Characterization of half-radial matrices

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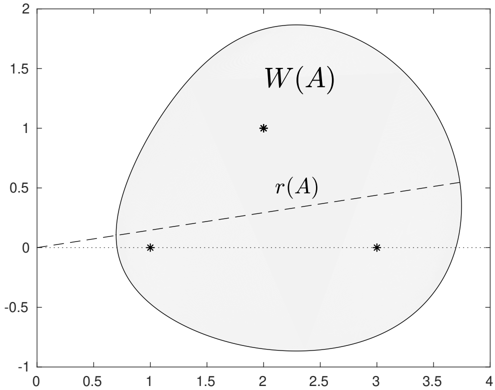
Field of values (numerical range)

and numerical radius of $A \in \mathbb{C}^{n\times n}$



$$W(A) \equiv \{z^*Az : z \in \mathbb{C}^n,\ \|z\| = 1\}$$

$$r(A) \equiv \max_{\zeta \in W(A)} |\zeta|$$

Numerical radius and the matrix 2-norm

It holds that

$$r(A) \;\le\; \|A\| \;\le\; 2\,r(A)\,,$$

- bounds attainable,
- $r(A) \;=\; \|A\| \longrightarrow$ **radial** matrices, well understood, several equivalent characterization [Horn, Johnson '94, Gustafson, Rao '97],
- $r(A) \;=\; \frac{1}{2}\|A\| \longrightarrow$ **half-radial**, sufficient conditions:
  - $\mathcal{R}(A) \perp \mathcal{R}(A^*)$,
  - $W(A)$ is a circular disk centered at origin with radius $\frac{1}{2}\|A\|$,
  - A has a 2D reducing subspace on which it is the shift.

[Gustafson, Rao '97, Hogben '13]

Characterization of half-radial matrices using Θ_A set

Notation and assumptions

- Let $A \neq 0$ be square, $n \geq 2$. Denote $\langle Az, z \rangle \equiv z^* Az$.

- Consider unit norm vectors u, v such that $Av = \|A\| u$.

- **Maximum right and left** singular subspaces:

$$\mathcal{V}_{max}(A) \equiv \{v \in \mathbb{C}^n : \ \|Av\| = \|A\| \|v\|\} ,$$

  $\mathcal{U}_{max}(A)$ analogously.

- Any vector $z \in \mathbb{C}^n$ can be uniquely decomposed,

$$z = x + y, \qquad x \in \mathcal{R}(A^*), \ y \in \mathcal{N}(A),$$

  $\mathcal{R}(A^*)$ ... **range** of A^*, $\mathcal{N}(A)$ ... **null space** of A.

Θ_A set of maximizers

definition

$$r(A) = \max_{\|z\|=1} |\langle Az, z\rangle|$$

Define

$$\Theta_A \equiv \{z \in \mathbb{C}^n : \; \|z\| = 1, \; r(A) = |\langle Az, z\rangle|, \; \langle Ax, x\rangle = 0\}.$$

where

$$z = x + y, \qquad x \in \mathcal{R}(A^*), \; y \in \mathcal{N}(A).$$

Theorem [Hnětynková, T. '18]

$$\|A\| \; = \; 2r(A) \iff \Theta_A \neq \{\emptyset\}$$

In particular, if $\mathcal{R}(A) \perp \mathcal{R}(A^*)$, then Θ_A is non-empty.

Maximum singular subspaces

and half-radial matrices

Assume without loss of generality that $\|A\| = 1$.

Theorem [Hnětynková, T. '18]

$\|A\| = 2r(A) \iff \forall$ unit norm $v \in \mathcal{V}_{max}(A)$ it holds that

$$v \in \mathcal{N}(A^*), \qquad Av \in \mathcal{N}(A),$$

and

$$z = \frac{1}{\sqrt{2}}(v + Av) \quad \text{maximizes} \quad |\langle Az, z\rangle|.$$

- $v \in \mathcal{R}(A^*)$, $Av \in \mathcal{N}(A) \Rightarrow z \in \Theta_A$.
- $\|A\| = 2r(A) \Rightarrow \mathcal{V}_{max}(A) \subseteq \mathcal{N}(A^*)$, $\mathcal{U}_{max}(A) \subseteq \mathcal{N}(A)$.

$\mathcal{V}_{max}(A) \subseteq \mathcal{N}(A^*),\ \mathcal{U}_{max}(A) \subseteq \mathcal{N}(A)$.

is not sufficient for A to be half-radial

Consider A, $\|A\| = 1$, $\frac{1}{2} < \rho < 1$,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Obviously,

$$\mathcal{V}_{max}(A) = span\{e_1\} = \mathcal{N}(A^*),$$

$$\mathcal{U}_{max}(A) = span\{e_3\} = \mathcal{N}(A),$$

but A is not half-radial,

$$r(A) \geq |\langle Ae_2, e_2 \rangle| = \rho > \frac{1}{2}.$$

[Marie Kubínová]

Structure of Θ_A

From the previous (assuming $\|A\| = 1$)

$$\Theta_A \neq \{\emptyset\} \iff \|A\| = 2r(A) \;\Rightarrow\; z = \frac{1}{\sqrt{2}}(v + Av) \in \Theta_A,$$

where $v \in \mathcal{V}_{max}(A)$, $\|v\| = 1$.

Theorem [Hnětynková, T. '18]

Θ_A is either empty or

$$\Theta_A = \left\{ \frac{1}{\sqrt{2}} \left(e^{i\alpha} v + e^{i\beta} Av \right) : \; v \in \mathcal{V}_{max}(A), \; \|v\| = 1, \; \alpha, \beta \in \mathbb{R} \right\}.$$

Algebraic structure of half-radial matrices

Algebraic structure of half-radial matrices

A generalization of result by [Gustafson, Rao '97]

Theorem [Hnětynková, T. '18]

A is half-radial $\iff$ A is unitarily similar to the matrix

$$(I_m \otimes J) \oplus B = \begin{bmatrix} J & & & \\ & \ddots & & \\ & & J & \\ & & & B \end{bmatrix},$$

where $m = \dim \mathcal{V}_{max}(A)$,

$$J \;=\; \|A\| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and B is a matrix satisfying $\|B\| < \|A\|$, $r(B) \le \frac{1}{2}\|A\|$.

Half-radial matrices
and Crouzeix's conjecture

Crouzeix's conjecture

" *Where in the complex plane does a matrix live?* " [Nick Trefethen]

Try to determine sets $\Omega \subset \mathbb{C}$ associated with A such that

$$\| p(A) \| \sim \| p \|_{\Omega} = \max_{\zeta \in \Omega} | p(\zeta) | .$$

" *When eigenvalues do not tell the whole story,*
the field of values may give more info. " [Anne Greenbaum]

For any A and any polynomial p it holds that

$$\| p(A) \| \leq c \| p \|_{W(A)} ,$$

- **conjecture** $c = 2$ [Crouzeix '04],
- proof $c = 11.08$ [Crouzeix '07],
- proof $c = 1 + \sqrt{2}$ [Crouzeix, Palencia '17].

Crouzeix's inequality

holds in some cases

$$\| p(A) \| \leq 2 \| p \|_{W(A)}$$

- if A is normal (2 can be improved to 1),
- $n = 2$ [Crouzeix '04],
- $p(\zeta) = \zeta^k$ [Berger, Pearcy '66],
- if $W(A)$ is a disk [Badea '04, Okubo, Ando '75, von Neumann '51],
- $$A = \begin{bmatrix} \lambda & \alpha_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \alpha_{n-1} \\ \alpha_n & & & \lambda \end{bmatrix}.$$

[Choi, Greenbaum '12], [Choi '13]

$$\| p(A) \| \le 2 \| p \|_{W(A)}$$

Lemma [Hnětynková, T. '18]

Half-radial matrices satisfy Crouzeix's inequality.
The bound is attained for $p(\zeta) = \zeta$.

Lemma [Hnětynková, T. '18]

Let an integer $k \ge 1$ be given. It holds that

$$\| p(A) \| = 2 \| p \|_{W(A)}$$

for $p(\zeta) = \zeta^k \iff A^k$ is half-radial and $r(A^k) = r(A)^k$.

Crouzeix ratio

$$f(p, A) \; = \; \frac{\| \, p \, \|_{W(A)}}{\| \, p(A) \, \|} \; \geq \; \frac{1}{2} \; ?$$

[Greenbaum, Overton '18]

- Optimization problem $\rightarrow$ **properties**,
- use **BFGS** method, Matlab and Chebfun,
- fix p or A or none,
- **conjecture**: $\frac{1}{2}$ can be attained only for ζ^k,
- **conjecture**: only for the **Crabb-Choi-Crouzeix** matrix.

Crabb-Choi-Crouzeix matrix

Independently used by [Choi '13], [Crouzeix '15], [Crabb '71],

$$C_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad C_n = \begin{bmatrix} 0 & \sqrt{2} & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & \sqrt{2} \\ & & & & & 0 \end{bmatrix}.$$

C_n^n is half-radial and

$$\| C_n^n \| = 2\, r(C_n^n), \qquad r(C_n^n) = r(C_n)^n.$$

Crabb's theorem

Theorem [Crabb '71]

Let A be a bounded linear operator on a Hilbert space H, and let $v \in H$, $\|v\| = 1$. Suppose that $r(A) = 1$ and that $\|A^k v\| = 2$ for some integer k. Then $A^{k+1}v = 0$, $\|A^i v\| = \sqrt{2}$, $i = 1, 2, \ldots, k-1$, the elements $v, Av, \ldots, A^k v$ are mutually orthogonal, and their linear span is a reducing subspace of A.

$\Rightarrow$

Lemma [Hnětynková, T. '18]

Let $A \in \mathbb{C}^{(n+1)\times(n+1)}$, $r(A) = 1$. It holds that

$$\|\, p(A) \,\| \;=\; 2 \,\|\, p \,\|_{W(A)}$$

for $p = \zeta^n \iff A$ is unitarily similar to C_n .

Exclusivity of the Crabb-Choi-Crouzeix matrix

Theorem [Hnětynková, T. '18]

Let $A \in \mathbb{C}^{(n+1)\times(n+1)}$. It holds that

$$\|\,p(A)\,\| \;=\; 2\,\|\,p\,\|_{W(A)}$$

for $p = \zeta^k$ and $1 \leq k \leq n \iff$ A is unitarily similar to

$$r(A)\begin{bmatrix} C_k & \\ & B \end{bmatrix},$$

where $r(B) \leq 1$ and $\|B^k\| \leq 2$.

Conjectures [Greenbaum, Overton '18]:

- $\frac{1}{2}$ can be attained only for ζ^k ?
- only for the **Crabb-Choi-Crouzeix** matrix $\rightarrow$ **yes**.

Summary

$$\|A\| \ \le\ 2\, r(A)$$

- A is **half-radial** $\Longleftrightarrow \Theta_A \neq \{\emptyset\}$.
- **Structure** of the set Θ_A ,

$$\frac{1}{\sqrt{2}}\left(e^{\boldsymbol{i}\alpha} v + e^{\boldsymbol{i}\beta} A v\right)$$

  where $v \in \mathcal{V}_{max}(A),\ v \in \mathcal{N}(A^*),\ Av \in \mathcal{N}(A)$.

- **Algebraic** characterization of half-radial matrices .
- If the upper bound in Crouzeix's inequality

$$\|\, p(A)\,\| \ \le\ 2\,\|\, p\,\|_{W(A)}$$

  can be attained **only for a monomial**, then our theorem **characterizes all matrices** for which the bound is attained.

References

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- A. Greenbaum and M. L. Overton, [Numerical investigation of Crouzeix's conjecture, LAA 542 (2018), pp. 225–245]
- K. E. Gustafson, D. K. M. Rao, [Numerical range: The field of values of linear operators and matrices, Universitext, Springer-Verlag, New York, 1997]
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Thank you for your attention!