THEORY OF LIMITS OF SEQUENCES OF LATIN SQUARES (EXTENDED ABSTRACT)

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ABSTRACT. We build up a limit theory for sequences of Latin squares, which parallels the theory of limits of dense graph sequences. Our limit objects, which we call Latinons, are certain two variable functions whose values are probability distributions on $[0, 1]$. Left-convergence is defined using densities of $k \times k$ subpatterns in finite Latin squares, which extends to Latinons. We also provide counterparts to the cut distance, and prove a counting lemma, and an inverse counting lemma.

1. INTRODUCTION

In this extended abstract, we describe a theory of limits of sequences of Latin squares. Recall that a *Latin square of order n* is a matrix $L \in \mathbb{N}^{n\times n}$ such that each of the numbers $1, \dots, n$ appears precisely once in every row and column. We emphasise that our rows, columns, and symbols in the entries are ordered (by the natural order on $\mathbb{N}$) — for example, we can talk about one column being to the left of another one. One could also consider an "unordered" version of Latin squares, which would lead to an entirely different limit theory.

Our theory parallels similar theories for limits of sequences of dense graphs and of permutations. It is worthwhile to recall the framework underlying limits of sequences of discrete structures in general.

(1) There is a notion of density for the given class of finite discrete structures.
(2) There is a class of limit objects. The notion of density above is extended to these limit objects.
(3) The densities induce a totally bounded metric on the finite structures and the limit objects. Convergence with respect to this topology is called *left-convergence*.
(4) A key result is a compactness theorem: *For every sequence of finite structures in the class, there exists an accumulation point (with respect to left-convergence) in the space of limit objects.*
(5) The space of limit objects is minimal, that is, every limit object can be approximated (in the sense of left-convergence) by a finite object.
(6) There exists another metric, defined in more global terms, which generates the same topology as the left-convergence topology. This metric is called the *cut-distance*.

For graphs, this framework leads to densities $t(F, \cdot)$ and graphons, the compactness theorem (4) was famously proven by Lovász and Szegedy in [6], and the most streamlined approach for (5) is via so-called W-random graphs. We will assume some knowledge of this theory here, referring the reader to [5].

Restricting a Latin square L to any row, we get a bijection between the columns and the symbols, i.e., a permutation on $\{1, \dots, n\}$. Similarly, we get a permutation on $\{1, \dots, n\}$ by restricting to any column. Last, fixing any symbol and restricting only on entries of L with that symbol leads to a permutation between the rows and the columns, which is yet another way of obtaining a permutation on $\{1, \dots, n\}$. So, before working out a theory of limits of Latin squares, it is instructive to look one level lower on limits of permutations, which has been worked out in [2]. Similarly to our remark in the first paragraph, the theory from [2] is for "ordered" permutations, that is the bijection on the set $\{1, \dots, n\}$ inherits the natural linear order from $\mathbb{N}$. Given a permutation $\pi \in \mathbb{S}_n$, the density of a permutation $\sigma \in \mathbb{S}_k$ (for $k \le n$) is the probability that having taken a uniformly random k-tuple $\{v_1 < \ldots < v_k\} \subset \{1, \dots, n\}$ we have that $\sigma(i) < \sigma(j)$ if and only if $\pi(v_i) < \pi(v_j)$ for each $i, j \in \{1, \dots, n\}$. Limits of permutations, called *permutons*, turn out to be probability measures on $[0, 1]^2$ with uniform marginals (on both coordinates). The density of a permutation $\sigma \in \mathbb{S}_k$ in a permuton P, denoted by $t(\sigma, P)$, can be defined in the following way, referred to as *P-sampling*. Sample independently points $p_1, \dots, p_k \in [0, 1]^2$ according to P. Now, reorder the points so that their x-coordinates are increasing, and call the reordered sequence $q_1, \dots, q_k$ (due to the uniform marginals condition, it happens with probability zero that two points would have the same x-coordinate). Then $t(\sigma, P)$ is the probability that for every $i, j \in \{1, \dots, k\}$ we have that $\sigma(i) < \sigma(j)$ if and only if the y-coordinate of q_i is less than that of q_j. Let us also quickly comment on (4) and (5) in this setting. There is a natural way to transform a finite permutation into a measure on $[0, 1]^2$ with uniform marginals. The accumulation points in (4) are exactly accumulation points of the sequence of these transformed permutations with respect to the weak topology (which is well-known to be compact). To address (5), one can show that P-sampling yields with high probability (as $k \to \infty$) a permutation with similar subpermutation densities.

FG, RH, JH: Research supported by GAČR project 18-01472Y and RVO: 67985840; MSh: Research supported by the European Union's Horizon 2020 research and innovation programme under the Marie Curie Individual Fellowship agreement No 752426. RH: This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 648509) and from the MUNI Award in Science and Humanities of the Grant Agency of Masaryk University. This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains.

2. Densities in Latin squares and some examples

An example of a Latin square of order n which easily comes to mind is defined by $L_n(i,j) := i+j \mod n$. A natural first guess at how a limit object of the sequence (L_n) could look like would hence be the map $W : [0,1]^2 \to [0,1]$ defined by $W(x,y) := x+y \mod 1$. However, the following examples show that such a notion cannot capture the richness of the limit object.

Example 2.1. *For convenience of this example, here $n \times n$ squares will have values in $[0, n-1]$ and we assume that n is divisible by 2. We define a Latin square H_n, and we give a randomised construction for a square P_n by*

$$H_n(i,j) := \begin{cases} i+j \mod n & \text{if } i+j \equiv 0 \mod 2, \\ -i-j \mod n & \text{if } i+j \equiv 1 \mod 2; \end{cases} \qquad P_n(i,j) := \begin{cases} i+j \mod n & \text{with probability 1/2,} \\ -i-j \mod n & \text{with probability 1/2.} \end{cases}$$

First, note that it is indeed possible to construct a Latin square P_n. We can choose a pseudo-random red-blue-colouring of the $n \times n$ grid and insert $i+j \mod n$ if the entry is red and $-i-j \mod n$ if the entry is blue. This way we can almost completely fill the grid to obtain a partial Latin square. By a result of Keevash [3] this partial Latin square can then be completed to a Latin square. We omit the details here, but a similar process is described later in the abstract. Now, (P_n) exhibits the problem that $\lim_{n\to\infty} P_n(xn, yn)/n$ does not exist for $x, y \in [0,1]$. We expect the two accumulation points to be $x+y \mod 1$ and $-x-y \mod 1$. A solution to this problem is to let the limit object be a distribution-valued graphon. A candidate for the limit of (P_n) is therefore $P(x,y) := dirac(x+y \mod 1)/2 + dirac(-x-y \mod 1)/2$. However, the sequence (H_n) shows that such a definition is still not general enough. In the limit and for values $(x,y) \in [0,1]^2$ we cannot distinguish the odd from the even case. Since in a square $[(x-\varepsilon)n, (x+\varepsilon)n] \times [(y-\varepsilon)n, (y+\varepsilon)n]$ half of the time the value $H_n(i,j)$ will be $i+j \mod n$ and half of the time $-i-j \mod n$, one could guess that H_n has the same limit as P_n. This is however not correct, as the densities of substructures in (H_n) and (P_n) converge to different limits. To continue the discussion we therefore need to define the notion of density. Let $\mathcal{R}(k,\ell)$ be the set of all $k \times \ell$ matrices such that every $x \in [k\ell]$ appears as an entry precisely once. One natural generalisation of densities in permutations $t(\sigma, \cdot)$ to Latin squares is the following; for a matrix $A = (a_{i,j}) \in \mathcal{R}(k,\ell)$ and a Latin square L, define $t(A, L)$ to be the probability that the restriction of L to a random $k \times \ell$ submatrix $B = (b_{i,j})$ (induced by a random k-set of rows and ℓ-set of columns) has the property that, for every $i \in [k]$ and $j \in [\ell]$, the entry $b_{i,j}$ has the same position in the ordered $k\ell$-tuple of the entries of B as the entry $a_{i,j}$ in the ordered $k\ell$-tuple of the entries of A — in such a case we say that A and B represent the same *pattern*. We now say that a sequence of Latin squares (L_n) (left-)*converges*, if the densities $(t(A, L_n))$ converge for all $A \in \mathcal{R}(k,\ell)$ and $k, \ell \in \mathbb{N}$. One can now check that for the 2×3 pattern A defined by $A_{1,1} := 1, A_{1,2} := 2, A_{1,3} := 3, A_{2,1} := 4, A_{2,2} := 5, A_{2,3} := 6$ we have that $\lim_{n\to\infty} t(A, H_n) > \lim_{n\to\infty} t(A, P_n)$. We therefore need to be able to encode "local information" in the limit object, which suggests to consider a limit object like $H : \Gamma^2 \to \mathfrak{P}([0,1])$, where $\Gamma = [0,1] \times \{\text{odd}, \text{even}\}$. This leads to a different treatment of columns and rows compared to the values. This is justified by the fact that we can find a sequence of Latin squares (L_n) which is convergent, such that the sequence of Latin squares (L'_n) formed by column-value swaps (that is, for all x, y, z we have that if $L(x,y) = z$, then $L'(x,z) = y$) does not converge.

3. The theory of Latinons

3.1. **Defining Latinons.** We now want to precisely define the limit of a sequence of Latin squares of order n, as $n \to \infty$. The first step to do this is to rescale by $\frac{1}{n}$, that is, to transplant the rows, the columns and the symbols into the interval $[0,1]$. So, the most naive guess would be that a limit of Latin squares is a function from $[0,1]^2$ to $[0,1]$ with some sort of "uniform marginal property" (which would reflect that each symbol appears exactly once in each row and each column). In the previous section, however, we saw that several additional properties need to be recorded in the limit.

(i) There must be a way of recording not only global row and column position, but also local behaviour.
(ii) Limits of entries will be probability distributions on $[0,1]$ rather than just values in $[0,1]$.

We are now ready to define our limit objects, which we call *Latinons*. Let $\mathfrak{P}([0,1])$ be the set of all Borel probability measures on $[0,1]$, and let $\lambda^{\otimes d}$ be the Lebesgue measure on $\mathbb{R}^d$. Let (Γ, γ) be a fixed atomless separable probability space.

Definition 3.1 (Latinons). *Suppose that $f : \Gamma \to [0,1]$ is a measure preserving map and $K : \Gamma^2 \to \mathfrak{P}([0,1])$ is a Borel map. We say that the pair (K, f) is a Latinon if for almost every $s \in \Gamma$ and for every measurable set $C \subset [0,1]$ we have*

$$\int_{t\in\Gamma} K(s,t)(C)\mathrm{d}\gamma = \lambda^{\otimes 1}(C) = \int_{t\in\Gamma} K(t,s)(C)\mathrm{d}\gamma \,. \tag{1}$$

That is, in Definition 3.1 the rows are indexed by Γ, and the global position of a row $s \in \Gamma$ corresponds to $f(s)$. The possible non-injectivity of f allows us to accommodate information about the local behaviour. Note that the formulae in Definition 3.1 are counterparts to the uniform marginals property of permutons.

3.2. **Densities in Latinons and limits of examples from Section 2.** Given $A = (a_{i,j}) \in \mathcal{R}(k,\ell)$ and a Latinon (K, f) we want to generalise the notion of density from Latin squares to Latinons. We can introduce a strict partial order $<_f$ on Γ by defining $x <_f y$ if and only if $f(x) < f(y)$. Then let $t(A,(K,f))$ be the probability that, given a random selection of $\mathbf{x} = (x_1, \cdots, x_k) \in \Gamma^k_{<_f}$ and $\mathbf{y} = (y_1, \cdots, y_\ell) \in \Gamma^\ell_{<_f}$, and sampling for each (i, j) with $i \in [k]$, $j \in [\ell]$ a real-valued $b_{i,j}$ from the distribution $K(x_i, y_j)$, the matrix $(b_{i,j})_{i\in[k],j\in[\ell]}$ represents the same pattern as A. For the sequences from Example 2.1 we can now define Latinons such that the densities of the Latin squares converge towards the densities of the corresponding Latinon. For the sequence (H_n), we set $\Gamma := [0, 1] \times \{\text{odd, even}\}$, γ is the uniform measure on Γ, and the functions f and K are defined by

$$f(x,a) := x \ , \quad K((x,a),(y,b)) := \begin{cases} x+y \mod n & \text{if } a = b, \\ -x-y \mod n & \text{if } a \neq b. \end{cases}$$

For the sequence (P_n), we set $\Gamma := [0, 1]$, γ is the uniform measure on Γ, f is the identity, and K is defined by

$$K(x,y) := \begin{cases} x+y \mod n & \text{with probability } 1/2, \\ -x-y \mod n & \text{with probability } 1/2. \end{cases}$$

3.3. **Our results.** Our two main results are compactness of the space of Latinons, and approximability of a Latinon by finite Latin squares. As is usual, such results are needed for any complete limit theory of a certain discrete structure.

Theorem 3.2. *For each sequence $(L_n)_n$ of finite Latin squares of growing orders there exists a subsequence $(L_{k_i})_i$ and a Latinon (K, f) such that $(L_{k_i})_i$ left-converges to (K, f).*

Theorem 3.3. *For each Latinon (K, f) there exists a sequence $(L_n)_n$ of finite Latin squares of growing orders that left-converges to (K, f).*

We also introduce an analogue of a cut-distance for Latinons. For this we first define a natural generalisation of the cut-norm to Latinons. To define the distance we then allow an additional reordering of the columns and rows when comparing the measures, however since our columns and rows are ordered by $<_f$ such a reordering must only be small or otherwise increase the distance. Let $S_{[0,1]}$ be the set of all measure-preserving bijections on the unit interval.

Definition 3.4. *Let $L_1 = (W, f)$ and $L_2 = (U, g)$ be Latinons. We define*

$$\delta_L(L_1, L_2) := \inf_{\varphi,\psi\in S_{[0,1]}} \left(\left\| O^f - O^{g\circ\varphi} \right\|_\square + \left\| O^f - O^{g\circ\psi} \right\|_\square + \left\| W - U^{\varphi,\psi} \right\|_D \right)$$

where $O : [0, 1]^2 \to [0, 1]$ is the digraphon such that $O(x,y) = \begin{cases} 1, & x < y \\ 0, & \textit{otherwise} \end{cases}$ *and*

$$\left\| W - U^{\varphi,\psi} \right\|_D := \sup_{\substack{R,C,V\subseteq[0,1] \\ V \, interval}} \int_{x\in R} \int_{y\in C} W(x,y)(V) - U(\varphi(x),\psi(y))(V) dy dx \ .$$

We show counting and inverse counting lemmas which demonstrate that the topologies induced by left-convergence and the cut-distance are equivalent. The proofs of both statements involve reducing the problem to "compression of digraphons", a notion which will be explained in a later section. The proof of the inverse counting lemma relies on a sampling lemma which states that sampling a pattern from a Latinon is close to the Latinon in cut-distance.

Lemma 3.5. *Let $k, \ell \in \mathbb{N}$. Then there exists a constant $c_{k,\ell}$ such that for every $d \in \mathbb{N}$, Latinons L_1, L_2 and $A \in \mathcal{R}(k, \ell)$ we have $|t(A, L_1) - t(A, L_2)| < c_{k,\ell}/2^d + c_{k,\ell} 2^{dk\ell} \cdot \delta_L(L_1, L_2)$.*

Lemma 3.6. *For every $\delta > 0$ there exist $k \in \mathbb{N}$ and $\varepsilon > 0$ such that for every two Latinons L_1 and L_2 with $\delta_L(L_1, L_2) > \delta$ there exists $A \in \mathcal{R}(k, k)$ such that $t(A, L_1) - t(A, L_2) > \varepsilon$.*

3.4. **Possible application I: Extremal questions.** Many extremal questions concern the concept of densities of substructures within discrete structures. Such questions have become prominent due to the recent development of limit concepts (e.g Razborov's flag algebras). Our theories here to the best of our knowledge would provide the first limit framework for attacking extremal problems in Latin squares.

3.5. **Possible application II: Quasirandomness.** The famous Chung-Graham-Wilson Theorem asserts (among others) that if a finite graph has density of the edge close to $\frac{1}{2}$ and density of the 4-cycle close to $\frac{1}{16}$ then its density of any given graph F is close to $2^{-e(F)}$. An analogous result for permutons was obtained by Král' and Pikhurko [4], thus answering a long-standing question of Graham: if a finite permutation has density of each pattern $\pi \in \mathbb{S}_4$ close to $\frac{1}{4!}$ then it has density of any given pattern $\sigma \in \mathbb{S}_k$ close to $\frac{1}{k!}$. The proof of Král' and Pikhurko uses in an essential way equivalence of left-convergence and "cut-distance convergence"[1], something that does not naturally translate to the finite setting. Actually, their proof was in the eyes of some researchers the first convincing argument

[1] which in the setting of permutations corresponds to just weak convergence of measures

about advantages of the limit approach to extremal problems in combinatorics. We pose an analogous conjecture about Latin squares below, in (a) in finite language and in (b) for Latinons; these two versions are equivalent by Theorems 3.5 and 3.6. Let us remark that we believe that the latter formulation may be admissible for a proof in the spirit of Král' and Pikhurko.

Conjecture 3.7. *There exists a number $k \in \mathbb{N}$ so that the following holds.*

- *(a) Suppose that $(L_n)_n$ is a sequence of Latin squares so that for each $A \in \mathcal{R}(k,k)$ we have $\lim_{n\to\infty} t(A, L_n) = \frac{1}{(k^2)!}$. Then for any $\ell \in \mathbb{N}$ and any $B \in \mathcal{R}(\ell,\ell)$ we have $\lim_{n\to\infty} t(B, L_n) = \frac{1}{(\ell^2)!}$.*
- *(b) Suppose that (K, f) is a Latinon for which we have $t\big(A, (K, f)\big) = \frac{1}{(k^2)!}$ for each $A \in \mathcal{R}(k,k)$. Then $K(s,t)$ equals to the Lebesgue measure on $[0,1]$ for almost every $(s,t) \in \Gamma^2$.*

3.6. **Possible application III: Counting Latin squares with given densities and a large deviation principle.** Chatterjee and Varadhan [1] realised that the graphon formalism can be used to approach some questions about large deviations in Erdős–Rényi random graphs. Due to space constraints we are unable to introduce this beautiful field in more detail, and jump directly to the setting of Latinons. Suppose that we are given a Latinon $L = (K, f)$ as in Definition 3.1, and suppose that for each $(s,t) \in \Gamma^2$, the measure $K(s,t)$ has a Radon-Nikodym derivative with respect to the Lebesgue measure, say $d_{s,t}$. We then define the *entropy* of L, $H(L) := \int_{(s,t)\in\Gamma^2} \int_{x\in[0,1]} -d_{s,t}(x) \ln d_{s,t}(x)\, \mathrm{d}x\, \mathrm{d}\gamma^{\otimes 2}$. We believe that this quantity captures how many finite Latin squares there are close to a given Latinon. More precisely, we believe that there are $\left((1 \pm o(1)) \cdot \frac{\exp(H(L))n}{e^3}\right)^{n^2}$ Latin squares of order n close to L in the cut-distance. The precise definition of "close" is similar as in [1].

By similar reasoning as in [1], having such a counting result would allow us to approach questions such as the following. *Fix patterns A and B and a number $t_A \in [0,1]$. Pick a random Latin square L of order n (large), conditioned on $t(A, L) = t_A \pm o(1)$. How does the random variable $t(B, L)$ behave?*

4. Approaching Latinons via compressions of digraphons

The key for proving Theorems 3.2, 3.5, and 3.6 is to represent a Latin square or a Latinon using what we call a "compression of digraphons". This allows us to access tools available in the theory of graphons. To illustrate the method, let us sketch a proof of Theorem 3.2. Suppose that L is a Latin square of order n. Then we associate to L the following *compression*

$$\mathbf{L} = (O, W^{1,1}, W^{1,2}, W^{2,1}, \cdots, W^{2,4}, W^{3,1}, \cdots, W^{3,8}, \ \cdots\ , W^{d,1}, \cdots, W^{d,2^d}, \ \cdots) \,,$$

where O and W's are digraphons from Γ^2 to $\{0,1\}$, defined as follows.

- Partition Γ arbitrarily into n sets $\Gamma_1, \ldots, \Gamma_n$ of measure $\frac{1}{n}$ each (so, $\mathbf{L}$ will depend on this partition).
- Define O to be 1 on each set $\Gamma_i \times \Gamma_j$ with $i < j$, and 0 on the rest.
- Given $d \in \mathbb{N}$ and $k \in [2^d]$, define $W^{d,k}$ to be 1 on each set $\Gamma_i \times \Gamma_j$ with $\frac{L(i,j)}{n} \in [\frac{k-1}{2^d}, \frac{k}{2^d})$ (the right end of this interval is taken closed if $k = 2^d$), and 0 on the rest.

A crucial property of our construction is that $W^{d,k}$'s are nested in the sense that for each $d \in \mathbb{N}$ and for each $k \in [2^d]$ we have $W^{d,k} = W^{d+1,2k-1} + W^{d+1,2k}$.

Now, given a sequence of Latin squares $L_1, L_2, \ldots$ in Theorem 3.2, we consider a sequence of associated compressions $\mathbf{L}_1 = (O_1, W_1^{1,1}, W_1^{1,2}, \cdots), \mathbf{L}_2 = (O_2, W_2^{1,1}, W_2^{1,2}, \cdots), \ldots$. On this sequence, we apply an extension of the Lovász–Szegedy compactness theorem for tuples of (di)graphons, stated in Theorem 4.1 below. Let $\mathcal{W}_0$ be the space of measurable, not necessarily symmetric, functions $W : \Gamma^2 \to [0,1]$ together with the cut metric $d_\square(U, W) := \sup_{S,T\subseteq\Gamma} \left|\int_{S\times T} (U - W)(s,t) \mathrm{d}s\mathrm{d}t\right|$. We can define a metric $\delta_\square^{\mathbb{N}}$ on $\mathcal{W}_0^{\mathbb{N}}$ by setting

$$\delta_\square^{\mathbb{N}}((U_n)_{n\in\mathbb{N}}, (W_n)_{n\in\mathbb{N}}) := \inf_{\varphi:[0,1]\to[0,1]} \sum_{n=1}^{\infty} \tfrac{1}{2^n} d_\square(U_n, W_n^\varphi) \,,$$

where the infimum is taken over all invertible measure preserving maps and $W^\varphi(x,y) = W(\varphi(x), \varphi(y))$. Note that this is not the same as $\sum_{n=1}^{\infty} \frac{1}{2^n} \delta_\square(U_n, W_n)$ and therefore compactness of $\delta_\square^{\mathbb{N}}$ does not follow immediately by combining Tychonoff's theorem with the standard Lovász–Szegedy compactness theorem.

Theorem 4.1. *The space $(\mathcal{W}_0^{\mathbb{N}}, \delta_\square^{\mathbb{N}})$ is (sequentially) compact.*

So, using Theorem 4.1, we can find a $\delta_\square^{\mathbb{N}}$-limit point $\mathbf{L}_* = (O_*, W_*^{1,1}, W_*^{1,2}, \cdots)$ of some subsequence $\mathbf{L}_{k_1}, \mathbf{L}_{k_2}, \ldots$. Crucially, note that the nestedness property transfers to $(W_*^{d,k})_{d,k}$. Let us show how to interpret $\mathbf{L}_*$ as the Latinon (K, f). Firstly, as O_* arises as a cut-distance limit of full upper-triangular matrices, it must be weakly isomorphic to the upper-triangular digraphon. In particular, the function $f : \Gamma \to [0,1]$, $f(s) := \int_t O_*(s,t)\ \mathrm{d}t$ is measure preserving. Let us now show how to construct $K(s,t)$ at a given $(s,t) \in \Gamma^2$. We want that for each $d \in \mathbb{N}$ and for each $k \in [2^d]$, the $K(s,t)$-measure of the interval $[\frac{k-1}{2^d}, \frac{k}{2^d})$ (again, with a closed interval for $k = 2^d$) equals to $W_*^{d,k}(s,t)$. Indeed, thanks to the nestedness property above, Carathéodory's extension theorem guarantees that such a measure exists (and is unique). It can now be verified that (K, f) satisfies Definition 3.1, and that it is indeed a left-limit of $(L_{k_i})_i$.

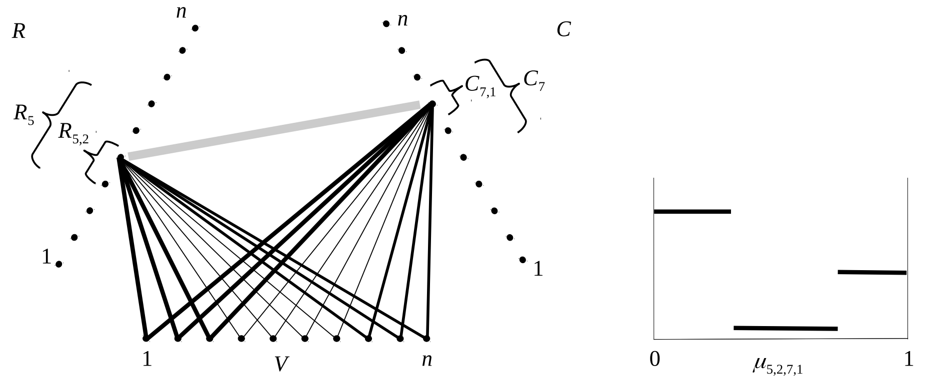

Figure 1. Different weight (depicted by thickness) on triangles from a fixed pair of vertices in $R_{5,2} \times C_{7,1}$ corresponds to intensities of $\mu_{5,2,7,1}$ around scaled-down labels on V. Choosing a different pair in $R_{5,2} \times C_{7,1}$ would does not change the weights.

5. Approximating Latinons via finite Latin squares

Let us briefly sketch the proof of Theorem 3.3. Our proof depends on the recent breakthrough results of Keevash [3] on combinatorial designs. As a matter of fact, Keevash used a very similar approach to obtain lower bounds on generalisations of Latin squares, so-called higher-dimensional permutations. So, suppose that our task is to approximate a given graphon (K, f) by a finite Latin square. Firstly, using some basic approximation properties of Borel measures, we can find a partition $[0,1] = J_1 \sqcup \ldots \sqcup J_m$ into short intervals, and for each $i \in [m]$ a partition $f^{-1}(J_i) = \Gamma_{i,1} \sqcup \ldots \sqcup \Gamma_{i,p_i}$ and for each $i, j \in [m]$ and $q \in [p_i]$, $r \in [p_j]$, measures $\mu_{i,q,j,r} \in \mathfrak{P}([0,1])$ which are constant multiples of the Lebesgue measure on each step $J_1, \ldots, J_m$ such that for most of $(s,t) \in \Gamma_{i,q} \times \Gamma_{j,r}$ we have that $K(s,t)$ is similar (in a suitable metric) to $\mu_{i,q,j,r}$.

We say that a collection of edge-disjoint triangles in a given graph forms a *triangle decomposition* if each edge is covered by some triangle from that collection. The importance of this concept in this work stems from the fact that each Latin square of order n can be encoded as a triangle-decomposition of the tripartite graph $K_{n,n,n}$, where one part (say R) represents row indices, another one column indices (say C), and the last one values in entries (say V); so each of these parts is labelled by $1, \ldots, n$. We now put a certain weighting on these triangles, as follows. Partition R as $\bigsqcup_{i=1}^{m} \bigsqcup_{q=1}^{p_i} R_{i,q}$ so that the ordering reflects the positions of $\bigsqcup_{i=1}^{m} \bigsqcup_{q=1}^{p_i} \Gamma_{i,q}$, and similarly partition $C = \bigsqcup_{i=1}^{m} \bigsqcup_{q=1}^{p_i} C_{i,q}$. Partition $V = \bigsqcup_{i=1}^{m} V_i$ so that the ordering reflects the positions of $\bigsqcup_{i=1}^{m} J_i$. Now, assume T is a triangle, say between $R_{i,q}$, $C_{j,r}$ and V_ℓ which we assume to be labelled by labels around αn. Assign T the weight corresponding to the Radon-Nikodym derivative of $\mu_{i,q,j,r}$ around α. See Figure 1.

Now, run the usual Rödl nibble for producing an approximate triangle decomposition of the $K_{n,n,n}$. That is, we start randomly extracting small batches of triangles in rounds, deleting the edges in the extracted triangles. The only difference is that this time, each triangle is selected with probability proportional to its weight. Quasirandomness conditions needed for the Rödl nibble can be controlled in the usual way. We end up with a very sparse subgraph of $K_{n,n,n}$, which corresponds to a partial Latin square. Now, Keevash's machinery can be used to complete this partial Latin square into a complete Latin square. It can be shown that with high probability this Latin square is close to (K, f) in densities.

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