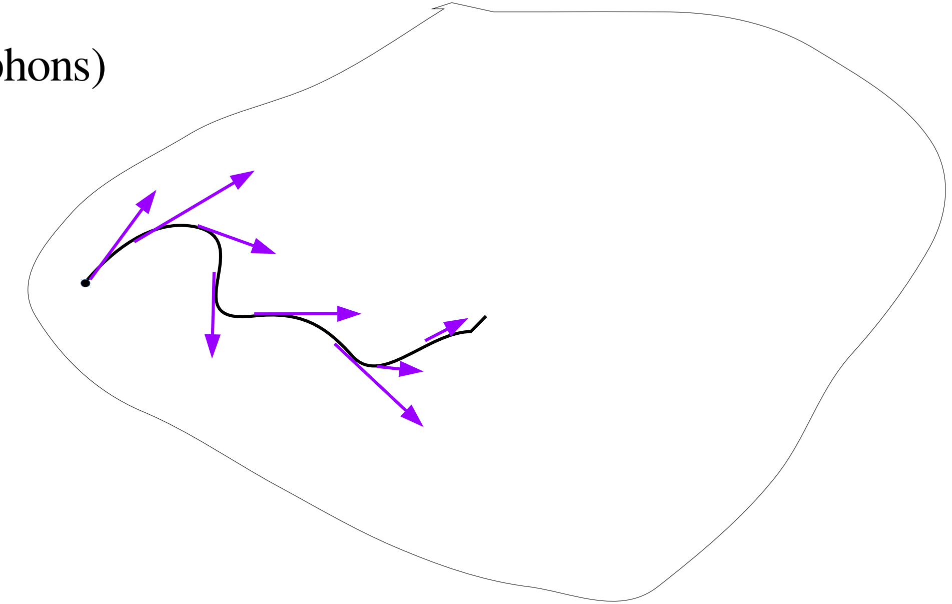
$$V(W) = \lim_{\epsilon \to 0} \frac{\Phi^{\epsilon}(W) - W}{\epsilon}$$

- Integral equation

$$\Phi^T W = -W + \oint_{t=0}^{T} V(\Phi^t W)$$

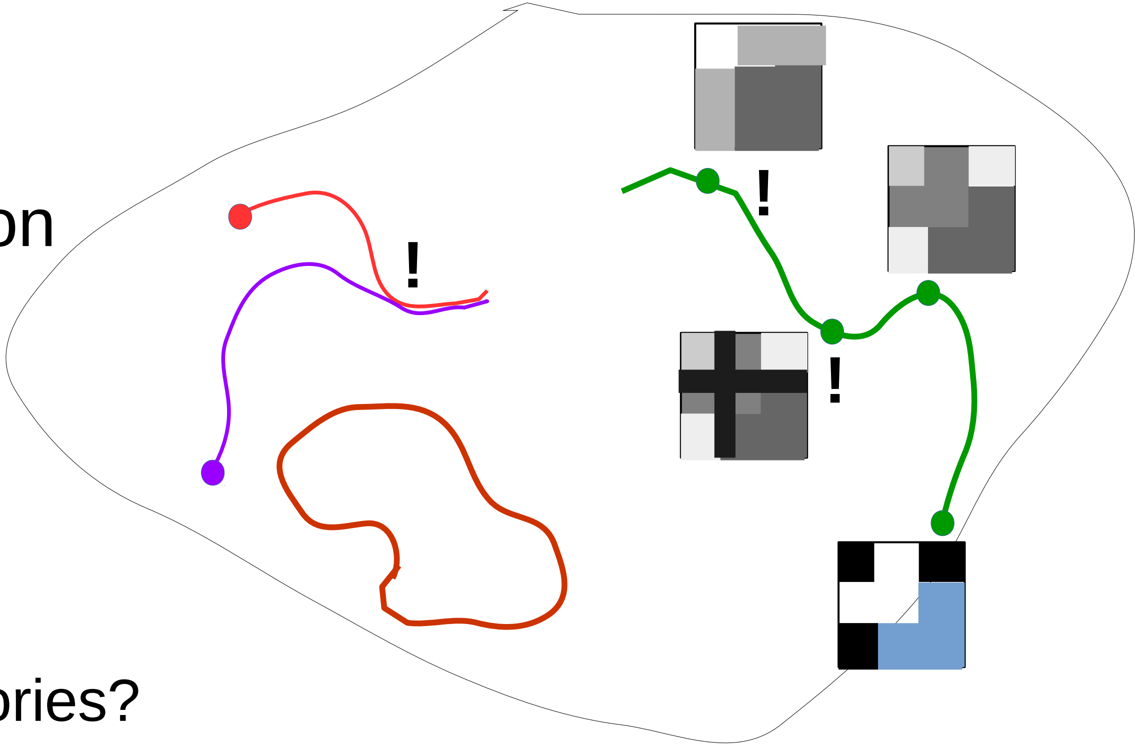
- Triangle removal flip process

$$V(W)(x,y) = -W(x,y)\int_z W(x,z)W(y,z)$$

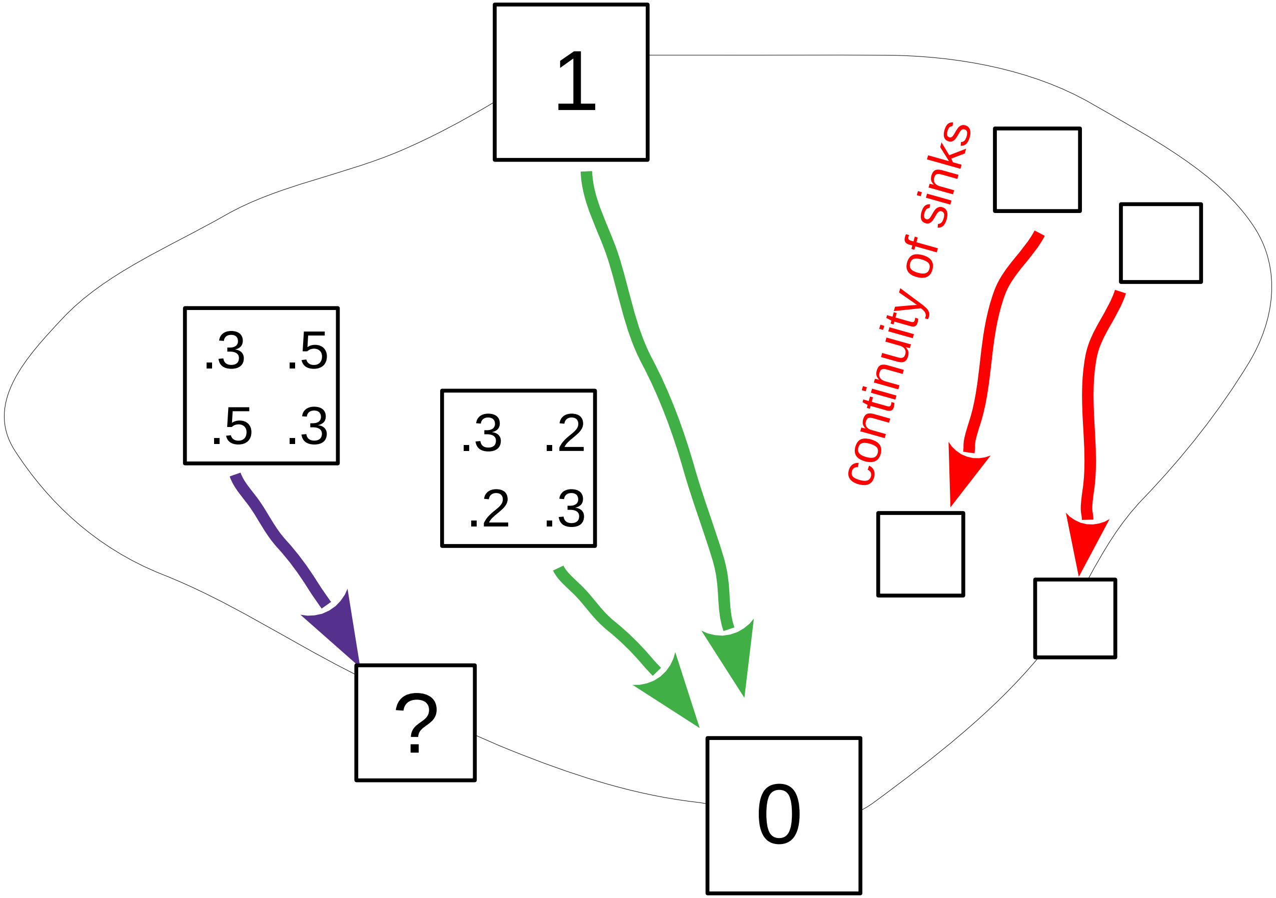


What is all this good for ?

- No confluences
- Going back in time
- Block structure preservation
- Limits $t \to \infty$:
  - Stable and unstable fixed points (often constants)
  - Periodic trajectory
  - Really complicated trajectories?
- Speed of convergence



On the triangle removal process



Behavior of individual trajectories

Fix a rule $\mathcal{R}$. X is a graphon.

- "Typically", trajectory ($\Phi^t(X)$: t) converges to a graphon.
- We have an example of a periodic nonconstant trajectory,

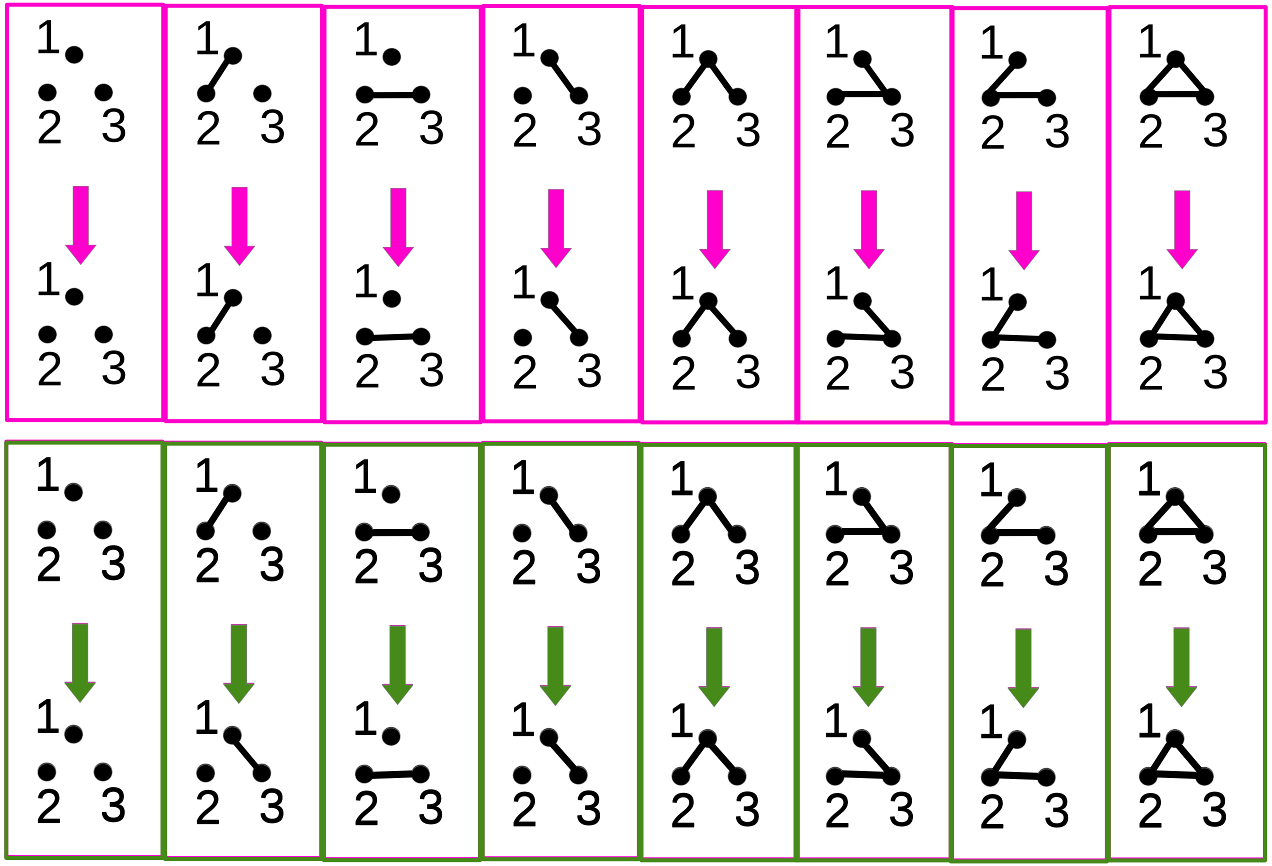
$$\Phi^t(X)=\Phi^{t+7}(X)$$

- Does there exist a really complicated trajectory?

$$S:=\{\Phi^t(X): t\in[0,+\infty)\}$$

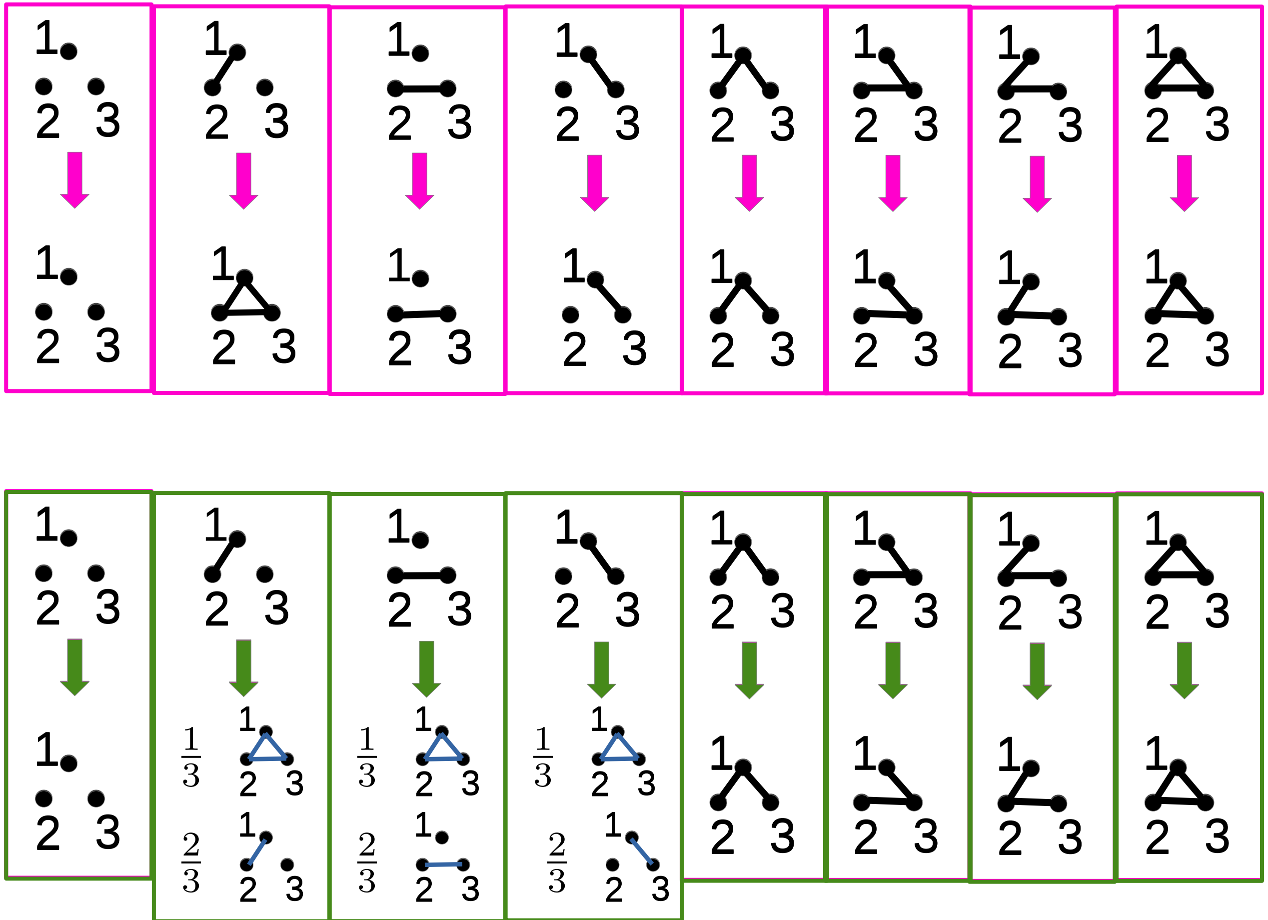
  - ➔ Can the set S be totally unbounded?
  - ➔ Can we have limsup $\Phi^t(X)=1$ and liminf $\Phi^t(X)=0$?

Uniqueness (Eng Keat Hng)



Labels matter!

Uniqueness (Eng Keat Hng)



Theorem:
If there is not an obvious reason for two rules of the same order to have the same trajectories, then (some) trajectories will be different.