A Turán-type theorem for large-distance graphs in Euclidean spaces, and related isodiametric problems (extended abstract)

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Abstract

A *large-distance graph* is a measurable graph whose vertex set is a measurable subset of $\mathbb{R}^d$, and two vertices are connected by an edge if and only if their distance is larger that 2. We address questions from extremal graph theory in the setting of large-distance graphs, focusing in particular on upper-bounds on the measures of vertices and edges of K_r-free large-distance graphs. For example, one of our main results says that if $A \subset \mathbb{R}^2$ is a measurable set such that the large-distance graph on A does not contain any complete subgraph on three vertices then the 2-dimensional Lebesgue measure of A is at most 2π.

The results presented in this extended abstract are motivated by the following classical question from extremal graph theory. Let $k \geq 3$ be a given natural number. If we know the number $|V|$ of vertices in a simple graph $G = (V, E)$, what can we say about the relationship between the number $|E|$ of edges in G and the number of complete subgraphs on k vertices in G? More precisely, we are motivated namely by the following two problems.

1. If we know the number $|V|$ of vertices in a graph $G(V, E)$ which does not contain any complete subgraph on k vertices, what is the optimal upper bound for the number $|E|$ of edges in G?

2. If we know both the number $|V|$ of vertices and the number $|E|$ of edges in a graph $G(V, E)$, what is the optimal lower bound on the number of complete subgraphs on k vertices in G?

The solution to the first problem was given by Turán in 1941:

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Theorem 1 (Turán [12]). *Let $k \geq 3$ be a fixed natural number. Let $G = (V, E)$ be a graph which does not contain any complete subgraph on k vertices. Then*

$$|E| \leq \frac{1}{2}\left(1 - \frac{1}{k-1}\right) \cdot |V|^2.$$

The second problem, known as the clique density problem, and first formulated by Lovász and Simonovits [6] in 1983, turned out to be much more difficult. After Razborov [9] provided a solution for $k = 3$ and Nikiforov [8] for $k = 4$, a complete solution was given by Reiher [10]:

Theorem 2 (Reiher [10]). *Let $k \geq 3$ be a fixed natural number, and let $\gamma \in \left[0, \frac{1}{2}\right)$. Suppose that a graph $G = (V, E)$ satisfies $|E| \geq \gamma \cdot |V|^2$. Then G contains at least*

$$\frac{1}{(s+1)^k}\binom{s+1}{k}(1+\alpha)^{k-1}\left(1-(k-1)\alpha\right) \cdot |V|^k$$

complete subgraphs on k vertices, where $s \geq 1$ is an integer with $\gamma \in \left[\frac{s-1}{2s}, \frac{s}{2(s+1)}\right]$ and $\alpha \in \left[0, \frac{1}{s}\right]$ is defined by the formula $\gamma = \frac{s}{2(s+1)}(1-\alpha^2)$.

We shall be interested in certain measure-theoretic counterparts of the two problems mentioned above. Instead of simple graphs, we consider so called *large-distance graphs* whose underlying set is a subset of the Euclidean space $\mathbb{R}^d$ for some natural number d, and where two vertices are connected by an edge whenever they are far apart. This is covered by the following definition.

Definition 1 (Large-distance graphs). Let A be a measurable subset of $\mathbb{R}^d$. Then we define the *large-distance graph $\mathcal{G}_A$* corresponding to A as follows: The vertex set of $\mathcal{G}_A$ is the set A. The edge set $\mathcal{E}_A$ of $\mathcal{G}_A$ is defined by

$$\mathcal{E}_A = \{(p, q) \in A \times A : \|p - q\| > 2\}.$$

Let us emphasize that the precise value of the distance threshold (which is set to 2 in the definition above) is not important. Only a very simple rescaling would be needed to reformulate all of our results for any other choice of the distance threshold.

Graphs (finite, or infinite such as here) with edges defined by similar metric conditions arise naturally in many real-life scenarios. For example, a finite version of large-distance graphs could be used for planning new train lines. Vertices would be cities and edges would then represent distant pairs of cities, where a high-speed train line would be desirable. A lot of research has been done on so-called *distance graphs* where two points of a subset of $\mathbb{R}^d$ are connected by an edge if and only if their distance equals 1, see e.g. [11]. Recall also that considering the $(d-1)$-dimensional unit sphere S^{d-1} in $\mathbb{R}^d$ instead of an arbitrary subset A of $\mathbb{R}^d$ and changing the distance threshold in Definition 1 to some $\alpha < 2$, leads to the so called *Borsuk graph* (see [7, p. 30], or [5]).

Similarly to the motivating question from extremal graph theory mentioned above, we ask the following. Let $k \geq 3$ and $d \geq 2$ be given natural numbers. If we know the d-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^d$, what can we say about the relationship between the $2d$-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$ of the large-distance graph $\mathcal{G}_A$ and the kd-dimensional Lebesgue measure of all k-tuples of elements of A which induce a complete subgraph of $\mathcal{G}_A$? That is, the number of edges from the classical problem is replaced by their

$2d$-dimensional Lebesgue measure, and the number of complete subgraphs on k vertices is replaced by their kd-dimensional Lebesgue measure. In a complete analogy to Problems 1 and 2, this suggests the following two problems (here and later, λ_s denotes the s-dimensional Lebesgue measure).

1′. Let A be a measurable subset of $\mathbb{R}^d$ of a known d-dimensional Lebesgue measure. Suppose that the large-distance graph $\mathcal{G}_A$ does not contain any complete subgraph on k vertices (or that the kd-dimensional Lebesgue measure of all k-tuples of elements of A which induce a complete subgraph of $\mathcal{G}_A$ is zero). What is the optimal upper bound on the $2d$-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$?

2′. Let A be a measurable subset of $\mathbb{R}^d$. Suppose that we know the ratio $\frac{\lambda_d(A)^2}{\lambda_{2d}(\mathcal{E}_A)}$. What is the optimal lower bound on the kd-dimensional Lebesgue measure of all k-tuples of elements of A which induce a complete subgraph of the large-distance graph $\mathcal{G}_A$?

Note that for Problem 1′ to make sense, we do not even need to have any knowledge of the d-dimensional Lebesgue measure of A. This is because each set A satisfying the assumptions is clearly a subset (up to points belonging to a set of measure zero) of k balls of radius 2, and so there is an upper bound on the measure of A depending only on the dimension d. This consideration also shows that, in contrast to the motivating Problem 1 (where the number of vertices is essential), it is also legit to ask for the maximum d-dimensional Lebesgue measure of a set A satisfying the assumptions of Problem 1′. That is, we ask the following.

3′. Let A be a measurable subset of $\mathbb{R}^d$. Suppose that the large-distance graph $\mathcal{G}_A$ does not contain any complete subgraph on k vertices (or that the kd-dimensional Lebesgue measure of all k-tuples of elements of A which induce a complete subgraph of $\mathcal{G}_A$ is zero). What is the optimal upper bound on the d-dimensional Lebesgue measure of the set A?

Note also that while all the previous problems were nontrivial only for $k \geq 3$, Problem 3′ is interesting even in the easiest case when $k = 2$. Indeed, the solution of this special case is very well known as the *isodiametric inequality* (see [4, Theorem 2.4] or [2, Theorem 11.2.1]):

Theorem 3 (Isodiametric inequality). *Let $A \subset \mathbb{R}^d$ be a measurable set and let $\operatorname{diam}(A)$ denote its diameter. Then*

$$\lambda_d(A) \leq \left(\frac{\operatorname{diam}(A)}{2}\right)^d \omega_d \ ,$$

where ω_d denotes the d-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^d$.

By the isodiametric inequality, it immediately follows that for $k = 2$, the optimal upper bound for the d-dimensional Lebesgue measure of the set A from Problem 3′ equals ω_d.

Problem 1′ has already been considered in [1], where the following result has been obtained.

Theorem 4 ([1]). *Let $k \geq 3$ be a given natural number. Let $A \subset \mathbb{R}^d$ be a measurable set such that the kd-dimensional Lebesgue measure of all k-tuples of elements of A which induce a complete subgraph of $\mathcal{G}_A$ is zero. Then the $2d$-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$ is at most $\left(1 - \frac{1}{k-1}\right) \cdot \lambda_d(A)^2$.*

We reprove this theorem by a different method which also allows us to characterize those sets for which the inequality reduces to an equality.

Next, we focus on Problem 3′. Unfortunately, we are not able to find a general solution, so we treat only the simplest case of $d = 2$ and $k = 3$. It turns out that the answer to the problem is very natural and predictable: the optimal upper bound is attained by the disjoint union of two unit balls that are far apart. However, the proof is very far from being trivial and contains some tedious computations. A precise formulation of the result follows.

Theorem 5. *Let $A \subset \mathbb{R}^2$ be a measurable set such that the large-distance graph $\mathcal{G}_A$ does not contain any complete subgraph on three vertices. Then the 2-dimensional Lebesgue measure of A is at most 2π.*

To prove this theorem, we distinguish three cases depending on the diameter $\operatorname{diam}(A)$ of the set A. The cases where either $\operatorname{diam}(A) \leq 2\sqrt{2}$ or $\operatorname{diam}(A) \geq 4$ are easy. In the last case where $2\sqrt{2} < \operatorname{diam}(A) < 4$ we proceed as follows. Suppose, without loss of generality, that A is compact and find two points $p, q \in A$ whose distance realizes the diameter of A. Then A is the disjoint union of the following three subsets: the set of points from A that are in a distance > 2 from p, the set of points from A that are in a distance > 2 from q, and the intersection of A and of the two balls centered in p and q, respectively, with radius 2. The measure of the intersection of the two balls centered in p and q, respectively, with radius 2 can be easily computed (it depends on $\operatorname{diam}(A)$, of course). The most difficult part is to obtain upper bounds on the measure of the set of those points from A that are in a distance > 2 from p (or from q). To this end, we use the observation that the diameter of such a set is at most 2, and that such a set is contained in an annulus with inner radius 2 and outer radius $\operatorname{diam}(A)$. Then we prove an analogous result to the isodiametric inequality but with the additional assumption that the set under consideration is contained in the annulus. Finally, summing all the obtained upper bounds yields the result. Although we believe that some ideas from our proof could be useful even in the cases $k > 3$ or $d > 2$, we were not able to straightforwardly adapt our argument to this more general setting. Let us remark that for $k = 3$ and $d \geq 5$, the optimal upper bound in Problem 3′ is, maybe a little bit surprisingly, strictly larger than the volume of the union of two disjoint unit balls. This can be easily verified by computing the volume of a ball with radius $\frac{2}{\sqrt{3}}$ which clearly satisfies the assumptions of the problem.

By combining Theorems 4 and 5, we obtain the following result.

Theorem 6. *Let $A \subset \mathbb{R}^2$ be a measurable set such that the large-distance graph $\mathcal{G}_A$ does not contain any complete subgraph on three vertices. Then the 4-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$ is at most $2\pi^2$.*

As far as Problem 2′ is concerned, we report the following partial solution.

Theorem 7. *Let $\gamma \in \left[0, \frac{1}{2}\right]$ be given. Suppose that A is a measurable subset of $\mathbb{R}^d$ such that the $2d$-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$ of the large-distance graph $\mathcal{G}_A$ equals $\gamma \cdot \lambda_d(A)^2$. Then the $3d$-dimensional Lebesgue measure of all triples of elements of A which induce a complete subgraph of $\mathcal{G}_A$ is at least $2\gamma\,(4\gamma - 1) \cdot \lambda_d(A)^3$.*

Last, let us turn to graphs which are not complete. Recall that the Erdős–Stone Theorem [3] states that the Turán density of a given graph H with chromatic number k is asymptotically the same as the Turán density of the complete graph on k vertices. To understand the relation between the Erdős–Stone-type setting and Turán-setting in large-distance graphs, the following proposition, based on the Lebesgue density theorem, is central.

Proposition 8. *Suppose that H is a graph of chromatic number k and of order n. Suppose that A is a measurable subset of $\mathbb{R}^d$ such that the kd-dimensional Lebesgue measure of all k-tuples of elements of A which induce a complete subgraph of the large-distance graph $\mathcal{G}_A$ is positive. Then the the nd-dimensional Lebesgue measure of all copies of H in $\mathcal{G}_A$ is positive as well.*

In particular, this implies that the upper bounds in our Problems 1′ and 3′ remain the same if we replace the complete graph on k vertices by any graph H of chromatic number k (although, in the general case, we do not know whether the bounds still remain optimal).

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