An introduction to Abstract Algebraic Logic

Part I

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Consequence relations and closure operators

- AAL is the study of logics understood as consequence relations.

Definition

A consequence relation on a set A is a relation $\vdash \subseteq \mathcal{P}(A) \times A$ s.t. for all $X \cup Y \cup \{x\} \subseteq A$,

R. If $x \in X$, then $X \vdash x$.

M. If $X \vdash x$ and $X \subseteq Y$, then $Y \vdash x$.

C. If $X \vdash x$ and $Y \vdash y$ for all $y \in X$, then $Y \vdash x$.

A closure operator on A is a map $C : \mathcal{P}(A) \to \mathcal{P}(A)$ s.t. for all $X, Y \in \mathcal{P}(A)$,

R. $X \subseteq C(X)$.

M. If $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

C. $CC(X) = C(X)$.

Consequence relations and closure operators

- Consequence relations and closure operators are two faces of the same coin:
- If $\vdash$ is a consequence relation on A, then define

$$C_\vdash : \mathcal{P}(A) \to \mathcal{P}(A)$$

setting, for every $X \subseteq A$,

$$C_\vdash(X) := \{x \in A : X \vdash x\}.$$

The map $C_\vdash$ is a closure operator.
- If C is a closure operator on A, then define

$$\vdash_C \subseteq \mathcal{P}(A) \times A$$

setting, for every $X \cup \{x\} \subseteq A$,

$$X \vdash_C x \iff x \in C(X).$$

Then $\vdash_C$ is a consequence relation.
- These transformations are indeed inverse one to the other.

Logics as substitution invariant consequence relations

Convention

From now on we work within a fixed (but arbitrary) algebraic language. In particular,

$$\boldsymbol{Fm} = \text{ term algebra built up with countably many variables.}$$

Definition

A logic is a consequence relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$, which is substitution-invariant in the sense that for every substitution $\sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma\Gamma \vdash \sigma\varphi.$$

- The idea is that logics are consequence relations whose inferences are valid in virtue of their form (as opposed to their content).

Examples: substructural logics

Example

- Let K be a variety of residuated lattices (with involution). Then set

$$\Gamma \vdash_{\mathsf{K}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and hom } v \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } 1 \leq v(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } 1 \leq v(\varphi).$$

- The relation $\vdash_{\mathsf{K}}$ is a logic in our sense, the substructural logic naturally associated to K:

$$\text{if } \mathsf{K} = \text{Heyting algebras, then } \vdash_{\mathsf{K}} = \text{intuitionistic logic}$$
$$\text{if } \mathsf{K} = \text{MV-algebra, then } \vdash_{\mathsf{K}} = \text{Łukasiwicz logic}$$
$$\text{if } \mathsf{K} = \text{De Morgan monoids, then } \vdash_{\mathsf{K}} = \text{relenvance logic } \boldsymbol{R}_t.$$

Examples: modal logics

Example

- Let F be a class of all Kripke frames $\boldsymbol{W} = \langle W, R \rangle$. Then set

$$\Gamma \vdash^{l} \varphi \iff \text{for every } \boldsymbol{W} \in \mathsf{F}, v \colon Var \to \mathcal{P}(W) \text{ and } w \in W, \text{ if } v, w \Vdash \gamma \text{ for all } \gamma \in \Gamma, \text{ then } v, w \Vdash \varphi.$$
$$\Gamma \vdash^{g} \varphi \iff \text{for every } \boldsymbol{W} \in \mathsf{F} \text{ and } v \colon Var \to \mathcal{P}(W), \text{ if } v, w \Vdash \gamma \text{ for all } \gamma \in \Gamma \text{ and } w \in W, \text{ then } v, w \Vdash \varphi \text{ for all } w \in W.$$

- $\vdash^{l}$ and $\vdash^{g}$ are respectively the local and global modal consequences of the system $\mathcal{K}$.
- They are different, since

$$x \nvdash^{l} \Box x \text{ while } x \vdash^{g} \Box x.$$

Matrices

- Logics may have different kind of semantics, e.g. relational, topological, game-theoretic, categorical and... matrix-based.

Definition

1. A (logical) matrix is a pair $\langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an algebra and $F \subseteq A$.
2. Every class of matrices M induces a logic as follows:

$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \boldsymbol{A}, F \rangle \in \mathsf{M} \text{ and hom } v \colon \boldsymbol{Fm} \to \boldsymbol{A} \text{ if } v[\Gamma] \subseteq F, \text{ then } v(\varphi) \in F.$$

Example

- If K is a variety of residuated lattices, then $\vdash_{\mathsf{K}}$ is the logic induced by the following class of matrices:

$$\{\langle \boldsymbol{A}, \uparrow 1 \rangle : \boldsymbol{A} \in \mathsf{K}\}.$$

Matrices as models of logics

Definition

Let $\vdash$ be a logic. A matrix $\langle \boldsymbol{A}, F \rangle$ is a model of a logic $\vdash$ when

$$\text{if } \Gamma \vdash \varphi, \text{ then for every hom } v \colon \boldsymbol{Fm} \to \boldsymbol{A} \text{ if } v[\Gamma] \subseteq F, \text{ then } v(\varphi) \in F.$$

Then we set $\mathrm{Mod}(\vdash) := \{\langle \boldsymbol{A}, F \rangle : \langle \boldsymbol{A}, F \rangle \text{ is a model of } \vdash\}$.

Completeness (1st version)

Every logic $\vdash$ coincides with the logic $\vdash_{\mathrm{Mod}(\vdash)}$ induced by its models $\mathrm{Mod}(\vdash)$.

- Drawback: $\mathrm{Mod}(\vdash)$ is a very artificial class of matrices, since

$$\langle \boldsymbol{A}, A \rangle \in \mathrm{Mod}(\vdash) \text{ for every algebra } \boldsymbol{A}.$$

Leibniz congruence

- We need a process to tame the matrices in $\mathrm{Mod}(\vdash)$:

Definition

Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$.

1. A congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is compatible with F when

$$\text{if } a \in F \text{ and } \langle a, b\rangle \in \theta, \text{ then } b \in F.$$

2. The largest such congruence (it exists!) is called the Leibniz congruence of F (over $\boldsymbol{A}$), and is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}F$.
3. The reduction of $\langle \boldsymbol{A}, F\rangle$ is $\langle \boldsymbol{A}, F\rangle^* := \langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F, F/\boldsymbol{\Omega}^{\boldsymbol{A}}F\rangle$.

Proposition

Every class of matrices M induces the same logic of M^*.

Reduced models

Definition

Let $\vdash$ be a logic. The class of reduced models of $\vdash$ is

$$\begin{aligned}\mathrm{Mod}^*(\vdash) :=& \mathbb{I}\,\mathrm{Mod}(\vdash)^* \\ =& \mathbb{I}\,\{\langle \boldsymbol{A}, F\rangle^* : \langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash)\} \\ =& \{\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash) : \boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathrm{Id}_{\boldsymbol{A}}\} \\ \mathrm{Alg}^*(\vdash) :=& \{\boldsymbol{A} : \exists F \subseteq A \text{ s.t. } \langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}^*(\vdash)\}.\end{aligned}$$

Completeness (2nd version)

Every logic $\vdash$ coincides with the logic $\vdash_{\mathrm{Mod}^*(\vdash)}$ induced by its reduced models $\mathrm{Mod}^*(\vdash)$.

Reduced models: examples

- In most cases, reduced models (as opposed to arbitrary models) of a logic are its intended matrix semantics.

Example: substructural logics

- If K is a variety of residuated lattices (with involution), then

$$\mathrm{Mod}^*(\vdash_{\mathrm{K}}) = \{\langle \boldsymbol{A}, \uparrow 1\rangle : \boldsymbol{A} \in \mathrm{K}\}.$$

Example: modal logics

- Let MA the variety of modal algebras, then

$$\begin{aligned}\mathrm{Mod}^*(\vdash^g) &= \{\langle \boldsymbol{A}, \{1\}\rangle : \boldsymbol{A} \in \mathrm{MA}\} \\ \mathrm{Mod}^*(\vdash^l) &= \{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathrm{MA} \text{ and } F \text{ is a lattice filter,} \\ &\qquad \text{which includes a single open filter, i.e. } \{1\}\}.\end{aligned}$$

Leibniz congruence again

- Reduced models have been defined thanks to the Leibniz congruence.

Definition

Let $\boldsymbol{A}$ be an algebra. A map $p \colon \boldsymbol{A} \to \boldsymbol{A}$ is a unary polynomial function of $\boldsymbol{A}$ if there is a term $\varphi(x, \vec{y})$ and elements $\vec{c} \in A$ such that for every $a \in A$,

$$p(a) = \varphi^{\boldsymbol{A}}(a, \vec{c}).$$

Theorem

Let $\boldsymbol{A}$ be an algebra, $F \subseteq A$, and $a, b \in A$.

$$\begin{aligned}\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F \iff & \text{for every unary pol. function } p \colon \boldsymbol{A} \to \boldsymbol{A}, \\ & p(a) \in F \text{ if and only if } p(b) \in F.\end{aligned}$$

Equational consequences

Convention

$Eq =$ set of equations in countably many variables.

Definition

Let K be a class of algebras and $\Theta \cup \{\alpha \approx \psi\} \subseteq Eq$.

$$\Theta \vDash_{\mathsf{K}} \varphi \approx \psi \iff \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and hom } v\colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } v(\alpha) = v(\beta) \text{ for every } \alpha \approx \beta \in \Theta, \text{ then } v(\varphi) = v(\psi).$$

The relation $\vDash_{\mathsf{K}}$ is the equational consequence relative to K.

- Remark: $\vDash_{\mathsf{K}}$ is not Birkhoff consequence of equational logic.

Generalized quasi-equations

Definition

1. A generalized quasi-equation is a formula

$$\Phi := \bigwedge_{i \in I} \alpha_i \approx \beta_i \to \varphi \approx \psi$$

written in at most countably many variables.

2. Let K be a class of algebras, then

$$\mathsf{K} \vDash \bigwedge_{i \in I} \alpha_i \approx \beta_i \to \varphi \approx \psi \iff \{\alpha_i \approx \beta_i : i \in I\} \vDash_{\mathsf{K}} \varphi \approx \psi$$

$$\mathsf{K} \vDash \forall \vec{x}\Big(\big(\bigwedge_{i \in I} \alpha_i \approx \beta_i\big) \to \varphi \approx \psi\Big)$$

3. A quasi-equation is a generalized quasi-equation whose antecendent is finite.

Generalized quasi-equations

Theorem

1. A class of algebras K is axiomatizable by generalized quasi-equations if and only if it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{U}$, where

$$\mathbb{U}(\mathsf{W}) := \{\boldsymbol{A} : \boldsymbol{B} \in \mathsf{W} \text{ for all countably generated } \boldsymbol{B} \in \mathbb{S}(\boldsymbol{A})\}.$$

2. For a generalized quasi-variety K TFAE:
   - K is axiomatizable by quasi-equations.
   - K is closed under $\mathbb{P}_{\mathrm{u}}$.
   - $\vDash_{\mathsf{K}}$ is finitary.

Definition

A class of algebras is a (generalized) quasi-variety if is axiomatizable by (generalized) quasi-equations.

Structural transformers

Definition

A structural transformer of formulas into equations is a map

$$\boldsymbol{\tau}\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$$

which commutes with unions and substitutions, i.e.

$$\boldsymbol{\tau}(\Gamma) = \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ and } \boldsymbol{\tau}(\sigma\Gamma) = \sigma\boldsymbol{\tau}\Gamma.$$

- If $\boldsymbol{\tau}\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ is a structural transformer, then $E(x) := \boldsymbol{\tau}(x)$ is only in variable x, and for every $\Gamma \subseteq Fm$,

$$\boldsymbol{\tau}(\Gamma) = \bigcup_{\gamma \in \gamma} E(\gamma).$$

- Structural transformers $\boldsymbol{\rho}\colon \mathcal{P}(Eq) \to \mathcal{P}(Fm)$ of equations into formulas are defined similarly.

Algebraizable logics

Definition

A logic $\vdash$ is algebraizable if there exist a generalized quasi-variety K and structural transformers

$$\boldsymbol{\tau}\colon \mathcal{P}(Fm) \longleftrightarrow \mathcal{P}(Eq)\colon \boldsymbol{\rho}$$

such that

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi) \qquad \text{(ALG1)}$$
$$\boldsymbol{\rho}(\Theta) \vdash \boldsymbol{\rho}(\varphi \approx \psi) \iff \Theta \vDash_{\mathsf{K}} \varphi \approx \psi \qquad \text{(ALG2)}$$
$$x \approx y \mathrel{=\!\mid\!\vDash_{\mathsf{K}}} \boldsymbol{\tau}\boldsymbol{\rho}(x \approx y) \qquad \text{(ALG3)}$$
$$x \dashv\vdash \boldsymbol{\rho}\boldsymbol{\tau}(x) \qquad \text{(ALG4)}$$

Algebraizable logics

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such that

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi) \qquad \text{(ALG1)}$$
$$x \approx y \mathrel{=\!\mid\!\vDash_{\mathsf{K}}} \boldsymbol{\tau}\boldsymbol{\rho}(x \approx y) \qquad \text{(ALG3)}$$

- Remark: Conditions (ALG2) and (ALG4) are redundant.

Theorem

If $\vdash$ is algebraizable, then the class K is uniquely determined and is called the equivalent algebraic semantics of $\vdash$.

Algebraizable logics: examples

Example: substructural logics

If K is a variety of residuated lattices, then $\vdash_{\mathsf{K}}$ is algebraizable with equivalent algebraic semantics K via:

$$\boldsymbol{\tau}(\Gamma) = \{1 \leq \gamma : \gamma \in \Gamma\}$$
$$\boldsymbol{\rho}(\Theta) = \{(\alpha \backslash \beta) \wedge (\beta \backslash \alpha) : \alpha \approx \beta \in \Theta\}.$$

- Exercise: Prove that the global modal consequence $\vdash^{g}$ is algebraizable with equivalent algebraic semantics the variety of modal algebras.

Non-Algebraizable logics: examples

Proposition

Algebraizable logics have theorems, i.e. if $\vdash$ is algebraizable, then there is φ such that $\emptyset \vdash \varphi$.

Example: non-algebraizable logics

- All logics without theorems, e.g.

  $\{\wedge, \vee\}$-fragment of classical logic
  Belnap-Dunn logic (without constants)
  Kleene 3-valued logics (without constants)

Algebraizable logics: syntactic characterization

- We need to investigate the definability of Leibniz congruence:

Theorem (definability of Leibniz congruence)

Let $\vdash$ be a logic and $\Delta(x, y)$ be a set of formulas. TFAE:

1. For every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$,
$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta^{\boldsymbol{A}}(a, b) \subseteq F.$$
2. The following inferences are valid in $\vdash$:
$$\emptyset \vdash \Delta(x, x) \quad \text{(Ref)}$$
$$x, \Delta(x, y) \vdash y \quad \text{(MP)}$$
$$\bigcup_{i \leq n} \Delta(x_i, y_i) \vdash \Delta(f(\vec{x}), f(\vec{y})) \quad \text{(Rep)}$$
for all connectives f of $\vdash$.

Algebraizable logics: syntactic characterization

Theorem (syntactic characterization of algebraizability)

A logic $\vdash$ is algebraizable if and only if there are a set of formulas $\Delta(x, y)$ and a set of equations $E(x)$ such that for all connectives f,
$$\emptyset \vdash \Delta(x, x) \quad \text{(Ref)}$$
$$x, \Delta(x, y) \vdash y \quad \text{(MP)}$$
$$\bigcup_{i \leq n} \Delta(x_i, y_i) \vdash \Delta(f(\vec{x}), f(\vec{y})) \quad \text{(Rep)}$$
$$\Delta E(x) \dashv\vdash x \quad \text{(ALG3)}$$

In this case,
$$\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}^*(\vdash) \iff F = \{a \in A : \boldsymbol{A} \vDash E(a)\} \text{ and }$$
$$\boldsymbol{A} \vDash E(\Gamma) \to E(\varphi) \text{ for every } \Gamma \vdash \varphi$$
$$\boldsymbol{A} \vDash E\Delta(x \approx y) \to x \approx y.$$

Algebraizable logics: syntactic characterization

Corollary

1. The equiv. alg. semantics of an alg. logic $\vdash$ is $\mathrm{Alg}^*(\vdash)$.
2. Algebriazability is preserved by extensions (not necessarily axiomatic).

Theorem

If $\vdash$ is an algebraizable logic with equivalent algebraic semantics K, then there is a dual isomorphism between the complete lattice of extensions of $\vdash$ and subgeneralized quasi-varieties of K.

Example

The typical correspondence between axiomatic extensions and subvarieties (e.g. normal modal logics, superintuitionistic logics etc.) is a special instance of this phenomenon.

Algebraizable logics: semantic characterization

- Generalized quasi-varieties need not be closed under $\mathbb{H}$.

Definition

Let K be a generalized quasi-variety and $\boldsymbol{A}$ and algebra. A congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is a K-congruence if $\boldsymbol{A}/\theta \in \mathsf{K}$.
$$\mathrm{Con}_{\mathsf{K}}\boldsymbol{A} := \{\theta \in \mathrm{Con}\boldsymbol{A} : \theta \text{ is a K-congruence}\}.$$

- $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is a complete lattice, since K is closed under subdirect products (and contains the trivial algebra).

Proposition

If K is a generalized quasi-variety, then $\mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}$ coincides with the set $\mathcal{T}h(\vDash_{\mathsf{K}})$ of closed sets of $C_{\vDash_{\mathsf{K}}}$.

Algebraizable logics: semantic characterization

Definition

Let $\vdash$ be a logic and $\boldsymbol{A}$ and algebra. A set $F \subseteq A$ is a deductive filter of $\vdash$ on $\boldsymbol{A}$, if $\langle \boldsymbol{A}, F \rangle \in \mathrm{Mod}(\vdash)$.

$$\mathcal{F}i_{\vdash}\boldsymbol{A} := \{F \subseteq A : \langle \boldsymbol{A}, F \rangle \in \mathrm{Mod}(\vdash)\}.$$

- $\mathcal{F}i_{\vdash}\boldsymbol{A}$ is a complete lattice.

Proposition

If $\vdash$ is a logic, then $\mathcal{F}i_{\vdash}\boldsymbol{Fm}$ coincides with the set $\mathcal{T}h(\vdash)$ of closed sets of $C_{\vdash}$.

Algebraizable logics: semantic characterization

Theorem (semantic characterization of algebraizability)

Let $\vdash$ be a logic and K a generalized quasi-variety. TFAE:

1. $\vdash$ is algebraizable with equivalent algebraic semantics K.
2. For every algebra $\boldsymbol{A}$ there is a lattice isomorphism $\Phi^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ that commutes with endomorphisms σ in the sense that $\Phi^{\boldsymbol{A}}\sigma^{-1}F = \sigma^{-1}\Phi^{\boldsymbol{A}}F$ for every $F \in \mathcal{F}i_{\vdash}\boldsymbol{A}$.
3. There is a lattice isomorphism $\Phi \colon \mathcal{T}h(\vdash) \to \mathcal{T}h(\vDash_{\mathsf{K}})$ that commutes with substitutions σ in the sense that $\Phi\sigma^{-1}\Gamma = \sigma^{-1}\Phi\Gamma$ for every $\Gamma \in \mathcal{T}h(\vdash)$.

Moreover, $\Phi^{\boldsymbol{A}}$ can be always taken to be $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$.

Algebraizable logics: semantic characterization

Example: semantic meaning of algebraizability

- Thus algebraizability abstracts the idea of a correspondence between congruences and special subsets of algebras (e.g. filters/ideals):

Boolean algebras	$\longleftrightarrow$	lattice filters
Heyting algebras	$\longleftrightarrow$	lattice filters
residuated lattices	$\longleftrightarrow$	lattice filters containing 1 and closed under fusion
modal algebras	$\longleftrightarrow$	open lattice filters
groups	$\longleftrightarrow$	normal subgroups
rings	$\longleftrightarrow$	two-sided ideals.

- The semantic description of algebraizability is also readily falsifiable, e.g. $\vdash^{l}_{\mathbf{K}}$ is not algebraizable!