On solving linear systems arising from Shishkin mesh discretizations

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joint work with
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October 20, 2016, Prague
Seminar of Numerical Mathematics, KNM MFF UK

Problem formulation

Convection-diffusion boundary value problem

$$-\epsilon\, u'' + \alpha\, u' + \beta\, u = f \quad \text{in} \quad \Omega = (0,1),$$

$$u(0) = u_0, \quad u(1) = u_1,$$

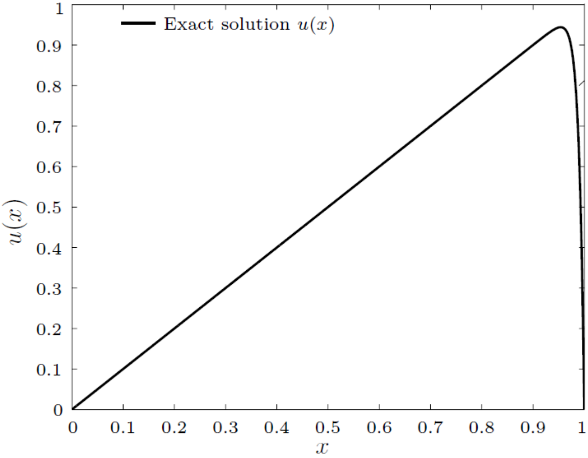
$\alpha > 0$, $\beta \geq 0$ constants, the problem is **convection dominated**

$$0 < \epsilon \ll \alpha.$$

[Stynes, 2005] (Acta Numerica)
[Roos, Stynes, and Tobiska, 1996, 2008] (book)
[Miller, O'Riordan, and Shishkin, 1996] (book)

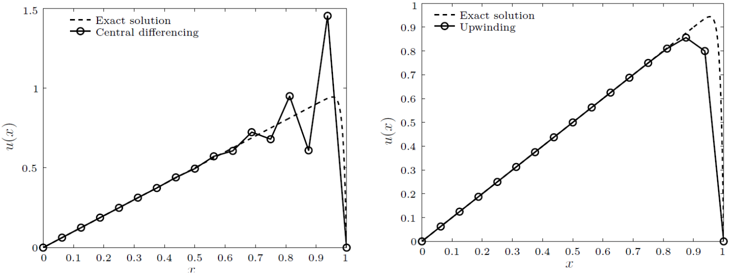
Solution and boundary layers

$\epsilon = 0.01$, $\alpha = 1$, $\beta = 0$, $u(0) = u(1) = 0$.



There are small subregions where the solution has a large gradient.

Numerical solution, equidistant mesh



Standard techniques:

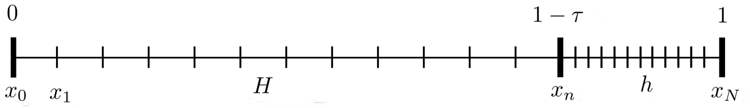
$$u'(ih) \approx \frac{u_{i+1} - u_{i-1}}{2h}, \qquad u'(ih) \approx \frac{u_i - u_{i-1}}{h}.$$

- Unnatural **oscillations** or **cannot resolve** the layers.
- Remedy: stabilization or **non-equidistant** mesh.
- We study discretizations for a **Shishkin mesh**.

Outline

Shishkin mesh on $[0, 1]$

Piecewise equidistant



N even, define the **transition point** $1 - \tau$ and n by

$$\tau \equiv \min\left\{\frac{1}{2}, \frac{\epsilon}{\alpha} 2 \ln N\right\}, \qquad n \equiv \frac{N}{2}.$$

If $\epsilon \ll \alpha$, then $1 - \tau$ is close to 1. Next define H and h by

$$H \equiv \frac{1-\tau}{n}, \qquad h \equiv \frac{\tau}{n}$$

and consider the Shishkin mesh, $x_0 = 0$,

$$x_i \equiv iH, \qquad x_{n+i} \equiv x_n + ih, \quad i = 1, \ldots, n.$$

Discretization on the Shishkin mesh - details

For simplicity $u(0) = u(1) = 0$

The **upwind** difference scheme is given by

$$-\epsilon\, \delta_x^2 u_i + \alpha\, D_x^- u_i + \beta u_i = f_i, \qquad u_0 = u_N = 0,$$

and the **central difference** scheme by

$$-\epsilon\, \delta_x^2 u_i + \alpha D_x^0 + \beta u_i = f_i, \qquad u_0 = u_N = 0,$$

where

$$\delta_x^2 u_i = \frac{2u_{i-1}}{(H+h)H} - \frac{2u_i}{Hh} + \frac{2u_{i+1}}{(H+h)h}, \qquad i = n,$$

and

$$D_x^- u_i = \frac{u_i - u_{i-1}}{H}, \quad 1 \le i \le n, \qquad D_x^0 u_i = \frac{u_{i+1} - u_{i-1}}{h+H}, \quad i = n.$$

surveys [Linss, Stynes, 2001], [Stynes, 2005], [Kopteva, O'Riordan, 2010]

Shishkin mesh and ϵ-uniform convergence

- The **upwind** difference scheme

$$-\epsilon\, \delta_x^2 u_i + \alpha\, D_x^- u_i + \beta u_i = f_i, \qquad u_0 = u_N = 0.$$

There exists a constant C such that

$$|u(x_i) - u_i| \le C \left(\frac{\ln N}{N} \right), \quad i = 0, \ldots, N,$$

see, e.g., [Stynes, 2005].

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There exists a constant C such that

$$|u(x_i) - u_i| \le C \left(\frac{\ln N}{N} \right)^2, \quad i = 0, \ldots, N,$$

[Andreev and Kopteva, 1996], a difficult proof, the scheme does not satisfy a discrete maximum principle. [Kopteva and Linss, 2001].

Outline

Structure of the matrix

$$A = \left[\begin{array}{cccc|c|cccc}
a_H & b_H & & & & & & & \\
c_H & \ddots & \ddots & & & & & & \\
 & \ddots & \ddots & b_H & & & & & \\
 & & c_H & a_H & b_H & & & & \\
\hline
 & & & c & a & b & & & \\
\hline
 & & & & c_h & a_h & b_h & & \\
 & & & & & c_h & \ddots & \ddots & \\
 & & & & & & \ddots & \ddots & b_h \\
 & & & & & & & c_h & a_h
\end{array}\right]$$

Entries

The upwind scheme

$$c_H = -\frac{\epsilon}{H^2} - \frac{\alpha}{H}, \qquad a_H = \frac{2\epsilon}{H^2} + \frac{\alpha}{H} + \beta, \qquad b_H = -\frac{\epsilon}{H^2},$$

$$c = -\frac{2\epsilon}{H(H+h)} - \frac{\alpha}{H}, \qquad a = \frac{2\epsilon}{hH} + \frac{\alpha}{H} + \beta, \qquad b = -\frac{2\epsilon}{h(H+h)},$$

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The central difference scheme

$$c_H = -\frac{\epsilon}{H^2} - \frac{\alpha}{2H}, \qquad a_H = \frac{2\epsilon}{H^2} + \beta, \qquad b_H = -\frac{\epsilon}{H^2} + \frac{\alpha}{2H},$$

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Matrix properties

- nonsymmetric
- A is M-matrix for the upwind scheme
- A is not an M-matrix for the central difference scheme
- A is highly **nonnormal**. Consider

$$-\epsilon u'' + u' = 1, \quad u(0) = 0, \quad u(1) = 0,$$

$\epsilon = 10^{-8}$ and $N = 46$, and the spectral decomposition

$$A = YDY^{-1}.$$

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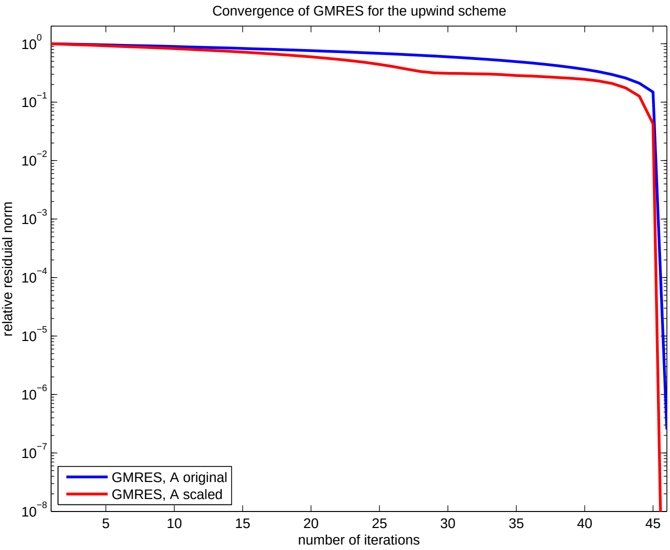
	upwind	upwind sc.	central	central sc.
$\kappa(A)$	4.05×10^{10}	2.96×10^{3}	6.23×10^{10}	2.95×10^{3}
$\kappa(Y)$	1.51×10^{17}	1.23×10^{19}	4.11×10^{3}	1.87×10^{2}

Solving linear system using GMRES



Outline

Multiplicative Schwarz method

Idea of solving $Ax = b$

- Given an approximation $x^{(k)}$, then $x = x^{(k)} + y$ and y satisfies

$$Ay = b - Ax^{(k)}.$$

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- Solve on the **first domain**

$$(R_1 A R_1^T)\,\tilde{y} \;=\; R_1(b - Ax^{(k)})$$

and approximate y by **prolongation** of $\tilde{y}$, i.e., by $R_1^T \tilde{y}$. Define

$$x^{(k+\frac{1}{2})} = x^{(k)} + R_1^T (R_1 A R_1^T)^{-1} R_1 (b - Ax^{(k)}).$$

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- Similarly, use $x^{(k+\frac{1}{2})}$ on the **second domain** and prolong,

$$x^{(k+1)} = x^{(k+\frac{1}{2})} + R_2^T(R_2 A R_2^T)^{-1} R_2\left(b - Ax^{(k+\frac{1}{2})}\right).$$

Multiplicative Schwarz method

Formalism

Define

$$P_i = R_i^T A_i^{-1} R_i A, \quad A_i \equiv R_i A R_i^T, \quad i = 1, 2.$$

The multiplicative Schwarz is the iterative scheme

$$x^{(k+1)} \;=\; T\,x^{(k)} + v, \quad T = (I - P_2)(I - P_1),$$

where v is defined such that the scheme is **consistent**.

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where v is defined such that the scheme is **consistent**. Hence,

$$x - x^{(k+1)} \;=\; T^{k+1}\,(x - x_0),$$

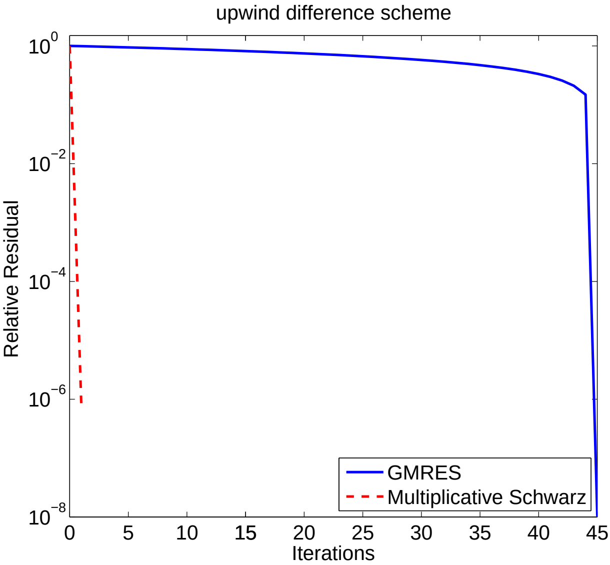
and

$$\|x - x^{(k+1)}\| \;\leq\; \|T^{k+1}\|\,\|x - x_0\|.$$

Is it convergent in our case?

Multiplicative Schwarz method

Experiment



Outline

Convergence analysis

Exploiting the structure

$$\frac{\|x - x^{(k+1)}\|}{\|x - x_0\|} \;\leq\; \|T^{k+1}\|\,.$$

Using $T = (I - P_2)(I - P_1)$ we are able to show that

$$T = \left[\begin{array}{c|c|c} & t_1 & \\ & \vdots & \\ 0\ldots0 & t_{n+1} & 0\ldots0 \\ & \vdots & \\ & t_{N-1} & \end{array}\right] \;=\; t\,e_{n+1}^T.$$

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Therefore, $T^2 = t\,(e_{n+1}^T\,t)\,e_{n+1}^T = t_{n+1}\,T$, and

$$\|T^{k+1}\| \;=\; |t_{n+1}|^k\,\|T\|\,.$$

How to bound $|t_{n+1}|$, and $\|T\|$ in a convenient norm ($\|\cdot\|_\infty$)?

$$A = \left[\begin{array}{c|c|c} A_H & & \\ & b_H & \\ \hline c & a & b \\ \hline & c_h & \\ & & A_h \\ & & \end{array}\right].$$

Let $m \equiv n-1$, $\rho \equiv |t_{n+1}|$ ... the **convergence factor**. Then,

$$\rho = \left| \frac{b b_H \left(A_H^{-1}\right)_{m,m}}{a - c b_H \left(A_H^{-1}\right)_{m,m}} \right| \left| \frac{c c_h \left(A_h^{-1}\right)_{1,1}}{a - b c_h \left(A_h^{-1}\right)_{1,1}} \right| .$$

A matrix $B = [b_{i,j}]$ is called a nonsingular **M-matrix** when

- B is nonsingular,
- $b_{i,i} > 0$ for all i, $b_{i,j} \leq 0$ for all $i \neq j$,
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If A_H and A_h are nonsingular **M-matrices**, then using [Nabben 1999],

$$\left(A_H^{-1}\right)_{m,m} \leq \min\left\{\frac{1}{|b_H|}, \frac{1}{|c_H|}\right\},$$

$$\left(A_h^{-1}\right)_{1,1} \leq \min\left\{\frac{1}{|b_h|}, \frac{1}{|c_h|}\right\}.$$

A sufficient condition: The sign conditions & irreducibly diagonal dominant $\Rightarrow$ nonsingular M-matrix. [Meurant, 1996], [Hackbusch, 2010]

The matrices **A_H and A_h are M-matrices**, and we know that

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Theorem (the upwind scheme) [Echeverría, Liesen, T. , Szyld, 2016]

For the upwind scheme we have

$$\rho \leq \frac{\epsilon}{\epsilon + \alpha H} \leq \frac{\epsilon}{\epsilon + \frac{\alpha}{N}},$$

and

$$\|T\|_\infty \leq \frac{\epsilon}{\epsilon + \alpha H}.$$

Convergence analysis

The central difference scheme

- A_h is still an M-matrix.
- If $\alpha H > 2\epsilon$, i.e. $b_H > 0$, then A_H is not an M-matrix
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- How to bound $(A_H^{-1})_{m,m}$? ... results by [Usmani 1994]
- We proved: If $m = N/2 - 1$ is even, then

$$b_H (A_H^{-1})_{m,m} \leq \frac{1 - \left|\frac{b_H}{c_H}\right|^m}{\left|\frac{c_H}{b_H}\right| + \left|\frac{b_H}{c_H}\right|^m} < \frac{2m\epsilon}{\epsilon + \frac{\alpha H}{2}} .$$

A_h **is** M**-matrix**, if $\alpha H > 2\epsilon$, A_H is not an M-matrix.

Theorem (the central diff. scheme) [Echeverría, Liesen, T. , Szyld, 2016]

Let $m = N/2 - 1$ be even, and let $\alpha H > 2\epsilon$. For the central differences we have

$$\rho < \frac{2m\epsilon}{\epsilon + \frac{\alpha H}{2}} < N \frac{\epsilon}{\epsilon + \frac{\alpha}{N}},$$

$$\|T\|_\infty < 2.$$

Thus, the error of the multiplicative Schwarz method satisfies

$$\frac{\|e^{(k+1)}\|_\infty}{\|e^{(0)}\|_\infty} < 2 \left(\frac{2m\epsilon}{\epsilon + \frac{\alpha H}{2}} \right)^k .$$

Remarks on diagonally scaled linear algebraic systems

$DAx = Db$

The ill-conditioning can be avoided by diagonal scaling [Roos 1996]: $Ax = b$ is multiplied from the left by

$$D = \left[\begin{array}{c|c|c} d_H I_m & & \\ \hline & d & \\ \hline & & d_h I_m \end{array}\right].$$

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- Such a scaling **preserves** the **Toeplitz** structure and the M-**matrix** property of the submatrices.
- Analysis depends on these properties and on ratios between elements in the same row such as $|b/a|$ and $|b_H/c_H|$. These **ratios are invariant** under diagonal scaling.

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- Analysis depends on these properties and on ratios between elements in the same row such as $|b/a|$ and $|b_H/c_H|$. These **ratios are invariant** under diagonal scaling.
- Consequently, all **convergence bounds hold** for the multiplicative Schwarz method applied to $DAx = Db$.

Outline

Schwarz method as a preconditioner

We have consistent scheme

$$x^{(k+1)} \; = \; T\,x^{(k)} + v.$$

Hence, x solves $Ax = b$ and also "the **preconditioned** system"

$$(I - T)x = v.$$

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We can formally define a **preconditioner** M such

$$Ax = b \quad \Leftrightarrow M^{-1}Ax = M^{-1}b \quad \Leftrightarrow (I - T)x = v.$$

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$$\begin{aligned} x^{(k+1)} &= x^{(k)} + (I - T)(x - x^{(k)}) \\ &= x^{(k)} + M^{-1}r^{(k)}. \end{aligned}$$

Schwarz method as a preconditioner
for GMRES

- The multiplicative **Schwarz** method as well as **GMRES** applied to the preconditioned system obtain their approximations from **the same Krylov subspace**.
- In terms of the residual norm, the preconditioned **GMRES** will **always** converge **faster** than the multiplicative Schwarz.

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- In terms of the residual norm, the preconditioned **GMRES** will **always** converge **faster** than the multiplicative Schwarz.
- Moreover, in this case, the iteration matrix T has **rank-one structure**, and

$$\dim\left(\mathcal{K}_k(I - T, r_0)\right) \leq 2.$$

- Therefore, GMRES converges in **at most 2 steps**,
  . . . a motivation for more dimensional cases.

How to multiply by T

Schwarz or preconditioned GMRES $\to$ only multiply by T,

$$T = (I - P_2)(I - P_1), \qquad P_i = R_i^T (R_i A R_i^T)^{-1} R_i A,$$

i.e., to solve systems of the form $\left(m = n - 1 = \frac{N}{2} - 1\right)$

$$\left[\begin{array}{c|c} A_H & b_H \\ \hline c & a \end{array}\right] \left[\begin{array}{c} y_{1:m} \\ \hline y_{m+1} \end{array}\right] = \left[\begin{array}{c} z_{1:m} \\ \hline z_{m+1} \end{array}\right].$$

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Using the **Schur complement**,

$$\left(A_H - \frac{b_H c}{a} e_m e_m^T\right) y_{1:m} = z_{1:m} - z_{m+1} \frac{b_H}{a_H} e_m.$$

Then apply **Sherman-Morrison** formula.

We need only to solve systems with A_H (Toeplitz)!

Outline

Numerical examples

Consider

$$-\epsilon u'' + u' = 1, \quad u(0) = 0, \quad u(1) = 0,$$

i.e.

$$\alpha = 1, \quad \beta = 0, \quad f(x) \equiv 1\,.$$

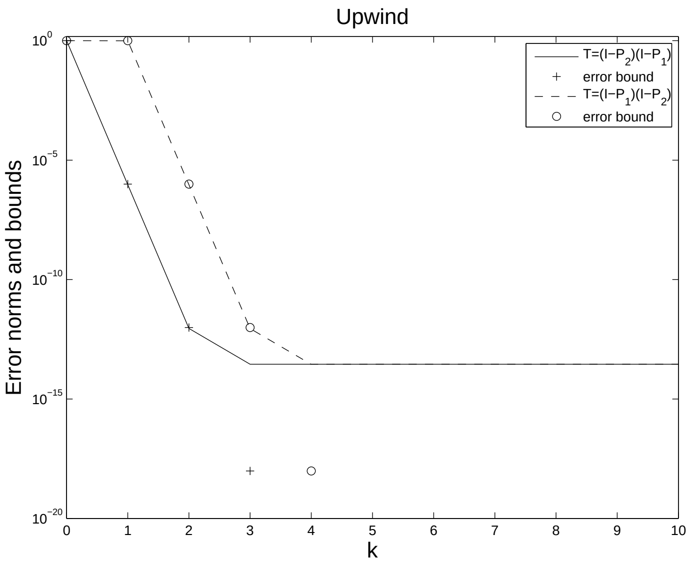
Choose $N = 198$, various values of ϵ.

	upwind		central differences	
ϵ	ρ_{up}	our bound	ρ_{cd}	our bound
10^{-8}	9.4×10^{-7}	9.9×10^{-7}	1.8×10^{-4}	3.9×10^{-4}
10^{-6}	9.4×10^{-5}	9.9×10^{-5}	1.8×10^{-2}	3.9×10^{-2}
10^{-4}	9.3×10^{-3}	9.8×10^{-3}	8.3×10^{-1}	3.8×10^{-0}

$$\rho_{up} < \frac{\epsilon}{\epsilon + \alpha H}, \qquad \rho_{cd} < \frac{2m\epsilon}{\epsilon + \frac{\alpha H}{2}}\,.$$

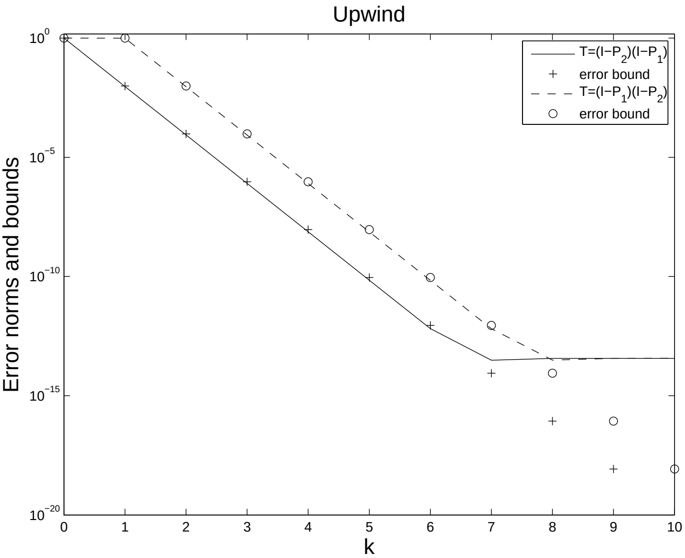
Numerical examples

Upwind, $\epsilon = 10^{-8}$



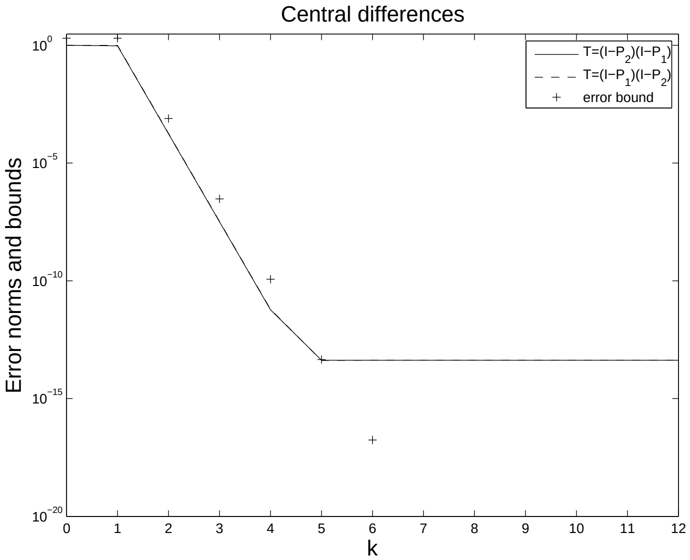
Numerical examples

Upwind, $\epsilon = 10^{-4}$



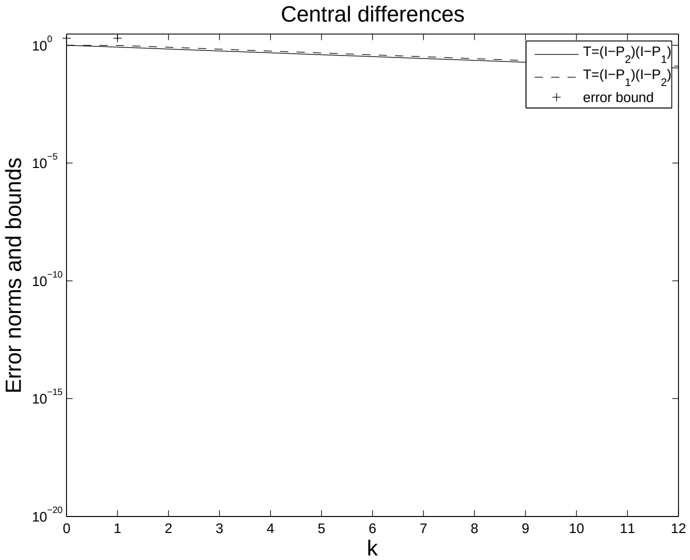
Numerical examples

Central differences, $\epsilon = 10^{-8}$



Numerical examples

Central differences, $\epsilon = 10^{-4}$



Conclusions and further work

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- Thanks to the **rank-one structure** of T, the preconditioned GMRES converges in **two steps**.
- Inspired by 1D case (**preconditioner**, **low-rank** structure), we can study **2D case**.

Related papers

- C. Echeverría, J. Liesen, P. Tichý, and D. Szyld, [Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on a Shishkin mesh, (2016), in preparation]
- J. Miller, E. O'Riordan, and G. Shishkin, [Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions, World Scientific, 1996.]
- H-G. Roos, M. Stynes, L. Tobiska, [Robust Numerical Methods for Singularly Perturbed Differential Equations, second edition, Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2008, 604 pp.]
- M. Stynes, [Steady-state convection-diffusion problems, Acta Numerica, 14 (2005), pp. 445–508.]

Thank you for your attention!